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Mean-Variance Ratio Test, A Complement to Coefficient of Variation Test and Sharpe Ratio Test

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Abstract To circumvent the limitations of the tests for coefficients of variation and Sharpe ratios, we develop the mean-variance-ratio statistic to test the equality of the mean-variance ratios and prove that our proposed statistic is uniformly most powerful unbiased. In addition, we illustrate the applicability of our proposed test to compare the performances of stock indices.

Keywords:
Coefficient of variation, Sharpe ratio, mean-variance ratio; test statistic; hypothesis testing. uniformly most powerful unbiased test
JEL classification: C12; G11

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Mean-Variance Ratio Test, A Complement to Coefficient of Variation Test and Sharpe Ratio Test

1 Introduction and Motivation

The coefficient of variation (CV), the ratio of the standard deviation and the mean of a random variable, provides a measure of relative variability. On the other hand, the Sharpe ratio (SR), the ratio of the excess expected return to its standard deviation, provides a measure of relative excess return. Populations can have the same relative variability or the same relative excess return even if the means and variances of the variables of interest are different.

Miller and Karson (1977) present the likelihood ratio test for the equality of two CVs whereas Jobson and Korkie (1981) develop a Sharpe-ratio statistic to test for the equality of two SRs. The CV test statistic is very important in medical sciences, physical sciences, and social sciences whereas the SR test statistic is important in finance as it provides a formal statistical comparison for the performances among portfolios. However, as both the CV and SR statistics possess only the asymptotic distributions, one could only obtain their properties for large samples, but not for small samples. Nevertheless, it is important to compare the performances of variables by using small samples as sometimes large samples are not available. Also it is, sometimes, not so meaningful to measure CVs and SRs for too large samples as the means and standard deviations of the underlying variables could change when the samples become large (Hamilton and Susmel, 1994). The main obstacle to developing the CV and SR tests for small samples is that it is impossible to obtain a uniformly most powerful unbiased (UMPU) statistic to test the equality of CVs or SRs in case of small samples. To circumvent this problem, in this paper we propose to use the mean-variance ratio (MVR) test for the comparison.

The MVR has its own importance in many areas. We illustrate its importance to Markowitz optimal portfolio theory. Let us consider a random p-vector $\mathbf{r}$ with mean vector $\boldsymbol{\mu}$ and covariance matrix V, a positive definite matrix. The random vector $\mathbf{r}$ can be regarded as the returns of p branches of stocks. According to Markowitz optimal portfolio theory (Markowitz, 1952; Bai et al., 2009), investors' objective is to find the optimal portfolio $\mathbf{a} = (a_, \ldots, a_p)^T$ such that the overall expected return $\mathbf{a}^T\boldsymbol{\mu}$ reaches its maximum subject to a given risk value $\sigma_0^2 = \mathbf{a}^T V \mathbf{a}$ or, equivalently, the risk $\mathbf{a}^T V \mathbf{a}$ reaches its minimum subject to a proleptic return $R_0 = \mathbf{a}^T\boldsymbol{\mu}$. For both of the two problems, the solutions are $\hat{\mathbf{a}} = bV^{-1}\boldsymbol{\mu}$ where $b = \sigma_0/\sqrt{\boldsymbol{\mu}^T V^{-1}\boldsymbol{\mu}}$ and $R_0/\boldsymbol{\mu}^T V^{-1}\boldsymbol{\mu}$, respectively. When the components of $\mathbf{r}$ are independent and have means μ_i and variances σ_i^2, the optimal solution for each component is equal to $\hat{a}_i = b\mu_i/\sigma_i^2$. This shows that the optimal portfolio component is proportional to its mean-variance ratio, and thus, the MVR is useful in making investment decision.

In the remainder of the paper, we shall discuss the evaluation of MVRs for small samples by providing a theoretical framework and then invoking both one-sided and two-sided UMPU MVR tests.

