2016

On generating discrete integrable systems via Lie algebras and commutator equations

Yu-Feng Zhang

Hon Wah Tam
Hong Kong Baptist University, tam@comp.hkbu.edu.hk

Citation
Zhang, Yu-Feng, and Hon Wah Tam. "On generating discrete integrable systems via Lie algebras and commutator equations." *Communications in Theoretical Physics* 65.3 (2016): 335-340.

On generating discrete integrable systems via Lie algebras and commutator equations

Yufeng Zhang[1] and Honwah Tam[2]

[1] College of Sciences, China University of Mining and Technology, Xuzhou 221116, P. R. China;

[2] Department of Computer Science, Hong Kong Baptist University, Hong Kong, P. R. China.

Abstract

In the paper, we introduced the Lie algebras and the commutator equations to rewrite the Tu-d scheme for generating discrete integrable systems regularly. By the approach the various loop algebras of the Lie algebra A_1 were defined so that the well-known Toda hierarchy and a novel discrete integrable system were obtained, respectively. A reduction of the later hierarchy is just right the famous Ablowitz-Ladik hierarchy. Finally, via two different enlarging Lie algebras of the Lie algebra A_1, we derived two resulting differential-difference integrable couplings of the Toda hierarchy, of course, they are all various discrete expanding integrable models of the Toda hierarchy. When the introduced spectral matrices are higher degrees, the way presented in the paper is more convenient to generate discrete integrable equations than the Tu-d scheme by using the software Maple.

Keywords: discrte integrable system, Lie algebra, integrable coupling

PACS Numbers: 05.45.Yv, 02.30.Jr, 02.30.Ik

1 Introduction

In recent years, search for discrete integrable systems and their solutions, symmetries, Hamiltonian structures, Bäcklund transformations, conservation laws, and so on, has made rapidly developed[1-16]. A common approach for generating discrete integrable systems usually starts from the following discrete spectral problem

$$E\psi = U\psi, \tag{1}$$

where $\psi = (\psi_1, \ldots, \psi_N)^T$ is an N-vector and $U = U(u, t, \lambda)$ is an $N \times N$ matrix which is dependent of a filed vector $u = (u_1, \ldots, u_p)^T$, the time variable t and a spectral parameter λ, and $Ef(n,t) = f(n+1,t)$. To generate differential-difference integrable systems, a $t-$evolution part corresponding to Eq. (1) is introduced for some matrix V as follows

$$\psi_t = V\psi. \tag{2}$$

The compatibility condition of Eqs. (1) and (2) gives rise to a differential-difference equation

$$U_t = (EV)U - UV \tag{3}$$

which is called a discrete zero-curvature equation. In terms of the scheme called the Tu-d scheme[3], we should need introducing a modified term Δ for V if necessary, and denote by $V_{(n)} = V + \Delta$ so that the discrete zero-curvature equation

$$U_{t_n} = (EV_{(n)})U - UV_{(n)} \tag{4}$$

leads to novel integrable discrete hierarchies. It is easy to see that Eq. (4) is the compatibility condition of the Lax pair

$$E\psi = U\psi, \psi_{t_n} = V_{(n)}\psi. \tag{5}$$

Compared with the Tu scheme[17], there is no commutator in the discrete zero-curvature equations (3) or (4). If we could construct a commutator appearing in (4), then we would follow the very familiar Tu scheme to generate discrete integrable systems, that is, we could imitate the well-known ideas of Tu scheme to investigate discrete integrable systems. An obvious difference between the Tu scheme and the Tu-d scheme reads that we could regularly construct the U and the V in the Lax pair (1) and (2) through Lie algebras. That is to say, set G to be a Lie algebra, and $\{e_1, \ldots, e_p\}$ is a basis of G. Assume again $\tilde{G} = G \otimes C([\lambda, \lambda^{-1}]$, where $C[\lambda, \lambda^{-1}]$ represents the set of Laurent polynomials in λ. A basis of $\tilde{G}$ is denoted by $\{e_1(n), \ldots, e_p(n), n \in \mathbf{Z}\}$. An element $R \in \tilde{G}$ is called pseudo-regular if for
$ker\, ad\, R = \{x | x \in \tilde{G}, [x, R] = 0\}$,
$Im\, ad\, R = \{x | \exists y \in \tilde{G}, x = [y, R]\}$,
it holds that

$$\tilde{G} = ker\, ad\, R \oplus Im\, ad\, R,$$

and $ker\, ad\, R$ is commutative. In addition, we define gradations of $\tilde{G}$ to be as follows

$$deg\,(X \otimes \lambda^n) = n, X \in \tilde{G}.$$

Assume

$$U = R + u_1 e_1 + \ldots + u_p e_p,$$

where $u_i (i = 1, \ldots, p)$ are component potential functions of the function $u = (u_1, \ldots, u_p)^T$. Denote $\alpha = deg(R), \epsilon_i = deg(e_i), i = 1, \ldots, p$. If α and ϵ_i satisfy