2 The UMPU Test for Mean-Variance Ratio Ordering

Let X_i and Y_i $(i = 1, 2, \cdots, n)$ be independent variables or excess returns drawn from the corresponding normal distributions $N(\mu, \sigma^2)$ and $N(\eta, \tau^2)$ with joint density $p(x, y)$ such that

$$p(x,y) = k \times \exp(\frac{\mu}{\sigma^2}\sum x_i - \frac{1}{2\sigma^2}\sum x_i^2 + \frac{\eta}{\tau^2}\sum y_i - \frac{1}{2\tau^2}\sum y_i^2) \qquad (1)$$

where $k = (2\pi\sigma^2)^{-n/2}(2\pi\tau^2)^{-n/2}\exp(-\frac{n\mu^2}{2\sigma^2})\exp(-\frac{n\eta^2}{2\tau^2})$. To evaluate the performances of X and Y, practitioners and academicians are interested in testing the hypotheses

$$H_0^{\#}: \frac{\sigma}{\mu} \leq \frac{\tau}{\eta} \quad \text{versus} \quad H_1^{\#}: \frac{\sigma}{\mu} > \frac{\tau}{\eta} \qquad (2)$$

to compare the performances of their corresponding CVs, σ/μ and τ/η; or in testing the hypotheses

$$H_0^{*}: \frac{\mu}{\sigma} \leq \frac{\eta}{\tau} \quad \text{versus} \quad H_1^{*}: \frac{\mu}{\sigma} > \frac{\eta}{\tau} \qquad (3)$$

to compare the performances of their corresponding SRs, μ/σ and η/τ.

Miller and Karson (1977) develop test statistic to test the hypotheses in (2) for large samples whereas Jobson and Korkie (1981) develop test statistic to test the hypotheses in (3) for large samples but their tests are not appropriate for testing small samples as the distributions of their test statistics are only valid asymptotically. However, it is important to test the hypotheses in (3) for small samples but it is impossible to obtain any UMPU test statistic to test the inequality of the CVs in (2) or the SRs in (3) for small samples. To circumvent this problem, in this paper we propose to alter the hypothesis to test the inequality of the mean-variance ratios as shown in the following:

$$H_0 : \frac{\mu}{\sigma^2} \leq \frac{\eta}{\tau^2} \quad \text{versus} \quad H_{11} : \frac{\mu}{\sigma^2} > \frac{\eta}{\tau^2} \, . \tag{4}$$

Rejecting H_0 suggests X to possess higher mean-variance ratio (or better investment prospect in finance) as X possesses either smaller variance or bigger mean (or higher excess mean return in finance) or both. As, sometimes, academics and practitioners do conduct the two-sided test to compare the MVRs, to complete the theory we also consider the following hypotheses in this context:

$$H_0 : \frac{\mu}{\sigma^2} = \frac{\eta}{\tau^2} \quad \text{versus} \quad H_{12} : \frac{\mu}{\sigma^2} \neq \frac{\eta}{\tau^2} \, . \tag{5}$$

The beauty of using the CV (SR) is that it provides a scale-free measure of the relative variability (relative excess return). Here, we illustrate the beauty of using the MVR in finance. One could easily illustrate the beauty of applying the MVR in other areas like physical sciences. We note that in some financial processes, the mean change in a short period of time is proportional to its variance change. For example, many financial processes Y_t could follow $dY_t = \mu^P(Y_t)dt + \sigma(Y_t)dW_t^P$, where μ^P is an N-dimensional function, σ is an $N \times N$ matrix, and W_t^P is an N-dimensional standard Brownian motion under the objective probability measure P. When $N = 1$, the SR will be close to 0 while the MVR will be independent of dt. Thus, when the time period dt is small, it is better to consider the MVR rather than the SR. In addition, one could easily modify the work from Bai et al. (2009) in the Markowitz mean-variance (MV) optimization (Markowitz, 1952) to show that the investment allocation is proportional to the MVR when the covariance matrix of stock returns is a diagonal matrix. Hence, when the asset is concluded to be superior in performance by utilizing the MVR test, its corresponding weight could then be computed based on the corresponding MVR test value, but not on the SR value. In this sense, our proposed test is better.

We first develop the one-sided UMPU test to test the hypothesis in (4):

Theorem 1 *Let X_i and Y_i $(i = 1, 2, \cdots, n)$ be independent random variables with joint distribution function defined in (1). For the hypotheses setup in (4), there exists a UMPU level-α test with the critical function $\phi(u, t)$ such that*

$$\phi(u,t) = \begin{cases} 1, & \text{when } u \geq C_0(t) \\ 0, & \text{when } u < C_0(t) \end{cases} \tag{6}$$

where C_0 is determined by

$$\int_{C_0}^{\infty} f^*_{n,t}(u)\ du = K_1\,; \tag{7}$$

with $f^*_{n,t}(u) = (t_2 - \frac{u^2}{n})^{\frac{n-1}{2}-1}(t_3 - \frac{(t_1-u)^2}{n})^{\frac{n-1}{2}-1}$ and $K_1 = \alpha \int_{\Omega} f^*_{n,t}(u)\ du$, in which $U = \sum_{i=1}^n X_i$, $T_1 = \sum_{i=1}^n X_i + \sum_{i=1}^n Y_i$, $T_2 = \sum_{i=1}^n X_i^2$, $T_3 = \sum_{i=1}^n Y_i^2$, and $T = (T_1, T_2, T_3)$; with $\Omega = \{u \mid \max(-\sqrt{nt_2}, t_1 - \sqrt{nt_3}) \leq u \leq \min(\sqrt{nt_2}, t_1 + \sqrt{nt_3})\}$ to be the support of the joint density function of (U, T).

Next, we introduce the two-sided UMPU test statistic as stated in the following theorem to test the equality of the MVRs listed in (5):

Theorem 2 *Let X_i and Y_i $(i = 1, 2, \cdots, n)$ be independent random variables with joint distribution function defined in (1). Then, for the hypotheses setup in (5), there exists a UMPU level-α test with critical function*

$$\phi(u,t) = \begin{cases} 1, & \text{when } u \leq C_1(t) \text{ or } \geq C_2(t) \\ 0, & \text{when } C_1(t) < u < C_2(t) \end{cases} \tag{8}$$

in which C_1 and C_2 satisfy

$$\begin{cases} \int_{C_1}^{C_2} f^*_{n,t}(u)\ du = K_2 \\ \int_{C_1}^{C_2} u f^*_{n,t}(u)\ du = K_3 \end{cases} \tag{9}$$

where $K_2 = (1-\alpha)\int_\Omega f^*_{n,t}(u)\ du$ *and* $K_3 = (1-\alpha)\int_\Omega u f^*_{n,t}(u)\ du$. *The terms* $f^*_{n,t}(u)$, T_i *(*$i = 1, 2, 3$*) and* T *are defined in Theorem 1.*

We call the statistic U in Theorem 1 (Theorem 2) to be the one-sided (two-sided) MVR test statistic for the hypotheses setup in (4) ((5)). One could easily apply numerical methods to look for the numerical solutions to equations in (9) as stated in the following problem:

Problem 3 *To compute the values of the constants* C_1 *and* C_2 *in* $\Omega = [I_d, I_u]$ *such that*

$$\int_{C_1}^{C_2} f^*_{n,t}(u)\ du = K_2 \tag{10}$$

and

$$\int_{C_1}^{C_2} u f^*_{n,t}(u)\ du = K_3 \tag{11}$$

where $f^*_{n,t}(u) = (t_2 - \frac{u^2}{n})^{\frac{n-1}{2}-1}(t_3 - \frac{(t_1-u)^2}{n})^{\frac{n-1}{2}-1}$, $K_2 = (1-\alpha)\int_\Omega f^*_{n,t}(u)\ du$, *and* $K_3 = (1-\alpha)\int_\Omega u f^*_{n,t}(u)\ du$.

3 Illustration

In this section, we demonstrate the superiority of the mean-variance ratio test developed in this paper over the traditional Sharpe ratio test by illustrating the applicability of our test to the decision making process of investing in Shanghai SE Composite (China) and Australia All Ordinary Index (Australia).[1] For simplicity, we only demonstrate the two-sided UMPU test. The data analyzed in this section are the monthly returns of China and Australia for the sample period from January 2003 to December 2007 in which the data from January 2005 to December 2005 are used to compute the mean-variance ratio in January 2006, while the data from February 2005 to January 2006 are used to compute the mean-variance ratio in February 2006, and so on.[2] However, using too short periods to compute the Sharpe ratio would not

[1] Similarly, one could conduct the CV test for comparison. We use SR test rather than the CV test here because it is more common for investors to use SR test in finance.