$$\alpha > 0, \alpha > \epsilon_i (i = 1, \ldots, p), \tag{6}$$

then the stationary zero-curvature equation

$$V_x = [U, V] \tag{7}$$

could have local solutions for the given spectral matrix U. Thus, under introducing the modified term Δ of $(\lambda^n V)_+$ which is denoted by $V^{(n)} = (\lambda^n V)_+ + \Delta$, the continuous zero-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{8}$$

generally could give rise to integrable hierarchies of evolution equations. Therefore, (6) is a guidance clue to construct U and V in (6) and (7) so that (7) could have differential solutions for V, and (8) could acquire integrable ideal equations. In order to make the Tu-d scheme match the Tu scheme as possible, we rewrite the Tu-d scheme according to the Tu scheme. First of all, a proper Lie algebra G and its loop algebra $\tilde{G}$ are introduced. Second, we apply the $\tilde{G}$ to introduce U and V in Eqs. (1) and (2). Third, we solve a stationary zero-curvature equation similar to Eq. (7):

$$(\Delta V)U = [U, V], \tag{9}$$

where $\Delta = E - 1, [U, V] = UV - VU$.
Taking a modified term Δ_n for $(V\lambda^n)_+$, denoted by $V_{(n)} = (V\lambda^n)_+ + \Delta_n$, the discrte zero-curvature equation

$$U_{t_n} = (\Delta V_{(n)})U - [U, V_{(n)}] \tag{10}$$

could lead to novel differential-difference equations. Actually, Eq. (10) is a rewritten formula of Eq. (4). Finally, with the help of the discrete trace identity proposed by Tu[3], we can derive the Hamiltonian structure of Eq. (10). In what follows, we shall apply the above version to some explicit applications by introducing two various loop algebras of a Lie algebra.

2 Generating two discrete integrable hierarchies

The simplest basis of the Lie algebra A_1 reads
$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$
whith the commutative relations as follows

$$[h_1, e] = e, [h_1, f] = -f, [h_2, e] = -e, [h_2, f] = f, [e, f] = h \equiv h_1 - h_2. \tag{11}$$

It follows from (11) that

$$\begin{aligned} &[h, e] = 2e, [h, f] = -2f, h_1h_1 = h_1, h_2h_2 = h_2, h_1h_2 = h_2h_1 = ee = ff = 0, h_1e = e, eh_1 = 0, \\ &h_1f = 0, fh_1 = f, h_2f = f, fh_2 = 0, h_2e = 0, eh_2 = e, ef = h_1, fe = h_2. \end{aligned} \tag{12}$$

We denote by G the above Lie algebra, that is,

$$G = \mathrm{span}\{h_1, h_2, e, f\}$$

equipped with the commutative relations (11) and (12).

2.1: A loop algebra of the Lie algebra G and the Toda hierarchy

A loop algebra of the Lie algebra G is the well-known form as follows

$$\tilde{G}_1 = \mathrm{span}\{h_1(n), h_2(n), e(n), f(n)\},$$

along with degrees $deg\, h_i(n) = deg\, e(n) = deg\, f(n) = n, i = 1, 2,$
where $h_i(n) = h_i\lambda^n, e(n) = e\lambda^n, f(n) = f\lambda^n, i = 1, 2; n \in \mathbf{Z}$.
We consider an isospectral problem by using $\tilde{G}_1$

$$\begin{cases} E\varphi = U\varphi, U = h_2(1) - ph_2(0) + e(0) - vf(0), \\ \varphi_t = \Gamma\varphi, \Gamma = \sum\limits_{n\geq 0}(a_n h_1(-n) - a_n h_2(-n) + b_n e(-n) + c_n f(-n)). \end{cases} \tag{13}$$

A set of solutions to Eq.(9) for V is given by

$$\begin{cases} c_n = -vb_n^{(1)}, \\ a_n^{(1)} + a_n + b_{n+1}^{(1)} - pb_n^{(1)} = 0, \\ va_n^{(1)} = c_{n+1} - va_n - pc_n, \\ c_n^{(1)} - a_{n+1}^{(1)} + a_{n+1} + p(a_n^{(1)} - a_n) = -vb_n, \end{cases} \tag{14}$$

where $x_n^{(i)} = E^i x_n, x = a, b, c; i = 1, 2, \ldots.$
Denote by
$\Gamma_+ = \sum\limits_{n=0}^{m}(a_n(h_1(m-n) - h_2(m-n)) + b_n e(m-n) + c_n f(m-n)) = \lambda^m\Gamma - \Gamma_-,$
then the stationary zero-curvature equation

$$(\Delta\Gamma)U = [U, \Gamma]$$

can be decomposed into the following form

$$(\Delta\Gamma_+)U - [U, \Gamma_+] = -(\Delta\Gamma_-)U + [U, \Gamma_-]. \tag{15}$$

We observe that the left-hand side of (15) contains terms with degree more than 0, while the right-hand side contains terms with degree less than 0. Hence both sides of (15) contain only terms with degree being 0. Therefore, we have

$$(\Delta\Gamma_+)U - [U, \Gamma_+] = (-b_{m+1}^{(1)} + b_{m+1})e(0) + (a_{m+1}^{(1)} - a_{m+1})h_2(0) + c_{m+1}f(0).$$

Denoting by $V_{(m)} = \Gamma_+ + \Delta_m, \Delta_m = b_{m+1}h_1(0)$, a direct calculation acquires

$$(\Delta V_{(m)})U - [U, V_{(m)}] = (\Delta a_{m+1})h_2(0) + (c_{m+1} + vb_{m+1})f(0).$$

Thus, the discrete zero-curvature equation (10) permits the lattice hierarchy

$$\begin{cases} p_{t_m} = -\Delta a_{m+1}, \\ v_{t_m} = -c_{m+1} - vb_{m+1} = vb_{m+1}^{(1)} - vb_{m+1}, \end{cases} \tag{16}$$

which is completely consistent with that in [3], the well-known Toda hierarchy.
Remark 1: It is easy to find that the pseduo-regular element in (13) is $h_2(1)$, whose degree reads $\deg h_2(1) = 1$ which is more than other elements, satisfying the condition (6). In addition, Eq. (15) is similar to the decomposed equation in the Tu scheme

$$(-\lambda^n V)_{+x} + [U, (\lambda^n V)_+] = (\lambda^n V)_{-x} - [U, (\lambda^n V)_-]. \tag{17}$$

The above steps for computing the lattice hierarchy (16) are completely same with that by the Tu scheme, which hints that we could imitate all thoughts of Tu scheme to generate lattice hierarchies by introducing various Lie algebras and their resulting loop algebras.

2.2: Another loop algebra of the Lie algebra G and a new lattice hierarchy

Another loop algebra of the Lie algebra G is defined as

$$\tilde{G}_2 = \text{span}\{h_1(n), h_2(n), e(n), f(n)\},$$

where $\deg h_i(n) = \deg e(n) = \deg f(n) = 2n + p, i = 1, 2; p = -1, 0$. An explicit loop algebra still denoted by$\tilde{G}_2$ satisfying the above requirements is given by

$$\tilde{G}_2 = \text{span}\{h_i(n) = h_i\lambda^{2n}, e(n) = e\lambda^{2n-1}, f(n) = f\lambda^{2n-1}, i = 1, 2\},$$

which possesses the following operating relations
$h_i(m)h_i(n) = h_i(m+n), i = 1, 2;$
$h_i(m)h_j(n) = e(m)e(n) = f(m)f(n) = 0, i \neq j;$
$h_1(m)e(n) = e(m+n), e(m)h_1(n) = h_1(m)f(n) = 0, f(m)h_1(n) = f(m+n) = h_2(m)f(m),$
$f(m)h_2(n) = h_2(m)e(n) = 0, e(m)h_2(n) = e(m+n), e(m)f(n) = h_1(m+n-1),$
$f(m)e(n) = h_2(m+n-1), [h_1(m), e(n)] = e(m+n), [h_1(m), f(n)] = -f(m+n),$
$[h_2(m), e(n)] = -e(m+n), [h_2(m), f(n)] = f(m+n),$
from the above appearances we can derive that
$[h(m), e(n)] = 2e(m+n), [h(m), f(n)] = -2f(m+n),$
$[e(m), f(n)] = h(m+n-1) = h_1(m+n-1) - h_2(m+n-1), mn \in \mathbf{Z}.$
Obviously, the loop algebra $\tilde{G}_2$ is different from the previous $\tilde{G}_1$. In the following, we shall apply the loop algebra $\tilde{G}_2$ to investigate a discrete integrable hierarchy.
Set

$$\begin{cases} U = h_1(1) + qh_2(0) + re(1) + sf(1), \\ \Gamma = \sum\limits_{n\geq 0} [a_n(h_1(-n) - h_2(-n)) + b_n e(-n) + c_n f(-n)]. \end{cases} \tag{18}$$

A direct calculation according to the stationary discrete equation

$$(\Delta\Gamma)U - [U, \Gamma] = 0 \tag{19}$$

gives that

$$\begin{cases} \Delta a_{n+1} + sb_n^{(1)} = rc_n, \\ q\Delta a_n = rc_n^{(1)} - sb_n, \\ r(a_{n+1}^{(1)} + a_{n+1}) = b_{n+1} - qb_n^{(1)}, \\ c_{n+1}^{(1)} - s(a_{n+1}^{(1)} + a_{n+1}) = qc_n. \end{cases} \tag{20}$$

The first equation in (20) can be derived from other three ones. In fact, we have $q(a_{n+1}^{(1)}+a_{n+1}) = rc_{n+1}^{(1)} - sb_{n+1} = r[qc_n + s(a_{n+1}^{(1)}+a_{n+1})] - s[qb_n^{(1)} + r(a_{n+1}^{(1)}+a_{n+1})] = qrc_n - qsb_n^{(1)}$.