[2] For this period, the p-values of the Jarque-Bera test are 0.7102 and 0.1063 for China and Australia, respectively, and the p-values of the Ljung-Box test for lag 6 and 12 are 0.5786 and 0.8843

be meaningful as discussed in our previous sections. Thus, we utilize a longer period from January 2003 to December 2005 to compute the Sharpe ratio in January 2006, from February 2003 to January 2006 to compute the in February 2006, and so on.[3]

Let X_t and Y_t be the returns of China and Australia at time t with means μ and η and variances σ^2 and τ^2, respectively. We test the hypothesis in (5). To test the hypothesis, we first compute the values of the test function U for the mean-variance ratio statistic as shown in Theorem 2 and display the values in Table 1. We then compute the corresponding critical values C_1 and C_2 under the test levels of 5% and 10% to test the hypothesis. In addition, in order to illustrate the performances of the indices and the corresponding test results visually, we exhibit the returns of the two indices and their difference in Figure 1A, and display their corresponding values of U with C_1 and C_2 in Figure 1B.

For comparison, we also compute the corresponding Sharpe ratio statistic, see, e.g. Leung and Wong (2008), such that

$$z_i = \frac{\hat{\tau}\hat{\mu} - \hat{\sigma}\hat{\eta}}{\sqrt{\hat{\theta}}} \tag{12}$$

which follows standard normal distribution asymptotically with

$$\theta = \frac{1}{T}\left[2\sigma^2\tau^2 - 2\sigma\tau Cov(X_t, Y_t) + \frac{1}{2}\mu^2\tau^2 + \frac{1}{2}\eta^2\sigma^2 - \frac{\mu\eta}{\sigma\tau}Cov(X_t, Y_t)^2\right]$$

to test the equality of the Sharpe ratios with the hypothesis in (5). Different from using one-year data to compute the values of our proposed statistic, we use the overlapping three-year data to compute the Sharpe ratio statistic as discussed before.

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for China, and 0.7347 and 0.6701 for Australia, respectively. These findings lead us not to reject the data are drawn from iid normal distribution.

[3]For this period, the p-values of the Jarque-Bera test are 0.4426 and 0.0117 for China and Australia, respectively, and the p-values of the Ljung-Box test for lag 6 and 12 are 0.5330 and 0.6147 for China, and 0.0780 and 0.1252 for Australia, respectively. The results lead us reject the China data are drawn from normal distribution at the 5% level and reject the Australia data are independent at the 10% level. This findings support our claim as discussed in the Introduction section and Conclusion section that large samples may not be suitable to be used.

The results are also reported in Table 1 next to the results for our proposed statistic while theirs plots are depicted in Figure 1C.

As shown in Table 1 and Figure 1C, we cannot detect any significant difference between the Sharpe ratios, implying that the performances of these two indices are indistinguishable. We note that the three-year monthly data being used to compute the Sharpe ratio statistic could be covering too short a period to satisfy the asymptotic statistical properties for the test but, still, we cannot find any significant difference between the performances of these two indices. If we use any longer period, the result is expected to be insignificant as the high means in some sub-periods could be offset by the low means in other sub-periods. Thus, a possible limitation of applying the Sharpe ratio test is that it would usually conclude indistinguishable performances among the indices, which may not be the situation in reality. In this aspect, looking for a statistic to evaluate the performances among assets for short periods is essential. In this paper, we adopt our proposed statistic to conduct the analysis. As shown in Table 1 and Figure 1B, we find that our proposed statistic does not disappoint us as it does show some significant differences in performances between these two indices. This information could be useful to investors for their decision making process.

4 Concluding Remarks

The limitation of the MVR test provided in this paper is that the test requires iid and normality assumptions for the underlying variables being examined. The MVR test could be used in medical sciences, physical sciences, social sciences, finance, and many other areas. Since it is not difficult to obtain iid observations following normality for data in medical sciences, physical sciences, social sciences, and many other areas, it is not a problem to apply the MVR test proposed in this paper to these areas.