Denoting by

$$\Gamma_+ = \sum_{n=0}^{m}[a_n h(-n) + b_n e(-n) + c_n f(-n)]\lambda^{2m} = \lambda^{2m}\Gamma - \Gamma_-,$$

then Eq. (19) can be decomposed into

$$(\Delta\Gamma_+)U - [U,\Gamma_+] = -(\Delta\Gamma_-)U + [U,\Gamma_-]. \tag{21}$$

The degrees of the left-hand side of (21) are more than -1, while the right-hand side less than 0. Therefore, both sides should be $-1, 0$. Thus, Eq. (21) permits that $(\Delta\Gamma_+)U - [U,\Gamma_+] = -(\Delta a_{m+1})h_1(0) + [b_{m+1} - 2ra_{m+1} - (\Delta a_{m+1})r]e(0) + [-c_{m+1} + 2sa_{m+1} + s(\Delta a_{m+1}) - \Delta c_{m+1}]f(0)$.

We take a modified term $\Delta_m = \delta h_2(0) - b_m e(0) - c_m f(0)$, and denote by $V_{(m)} = \Gamma_+ + \Delta_m$, it can be computed that

$(\Delta V_{(m)})U - [U, V_{(m)}] = (q\Delta\delta - rc_m^{(1)} + sb_m)h_2(0) + (b_m - r\delta)e(1) + (s\Delta\delta - c_m^{(1)} + s\delta)f(1)$.

Therefore, Eq. (10) admits that

$$\begin{cases} q_{t_m} = q\Delta\delta - rc_m^{(1)} + sb_m, \\ r_{t_m} = b_m - r\delta, \\ s_{t_m} = s\delta^{(1)} - c_m^{(1)}, \end{cases} \tag{22}$$

where δ is an arbitrary function with respect to m, t. Some reductions of Eq. (22) can be considered.

Case 1: Taking $q = 0, \delta = a_m$, then (22) reduces to the famous Ablowtiz-Ladik hierarchy

$$\begin{cases} r_{t_m} = b_m - ra_m, \\ s_{t_m} = sa_m - c_m^{(1)}. \end{cases}$$

Case 2: Taking $\delta = 0$, then (22) becomes

$$\begin{cases} q_{t_m} = -q\Delta a_m, \\ r_{t_m} = b_m, \\ s_{t_m} = -c_m^{(1)}, \end{cases}$$

which is just the main result in Ref.[11].

Case 3: Taking $\delta = a_m^{(1)} + a_m$, then (22) can reduce to

$$\begin{cases} q_{t_m} = 2qa_m, \\ r_{t_m} = qb_{m-1}^{(1)}, \\ s_{t_m} = -c_m^{(1)} + s(a_m^{(1)} + a_m^{(2)}). \end{cases} \tag{23}$$

Again set $q = 0$, Eq. (23) reduces to a new lattice hierarchy:

$$s_{t_m} = -c_m^{(1)} + s(a_m^{(1)} + a_m^{(2)}).$$

As similar to the case where the Hamiltonian structure of the Toda hierarchy (16) was derived from the discrete trace identity in Ref.[3], the Hamiltonian structures of (22) and (23) could be investigated by the discrete trace identity, here we do not further discuss them.

3 The linear and nonlinear discrete integrable models of the Toda hierarchy

As we know that some continuous expanding integrable models of the known integrable systems, such as the AKNS system, the KN system,the KdV system, and so on, were obtained by enlarging the Lie algebra A_1,, e.g. see[18-20]. In what follows, we want to extend the approach to the case of discrete integrable hierarchies. That is, we extend the Tu scheme for generating continuous expanding integrable models to the case by introducing commutators, as presented above, so that a great number of discrete expanding integrable systems could be readily generated just like generating expansion integrable models of continuous integrable systems. In the section, we only investigate the linear and nonlinear discrete expanding integrable models of the Toda hierarchy so that our method will be illustrated.

3.1: A linear discrete integrable coupling

Set[18]

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$g_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and denote L by

$$L = \text{span}\{h_1, h_2, e, f, g_1, g_2\}.$$

Assume $L_1 = \text{span}\{h_1, h_2, e, f\}, L_2 = \text{span}\{g_1, g_2\}$, then it is easy to see that $L = L_1 \oplus L_2, L_1 \cong G, [L_1, L_2] \subset L_2, [L_1, L_1] \subset L_1, [L_2, L_2] = \{0\}$.

Obviously, the linear space L is an enlarging Lie algebra of the Lie algebra G presented before. Since the Lie subalgebra L_1 is isomorphic to the Lie algebra G, then of course have the common operation relations. Therefore, we only consider the operating relations among $\{g_1, g_2\}$ with $\{h_1, h_2, e, f\}$. It is easy to compute by Maple that

$$\begin{aligned}
&g_1g_1 = g_2g_2 = g_1g_2 = g_2g_1 = 0, h_1g_1 = g_1, g_1h_1 = 0, h_2g_1 = g_1h_2 = 0, eg_1 = g_1e = 0, fg_1 = g_2, \\
&g_1f = h_1g_2 = g_2h_1 = 0, h_2g_2 = g_2, g_2h_2 = 0, eg_2 = g_1, g_2e = 0, fg_2 = g_2f = 0, \\
&[h_2, g_1] = 0, [h_2, g_2] = g_2, [e, g_1] = 0, [f, g_1] = g_2, [e, g_2] = g_1, [f, g_2] = 0, \\
&[h_1, g_1] = g_1, [h_1, g_2] = 0. \qquad (24)
\end{aligned}$$

A loop algebra of the Lie algebra L is defined as

$$\tilde{L} = \mathrm{span}\{h_1(n), h_2(n), e(n), f(n), g_1(n), g_2(n)\},$$

where $\deg X(n) = n, X \in L$. Applying the loop algebra $\tilde{L}$ introduces the following isospectral problems

$$\begin{cases} E\psi = U\psi, \\ U = h_2(1) - ph_2(0) + e(0) - vf(0) + qg_1(0) + rg_2(0), \end{cases} \tag{25}$$

and

$$\begin{cases} \psi_t = \Gamma\psi, \\ \Gamma = \sum\limits_{n\geq 0} [a_n(h_1(-n) - h_2(-n)) + b_n e(-n) + c_n f(-n) + d_n g_1(-n) + e_n g_2(-n)]. \end{cases} \tag{26}$$

The stationary discrete zero-curvature equation (9) permits the following equatiosn by using (11)-(12) and (24)-(26):

$$\begin{cases} c_n = -vb_n^{(1)}, \\ a_n^{(1)} + a_n + b_{n+1}^{(1)} - pb_n^{(1)} = 0, \\ va_n^{(1)} = c_{n+1} - va_n - pc_n, \\ c_n^{(1)} - a_{n+1}^{(1)} + a_{n+1} + p\Delta a_n = -vb_n, \\ a_n^{(1)}q + b_n^{(1)}r = e_n, \\ e_{n+1} = a_n^{(1)}r - c_n^{(1)}q - vd_n + pe_n. \end{cases} \tag{27}$$

The first four equations in (27) are the same with (14). Set $\Gamma_+ = \sum\limits_{n=0}^{m} [a_n(h_1(-n) - h_2(-n)) + b_n e(-n) + c_n f(-n) + d_n g_1(-n) + e_n g_2(-n)]\lambda^{2m} = \lambda^{2m}\Gamma - \Gamma_-$, similar to the previous discussions, we obtain that

$$(\Delta\Gamma_+)U - [U, \Gamma_+] = -b_{m+1}^{(1)}e(0) + (\Delta a_{m+1})h_2(0) + c_{m+1}f(0) + e_{m+1}g_2(0).$$

Taking $V_{(m)} = \Gamma_+ + \Delta_m, \Delta_m = b_{m+1}h_1(0)$, a direct computation yields that

$$(\Delta V_{(m)})U - [U, V_{(m)}] = (\Delta a_{m+1})h_2(0) + (c_{m+1} + vb_{m+1})f(0) + b_{m+1}^{(1)}g_1(0) + e_{m+1}g_2(0).$$

Thus, the discrete zero-curvature equation (10) permits the integrable hierarchy

$$\begin{cases} p_{t_m} = -\Delta a_{m+1}, \\ v_{t_m} = vb_{m+1}^{(1)} - vb_{m+1}, \\ q_{t_m} = qb_{m+1}^{(1)}, \\ r_{t_m} = e_{m+1}. \end{cases} \tag{28}$$