Nonetheless, it is common that observations from economics and finance are not iid and they do not satisfy normality assumption either. Thus, one may doubt the applicability of the MVR test in economics and finance. Nevertheless, we note that it is still possible to obtain iid observations following normality assumption in economics and finance, especially for weekly data, monthly data, and annual data. For example, it is well-known that some daily (stock or exchange rate) returns may not be white noise but contain a weak lag-one autocorrelation and thus they may not be iid. In addition, it is well-known that most daily (stock or exchange rate) returns are not normally distributed. However, in many situations, daily observations are not available and only weekly, monthly, or annual observations are available, see Wong et al (2006) for example of these types of data. In addition, we note that academics recommend not to use daily data to avoid day-of-the-week effects, nonsynchronous trading, and noises that can result from using daily data, see, for example, Chen, et al (2009) and the references therein for more information. In this situation, their autocorrelations become insignificant and thus, very likely, they are not being rejected to be independent. In addition, by central limited theorem, weekly, monthly, and annual observations could follow normality assumption. Thus, it is possible to get financial data satisfying iid and not being rejected to be normally distributed.[4] This infers that the MVR test could still be used in economics and finance.

We also note that the MVR test statistic recommended in this paper is not only suitable for small samples but also for large samples. On the other hand, the CV and SR tests are only suitable for large samples but may not be suitable for small samples in many situations. In this sense, our proposed MVR test is better. We note that it is not uncommon that getting large samples in medical sciences, physical sciences, and social sciences could be expensive or even impossible and thus academics and practitioners could consider to apply the MVR test instead of employing the CV or SR tests.

However, it is not difficult and usually not expensive to obtain large samples in finance. One may thus argue that it is not necessary to apply the MVR test in finance. We note that in many situations, though it is possible to obtain large samples but one could only use small samples in their analysis. A simple example is during the financial crisis in 2008, many fund managers and policy makers would like to analyze how serious the market crash. In this situation, the data before 2008 are available but

[4]We have found many observations in finance which are not rejected to be iid and not rejected to be normally distributed either. We have displayed one example as shown in our Illustration section. Other results are available on request.

cannot be used as the nature of the data have changed since the crisis. Thus, only observations starting from the downturn could then be used in the analysis and it becomes a small sample. In this situation, the CV and SR tests are not recommended. Fund managers and policy makers may consider to apply our proposed MVR test for their analysis.

In addition, we note that there are two basic approaches to the problem of portfolio selection under uncertainty. One approach is based on the concept of expected utility maximization (see, e.g., Egozcue and Wong, 2010) and stochastic dominance (SD) test statistics (see, e.g., Linton et al., 2005). The other is the mean-risk (MR) analysis in which the portfolio choice is made with respect to two measures – the expected portfolio mean return and portfolio risk. A disadvantage of the latter approach is that it is derived by assuming the Von Neumann-Morgenstern quadratic utility function and that returns are normally distributed (Hanoch and Levy, 1969). However, Wong and Ma (2008) and others have shown that, under some regularity conditions, the conclusion drawn from the MR comparison could be equivalent to the comparison of expected utility maximization for any risk-averse investor, not necessarily with only quadratic utility function.

In addition, the major limitation of the MR and SD statistics is that up to now academics can only develop their asymptotic distributions, but not their distribution for small samples. Nonetheless, the MVR test developed in this paper circumvents their limitations. Thus, the test developed in this paper sets a milestone in the literature of financial economics as our test is the first test making such comparison possible. Further study may extend the test developed in our paper to non-normal distribution, see for example, Bai and Guo (1999), Bai and Chow (2000), and Wong and Bian (2005) for more information on the non-normality.

Table 1: The results of the mean-variance ratio test and Sharpe ratio test for China versus Australia from 2006 to 2007

China V.S. Australia							
	Mean-Variance Ratio Test					Sharpe Ratio Test	
		$\alpha = 5\%$		$\alpha = 10\%$			
Time	U	C_1	C_2	C_1^*	C_2^*	Z	p-value
Jan-06	-0.0869	-0.1311	0.2382	-0.1058	0.211	-0.9890	0.3227
Feb-06	0.0541	-0.0043	0.3722	0.0241	0.3457	-1.0527	0.2925
Mar-06	-0.0054**	-0.0633	0.3081	-0.0370	0.2808	-1.0852	0.2778
Apr-06	0.0945	0.0438	0.3751	0.0687	0.3519	-1.0868	0.2771
May-06	0.2171*	0.2168	0.5094	0.2421	0.4918	-1.0349	0.3007
Jun-06	0.4365	0.3337	0.6071	0.3569	0.5910	-0.8973	0.3696
Jul-06	0.4363	0.3275	0.6015	0.3445	0.5793	-0.8400	0.4009
Aug-06	0.3982	0.2660	0.5730	0.2872	0.5503	-0.8209	0.4117
Sep-06	0.3552	0.2289	0.5401	0.2487	0.5141	-0.7679	0.4425
Oct-06	0.4164	0.2723	0.5673	0.2926	0.5441	-0.7112	0.4770
Nov-06	0.5199	0.4307	0.6427	0.4395	0.6216	-0.6764	0.4988
Dec-06	0.6470	0.5553	0.7685	0.5672	0.7501	-0.7066	0.4798
Jan-07	0.8348	0.7537	1.0284	0.7750	1.0097	-0.6796	0.4968
Feb-07	0.7952	0.7160	0.9949	0.7352	0.9729	-0.7301	0.4653
Mar-07	0.7965	0.7245	0.9998	0.7434	0.9782	-0.7223	0.4701
Apr-07	0.8971	0.8197	1.0698	0.8379	1.0516	-0.7111	0.4770
May-07	0.9810	0.9063	1.1695	0.9256	1.1501	-0.6167	0.5374
Jun-07	0.9178**	0.9307	1.1513	0.9433	1.1312	-0.5792	0.5625
Jul-07	0.8263**	0.8367	1.1056	0.8560	1.0849	-0.5102	0.6099
Aug-07	1.0197*	1.0161	1.2565	1.0290	1.2330	-0.3741	0.7083
Sep-07	1.1463	1.1348	1.3626	1.1448	1.3383	-0.2923	0.7701
Oct-07	1.1532*	1.1384	1.3722	1.1572	1.3578	-0.2911	0.7710
Nov-07	1.1755*	1.1646	1.3880	1.1806	1.3721	-0.2038	0.8385
Dec-07	0.8419	0.8159	1.1562	0.8404	1.1300	-0.3015	0.7630

** $p < 5\%$ and * $p < 10\%$, the MVR Test U and the corresponding critical values C_1 and C_2 (C_1^* and C_2^*) are defined in while the SR Test Z is defined in (12).

Appendix

Proof of Theorem 1

Let

$$\theta = \frac{\mu}{\sigma^2} - \frac{\eta}{\tau^2}, \quad \vartheta_1 = \frac{\eta}{\tau^2}, \quad \vartheta_2 = -\frac{1}{2\sigma^2}, \quad \text{and} \quad \vartheta_3 = -\frac{1}{2\tau^2}. \tag{13}$$

The joint density function of (X, Y) in (1) becomes:

$$\begin{aligned} p(x,y) &= (2\pi\sigma^2)^{-n/2}(2\pi\tau^2)^{-n/2} \exp(-\frac{n\mu^2}{2\sigma^2}) \exp(-\frac{n\eta^2}{2\tau^2}) \\ &\quad \times \exp(\theta u + \vartheta_1 t_1 + \vartheta_2 t_2 + \vartheta_3 t_3)\,, \end{aligned}$$

and the hypotheses in (4) become $H_0 : \theta \leq 0$ versus $H_1 : \ \theta > 0$. Apply the theorem in Lehmann (1986, pp. 145-149), the critical function

$$\phi(u,t) = \begin{cases} 1, & \text{when } u \geq C_0(t) \\ 0, & \text{when } u < C_0(t) \end{cases} \tag{14}$$