When we take $q = r = 0$, Eq. (28) just reduces to the well-known Toda hierarchy (16). Therefore, Eq. (28) is an integrable coupling of the Toda hierarchy, of course, also a discrete integrable expanding model of the Toda hierarchy. In what follows, we deduce a discrete integrable coupling of the Toda equation. For the sake, we take

$$b_0 = c_0 = 0, a_0 = \frac{1}{2}, b_1 = -1, d_n = 0, n = 0, 1, 2, \ldots$$

According to (27), we have
$a_1 = 0, a_2 = v, b_2 = -p^{(-1)}, a_3 = v(p + p^{(-1)}), b_3 = -v^{(-1)} - v - (p^{(-1)})^2, e_0 = \frac{1}{2}q,$
$e_1 = \frac{1}{2}(q + r), c_1 = v, e_2 = v^{(1)}r - v^{(1)}q + \frac{1}{2}p(q + r), c_2 = vp,$
$e_3 = v^{(1)}r - v^{(1)}p^{(1)}q + pv^{(1)}r - pv^{(1)}q + \frac{1}{2}p^2(q + r), \ldots$
Therefore, we get a discrete integrable coupling of the Toda equation as follows

$$\begin{cases} p_{t_2} = -\Delta[v(p + p^{(-1)})], \\ v_{t_2} = -v\Delta[v^{(-1)} + v + (p^{(-1)})^2], \\ q_{t_2} = -q[v^{(-1)} + v + (p^{(-1)})^2], \\ r_{t_2} = (p + 1)v^{(1)}r - (\Delta p)v^{(1)}q + \frac{1}{2}p^2(q + r). \end{cases} \tag{29}$$

It is easy to see that (29) is a linear discrete integrable coupling with respect to the new variables q and r. Therefore, Eq. (28) is known as a linear discrete integrable coupling of the Toda hierarchy.

3.2: A nonlinear discrete integrable coupling

To search for nonlinear discrete integrable coupling of the Toda hierarchy, we should enlarge the Lie algebra G to a Lie subalgebra of the Lie algebra A_3. Therefore, we set
$H_1 = \begin{pmatrix} h_1 & 0 \\ 0 & h_1 \end{pmatrix}, H_2 = \begin{pmatrix} h_2 & 0 \\ 0 & h_2 \end{pmatrix}, E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, F = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}, T_1 = \begin{pmatrix} 0 & h_1 \\ 0 & h_1 \end{pmatrix},$
$T_2 = \begin{pmatrix} 0 & h_2 \\ 0 & h_2 \end{pmatrix}, T_3 = \begin{pmatrix} 0 & e \\ 0 & e \end{pmatrix}, T_4 = \begin{pmatrix} 0 & f \\ 0 & f \end{pmatrix},$
where $h_1, h_2, e, f \in A_1$. We denote

$$Q = \mathrm{span}\{H_1, H_2, E, F, T_1, T_2, T_3, T_4, T_5\}, Q = Q_1 \oplus Q_2,$$

here $Q_1 = \mathrm{span}\{H_1, H_2, E, F\}, Q_2 = \mathrm{span}\{T_1, T_2, T_3, T_4\}$. If equipped with an operation for Q_1 as follows

$$[X, Y] = XY - YX, X, Y \in Q_1,$$

then Q_1 is a Lie algebra which is isomorphic to the previous Lir algebra G. Hence, Q_1 and G possess the same commutative relations. It is easy to compute by Maple that
$T_1T_1 = T_1, T_2T_2 = T_2, T_1T_2 = T_2T_1 = T_3T_3 = T_4T_4 = 0, T_3T_4 = T_1, T_4T_3 = T_2,$
$T_1T_3 = T_3, T_3T_1 = T_1T_4 = 0, T_4T_1 = T_4, T_2T_3 = 0, T_3T_2 = T_3, T_2T_4 = T_4, T_4T_2 = 0,$
$H_1T_1 = T_1, T_1H_1 = T_1, H_1T_2 = T_2H_1 = 0, H_1T_3 = T_3, T_3H_1 = 0, H_1T_4 = 0,$
$T_4H_1 = T_4, H_2T_1 = T_1H_2 = 0, H_2T_2 = T_2H_2 = T_2, H_2T_3 = 0, T_3H_2 = T_3, H_2T_4 = T_4,$
$T_4H_2 = 0, ET_1 = 0, T_1E = T_3, ET_2 = T_3, T_2E = 0, ET_3 = T_3E = 0, ET_4 = T_1,$
$T_4E = T_2, FT_1 = T_4, T_1F = 0, FT_2 = 0, T_2F = T_4, FT_3 = T_2, T_3F = T_1,$
$FT_4 = T_4F = 0, [T_4, H_1] = T_4, [T_4, T_1] = T_4, [T_4, T_2] = -T_4, [T_3, T_4] = T_1 - T_2,$
$[T_3, T_1] = -T_3, [T_3, T_2] = T_3, [H_2, T_3] = -T_3, [H_2, T_4] = T_4, [E, T_3] = 0,$
$[E, T_4] = T_1 - T_2, [F, T_3] = T_2 - T_1, [F, T_4] = 0, [H_1, T_3] = T_3, [H_1, T_4] = -T_4,$
$[T_3, T_4] = T_1 - T_2, [F, T_3] = T_2 - T_1, [E, T_1] = -T_3, [E, T_2] = T_3, [F, T_1] = T_4,$
$[F, T_2] = -T_4, \ldots$