is the UMPU test where C_0 is determined by

$$\int_{C_0(t)}^{\infty} f(u|t, \theta = 0)\; du = \alpha\,. \tag{15}$$

The value C_0 can be solved and thereafter the critical function can be obtained provided that we know the conditional distribution $f(u|t)$. Thus, we have to construct the conditional distribution. To do that, we first let $Z_1 = \frac{1}{n}\sum X_i$, $Z_2 = \frac{1}{n}\sum Y_i$, $Z_3 = \frac{\sum(X_i - \bar{X})^2}{\sigma^2}$, and $Z_4 = \frac{\sum(Y_i - \bar{Y})^2}{\tau^2}$, and define $U = nZ_1$, $T_1 = nZ_1 + nZ_2$, $T_2 = \sigma^2 Z_3 + nZ_1^2$, and $T_3 = \tau^2 Z_4 + nZ_2^2$, one could easily show that the joint density function of (U, T_1, T_2, T_3) can be expressed as:

$$\begin{aligned} f(u,t) &= \frac{1}{\sqrt{2\pi}\frac{\sigma}{\sqrt{n}}\sqrt{2\pi}\frac{\tau}{\sqrt{n}}2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}(\frac{t_2 - \frac{u^2}{n}}{\sigma^2})^{\frac{n-1}{2}-1} \\ &\quad \times \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}(\frac{t_3 - \frac{(t_1-u)^2}{n}}{\tau^2})^{\frac{n-1}{2}-1}\frac{1}{n^2\sigma^2\tau^2} \times e^{-\frac{n\mu^2}{2\sigma^2} - \frac{n\eta^2}{2\tau^2}} e^{\theta u + \vartheta_1 t_1 + \vartheta_2 t_2 + \vartheta_3 t_3} \end{aligned} \tag{16}$$

with $\big(-\sqrt{nt_2} \leq u \leq \sqrt{nt_2}\big)$ and $\big[(t_1 - \sqrt{nt_3}) \leq u \leq (t_1 + \sqrt{nt_3})\big]$.

Since

$$f(u|t) = \frac{f(u,t)}{f(t)} \tag{17}$$

where $f(t) = \int_\Omega f(u,t)\ du$, Equation (15) is equivalent to:

$$\frac{\int_{C_0(t)}^{\infty} f(u,t|\theta=0)\ du}{\int_\Omega f(u,t|\theta=0)du} = \alpha. \tag{18}$$

Equation (18) can further be simplified into (7) in which for every fixed t and n, the value of C_0 can then be determined by $\int_{C_0}^{\infty} f_{n,t}^*(u)\ du = K_1$ where $f_{n,t}^*(u) = (t_2 - \frac{u^2}{n})^{\frac{n-1}{2}-1}(t_3 - \frac{(t_1-u)^2}{n})^{\frac{n-1}{2}-1}$ and $K_1 = \alpha \int_\Omega f_{n,t}^*(u)\ du$. □

Proof of Theorem 2

Using the transformation in (13), the hypotheses setup in (5) become $H_0 : \theta = 0$ versus $H_1 :\ \theta \neq 0$. Applying the theorem in Lehmann (1986, pp. 145-149), we obtain the critical function

$$\phi(u,t) = \begin{cases} 1 & \text{when } u \leq C_1(t) \text{ or } \geq C_2(t) \\ 0 & \text{when } C_1(t) < u < C_2(t) \end{cases}$$

which is the UMPU test with C's determined by

$$E_{\theta=0}[\phi(U,T) \mid t] = \alpha \quad \text{and} \quad E_{\theta=0}[U\phi(U,T) \mid t] = \alpha E_{\theta=0}[U \mid t]\,.$$

Making use of the joint density function $f(u,t)$ in (16) and the conditional density function $f(u|t)$ in (17) derived in the proof of Theorem 1, the critical function then becomes:

$$\phi(u,t) = \begin{cases} 1, & \text{when } u \leq C_1(t) \text{ or } \geq C_2(t) \\ 0, & \text{when } C_1(t) < u < C_2(t) \end{cases}$$

where

$$\int_{C_1(t)}^{C_2(t)} f_{H_0}(u|t)\ du = 1-\alpha, \quad \text{and} \quad \int_{C_1(t)}^{C_2(t)} u f_{H_0}(u|t)\ du = (1-\alpha)\int_\Omega u f_{H_0}(u|t)\ du\,. \tag{19}$$