From the above computations,we find that

$$[Q_2, Q_2] \subset Q_1, [Q_1, Q_2] \subset Q_2,$$

which is apparently different from that of L_1 and L_2 presented before. Actually, the Lie algebra L is a Lie algebra of a homogeneous space of a Lie group, while the Q is a Lie algebra of a symmetric space of a Lie group. A corresponding loop algebra of the Lie algebra Q is defined as

$$\tilde{Q} = \text{span}\{H_1(n), H_2(n), E(n), F(n), T_1(n), T_2(n), T_3(n), T_4(n)\},$$

where $\deg X(n) = n, X \in Q$. In hat follows, we consider the isospectral problems by using the loop algebra $\tilde{Q}$:

$$\begin{cases} E\varphi = U\varphi, \\ U = (\lambda - p)H_2(0) + E(0) - vF(0) + uT_3(0) + sT_4(0) + wT_2(0), \end{cases} \tag{30}$$

and

$$\begin{cases} \psi_t = \Gamma\psi, \\ \Gamma = \sum\limits_{n\geq 0} [a_n(H_1(-n) - H_2(-n)) + b_nE(-n) + c_nF(-n) + d_nT_3(-n) + g_nT_4(-n) \\ \qquad +q_n(T_1(-n) - T_2(-n))], \end{cases} \tag{31}$$

where the symbol E in the first equation of (30) stands for forward operator $Ef(n) = f(n+1)$, which is different from the E in U and Γ in Eqs.(30) and (31). The compatibility condition of Eqs. (30) and (31) can be written as the form (10) along with a commutator operation, the corresponding stationary discrete zero-curvature equation (9) directly leads to

$$\begin{cases} c_n = -vb_n^{(1)}, \\ a_n^{(1)} + a_n + b_{n+1}^{(1)} - pb_n^{(1)} = 0, \\ va_n^{(1)} = c_{n+1} - va_n - pc_n, \\ c_n^{(1)} - a_{n+1}^{(1)} + a_{n+1} + p\Delta a_n = -vb_n, \\ sb_n^{(1)} - uc_n^{(1)} - vd_n^{(1)} - g_n^{(1)} = ug_n - 2sd_n \\ d_{n+1}^{(1)} + u(a_n^{(1)} + a_n) - pd_n^{(1)} + q_n^{(1)}(1+u) + g_n(1+u) + w(\Delta b_n + \Delta d_n + b_n + d_n) = 0, \\ g_{n+1}^{(1)} + s(a_n^{(1)} + a_n) = pg_n^{(1)} + (v-s)q_n^{(1)} + (v-s)q_n - w(c_n + g_n), \\ -\Delta q_{n+1} + p(\Delta q_n) + u(\Delta g_n) - w\Delta(a_n + q_n) = 0. \end{cases} \tag{32}$$

Similar to the previous discussions, we denote by
$\Gamma_+ = \sum\limits_{n=0}^{m} [a_n(H_1(m-n) - H_2(m-n)) + b_nE(m-n) + c_nF(m-n) + d_nT_3(m-n) + g_nT_4(m-n) + q_n(T_1(m-n) - T_2(m-n)) = \lambda^m\Gamma - \Gamma_-,$
after calculations one infers that
$(\Delta\Gamma_+)U - [U, \Gamma_+] = (b_{m+1} - b_{m+1}^{(1)})E(0) + (\Delta a_{m+1})H_2(0) + c_{m+1}F(0) - d_{n+1}^{(1)}T_3(0) + g_{m+1}^{(1)}T_4(0) + (\Delta q_{m+1})T_2(0).$
Taking $V_{(m)} = \Gamma_+ + b_{m+1}H_1(0)$, we can compute that

$(\Delta V_{(m)})U - [U, V_{(m)}] = (\Delta a_{m+1})H_2(0) + (c_{m+1} + vb_{m+1})F(0) + (\Delta q_{m+1})T_2(0) + (-d^{(1)}_{m+1} - ub_{m+1})T_3(0) + (g^{(1)}_{m+1} + sb_{m+1})T_4(0)$.
Hence, the discrete zero-curvature equation (10) admits

$$\begin{cases} p_{t_m} = -\Delta a_{m+1}, \\ v_{t_m} = vb^{(1)}_{m+1} - vb_{m+1} = v\Delta b_{m+1}, \\ u_{t_m} = -d^{(1)}_{m+1} - ub_{m+1}, \\ s_{t_m} = g^{(1)}_{m+1} + sb_{m+1}, \\ w_{t_m} = \Delta g_{m+1}. \end{cases} \tag{33}$$