In addition, equations in (19) are equivalent to the following:

$$\frac{\int_{C_1(t)}^{C_2(t)} f_{H_0}(u,t)\ du}{\int_\Omega f_{H_0}(u,t)\ du} = 1-\alpha\,, \quad \text{and} \quad \int_{C_1(t)}^{C_2(t)} u f_{H_0}(u,t)\ du = (1-\alpha)\int_\Omega u f_{H_0}(u,t)\ du\,. \tag{20}$$

Under H_0 (that is $\theta = 0$), equations in (20) can be further simplified as follows: For every fixed t and for each n, the values of (C_1, C_2) can be determined by

$$\begin{cases} \int_{C_1}^{C_2} f_{n,t}^*(u)\ du = K_2 \\ \int_{C_1}^{C_2} u f_{n,t}^*(u)\ du = K_3 \end{cases} \tag{21}$$

where $K_2 = (1-\alpha)\int_\Omega f_{n,t}^*(u)\ du$ and $K_3 = (1-\alpha)\int_\Omega u f_{n,t}^*(u)\ du$. □

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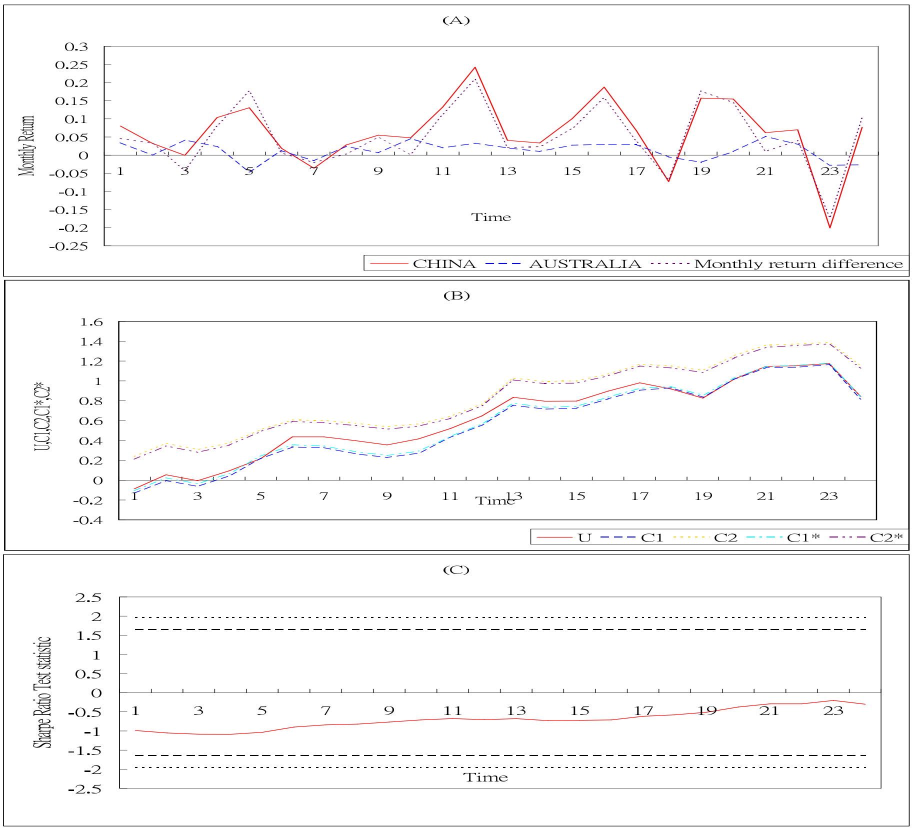
Figure 1: Plots of the monthly returns for China and Australia and corresponding mean-variance ratio test U and Sharpe ratio test statistic Z



The dotted lines in Figures 1B and 1C are critical values.

Note: The mean-variance ratio test statistic U is defined in Theorem 2 with C_1 C_1^*, C_2, and C_2^* defined in (9) and the Sharpe ratio test statistic Z is defined in (12). C_1 and C_2 are critical values of the 5% level while C_1^* and C_2^* are critical values of the 10% level.