When $u = s = w = 0$, (33) reduces to the Toda hierarchy. Hence, it is a discrete expanding integrable hierarchy of the Toda hierarchy, of course, a kind of discrete integrable coupling of the Toda hierarchy. In order to further recognize Eq. (33), we consider its a simple reduction when we take $m = 1$. Given the initial values $b_0 = c_0 = d_0 = g_0 = 0, a_0 = \frac{1}{2}, b_1 = -1, c_1 = v$, Eq. (33) reduces to

$$\begin{cases} p_{t_1} = -v^{(1)} + v, \\ v_{t_1} = -vp + vp^{(-1)}, \\ u_{t_1} = pu^{(-1)} - 2uw + wu^{(-1)} - w + up^{(-1)}, \\ s_{t_1} = -ps + ws^{(-1)} - wv - sp^{(-1)}, \\ w_{t_1} = us^{(-1)} - su, \end{cases}$$

which is a set of nonlinear differential-difference equations with respect to the new variables u, s, w. Therefore, we conclude that Eq. (33) is a nonlinear discrete integrable coupling of the Toda hierarchy, which is completely different from the discrete integrable coupling (28).
Remark 2: With the help of (9) and (10), generating discrete integrable hierarchies seems more complicated than the Tu-d scheme. However, we find that when the spectral matrices U and V in (1) and (2) are higher degrees, it is more convenient to deduce discrete integrable systems than the Tu-d scheme with the aid of software Maple. In addition, we do not again discuss the Hamiltonian structure of Eq. (33) in the paper.
Acknowledgement: This work was supported by and the National Natural Science Foundation of China (grant No. 11371361), the Innovation Team of Jiangsu Province hosted by China University of Mining and Technology (2014),and Hong Kong Research Grant Council (Grant No. HKBU202512)as well as the Natural Science Foundation of Shandong Province (grant No. ZR2013AL016).

References

[1] M.J. Ablowitz, J.F. Ladik. Nonlinear differential-difference equations. J. Math. Phys., 16(1975)598-603.

[2] I. Merola, O. Ragnisco and G.Z. Tu. A novel hierarchy of integrable lattices. Inverse Problems, 10(1994)1315-1334.

[3] G.Z. Tu. A trace identity and its applications to the theory of discrete integrable systems. J. Phys. A, 23(1990)3903-3922.

[4] M. Bruschi, O. Ragnisco, P.M. Sanitini and G.Z. Tu. Integrable sympletic maps. Physica D, 40(1991)273-294.

[5] H.W. Zhang, G.Z. Tu, W. Oevel, B. Fuchssteiner. Symmetries conserved quantities and hierarchies for some lattice systems with soliton structure. J. Math. Phys., 32(1991)1908-918.

[6] C.W. Cao, X.G. Geng and Y.T. Wu. From the special 2+1 Toda lattice to the Kadomtsev-Petriashvili equation. J. Phys. A, 32(1999)8059-8078.

[7] C.W. Cao and X.X. Xu. A finite genus solution of the Veselov's discrete Neumann system. Commun. Theor. Phys., 58(2012)469-474.

[8] C.W. Cao and X. Yang. A (2+1)-dimensional derivative Toda equation in the context of the Kaup-Newell spectral problem. J. Phys. A, 41(2008)025203(19pp).

[9] X.G. Geng, H.H. Dai, J.Y. Zhu. Decomposition of the discrete Ablowitz-Ladik hierarchy. Stud. Appl. Math., 118(2007)281-312.

[10] X.G. Geng. Algebro-geometric constructions of the discrete Ablowitz-Ladik flows and applications. J. Math. Phys., 44(2003)4573-4588.

[11] E.G. Fan, Z.H. Yang. A lattice hierarchy with a free function and its reductions to the Ablowtiz-Ladik and Volterra hierarchies. Int. J. Theor. Phys., 48(2009)1-9.

[12] Z.N. Zhu, H.W. Tam. Nonisospectral negative Volterra flows and mixed Volterra flows. J. Phys. A, 37(2004)3175-3187.

[13] D.J. Zhang, D.Y. Chen. Hamiltonian structure of discrete soliton systems. J. Phys. A, 35(2002)7225-7241.

[14] Y.B. Suris. On some integrable systems related to the Toda lattice. J. Phys. A, 30(1997)2235-2249.

[15] W.X. Ma, B. Fuchsteiner. Algebraic structure of discrete zero curvature equations and master symmetries of discrete evolution equations. J. Math. Phys., 40(1999)2400-2418.

[16] Z.J. Qiao. R-matrix and algebraic-geometric soliton for integrable symplectic map. Chin. Sci. Bull., 44(1999)114-118.

[17] G.Z. Tu. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems. J. Math. Phys., 30(1989)330-338.

[18] Y.F. Zhang and H.Q. Zhang. A direct method for integrable couplings of TD hierarchy. J. Math. Phys., 43(2002)466-472.

[19] W.X. Ma and M. Chen. Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebra. J. Phys. A, 39(2006)10787-10801.

[20] W.X. Ma, J.H. Meng and H.Q. Zhang. Integrable couplings, variational identities and Hamiltonian formulations. Global J. Math. Sciences, 1(2012)1-17.