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## Moment matrices in conditional heteroskedastic models under elliptical distributions with applications in AR-ARCH models

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**Abstract** It is well known that moment matrices play a very important rôle in econometrics and statistics. Liu and Heyde (2008) give exact expressions for two-moment matrices, including the Hessian for ARCH models under elliptical distributions. In this paper, we extend the theory by establishing two additional moment matrices for conditional heteroskedastic models under elliptical distributions. The moment matrices established in this paper implement the maximum likelihood estimation by some estimation algorithms like the scoring method. We illustrate the applicability of the additional moment matrices established in this paper by applying them to establish an AR-ARCH model under elliptical distribution.

**Keywords** Heteroskedasticity · likelihood · BHHH method · Newton-Raphson method · scoring method · AR-ARCH model

### 1 Introduction

The characteristics of asset returns have been of great concern to financial economists and statisticians. Are asset returns iid and normally distributed? We first discuss the distributions of asset returns. By examining the empirical data, Fama (1963; 1965a, b) conclude that the normality assumption in the distribution of a portfolio return is violated such that the distribution is ‘flat-tailed’ and suggest the family of stable Paretian distributions between normal and Cauchy distributions for the stock returns. Others suggest stock

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returns follow student- $t$  distribution (Blattberg and Gonedes, 1974), a mixture of normal distributions (Clark, 1973) or a mixture of non-normal stable distributions (Fielitz and Rozelle, 1983). The class of elliptical distributions has gained prominence to delineate to portfolio theory and risk management in recent years. Introduced by Kelker (1970) and further discussed by, for example, Fang et al. (1987) and Fang and Zhang (1990), elliptical probability distributions allow for the presence of heavy tails and asymptotic tail dependence. It contains the multivariate normal distribution as a special case; as well as many non-normal multivariate distributions including the multivariate Cauchy, multivariate exponential, multivariate elliptical analog of student- $t$  distribution, logistic, Bessel distribution, and non-normal variance mixtures of multinormal distributions. Thus, elliptical distributions should be an excellent candidate to model the distributions of portfolio returns.

Now, we turn to discuss the iid issue for asset returns. It is well known that stock returns typically possess heavy tails and time-varying volatility features that could be heteroskedastic in their variances, see, for example, Tse (1991), Rachev and Mittnik (2000), Tsay (2002) and McAleer (2005). To circumvent this limitation, Engle (1982) first introduces an autoregressive conditional heteroskedastic (ARCH) model with normality assumption to data.<sup>1</sup> Extension of the ARCH model includes a generalized autoregressive conditional heteroscedasticity (GARCH) innovation to allow both autoregressive and moving average components in the heteroskedastic variance of the returns to display a high degree of persistence.<sup>2</sup>

To overcome both “non-iid” and “non-normality” issues for stock returns, different studies have produced a range of different models. For example, to capture the characteristics of heavy tails and non-iid nature for stock returns, some non-normal and non-iid models have been proposed to replace the normality and iid assumptions. For example, Bollerslev (1987) and Spanos (1994) use GARCH models under student- $t$  distributions assumption while Rachev and Mittnik (2000) advocate models under stable distributions. Recent development includes models under the Laplace distributions investigated by Heyde and Kou (2004). For further distributions and studies reported, readers may refer to Hamilton (1994), Tiku et al. (2000), Knight and Satchell (2001) and Wong and Bian (2005) for more information. To improve the modeling technique further, the powerful elliptical distribution family is introduced in the system; see, for example, Bingham and Kiesel (2002), Liu (2004), McNeil et al. (2005), and Liu and Heyde (2008).

Estimation of (G)ARCH type models can be achieved by adopting the maximum likelihood estimation (MLE) method, a benchmark of the estimation. Solving the MLE by iterative methods is a formidable task and can be very problematic. One may encounter multiple roots, slow convergence, or convergence to wrong values or even divergence. Readers may refer to, for ex-

<sup>1</sup> Readers may refer to, for example, Gouriéroux (1997), Chan (2002), Lütkepohl (2005), and Tsay (2005) for more discussion.

<sup>2</sup> See, for example, Bollerslev (1986) and McAleer (2005) for more discussion.

ample, Engle (1982) and Fomby et al. (1984), for early discussion, estimation, and applications. Recent work include McAleer (2005), McNeil et al. (2005) and references therein. To obtain the MLE, it is common to use the Berndt, Hall, Hall and Hausman (BHHH)<sup>3</sup> and Newton-Raphson methods based on numerical derivatives; see e.g. Berndt et al. (1974), Bollerslev (1987), Hamilton (1994), Mak et al. (1997) and McNeil et al. (2005). Another important algorithm to get numerical optimisation is the scoring method.

Some econometricians prefer to use the BHHH algorithm because it requires only the first order of derivatives. Nonetheless, its weakness is that it may encounter a variety of numerical difficulties under certain situations, as pointed out by e.g. Li (2004, p. 98). On the other hand, Mak et al. (1997) recommend to use an iteratively reweighted least squares scheme which provides better convergence properties than the BHHH. In addition, studies demonstrate that the Newton-Raphson method performs better than the BHHH in the estimation. Thus, the Newton-Raphson has been recommended to use in a wide range of areas, including econometrics, statistics, engineering, etc.

However, it is found that the Newton-Raphson method is working well provided that the log-likelihood is smooth and convex around the maximum and provided that the initial value is reasonably close to the true value of the parameter. When the true parameters are less well behaved, estimation could go wrong by using the Newton-Raphson method. In this situation, the scoring method is found to yield superior results, especially in generalised linear modeling. It is also found to be equivalent to iteratively reweighted least squares, see, for example, Faraway (2006, pp 281-282). It is reasonable to expect that the scoring method performs very well because it uses the exact information matrix instead of numerical derivatives. In particular, in estimating the parameters in ARCH type models the scoring method has been long employed; see, for example, Engle (1982), a very important seminal paper. Readers may also refer to Liu and Polasek (1999) the uses of the theory in different cases of ARCH models.

In order to study ARCH type models with pseudo-maximum likelihood estimation, Gouriéroux (1997) and Liu and Neudecker (2009) develop results of moment matrices in these models. On the other hand, Liu and Heyde (2008) give exact expressions for two-moment matrices, including the Hessian for ARCH models, under elliptical distributions with MLE. The two-moment matrices can be used for BHHH and Newton-Raphson methods in the computations. Now, in order to employ the scoring method for MLE in conditional heteroskedastic models under elliptical distributions, the exact conditional moment matrices are required.

To achieve the target, in the present paper, we extend the theory further by establishing two additional moment matrices. These are the conditional expectations of the moment matrices developed by Liu and Heyde (2008). The newly found matrices can be used in applying the scoring method to obtain the maximum likelihood estimates for conditional heteroskedastic models un-

<sup>3</sup> BHHH is a MLE method proposed by Berndt, Hall, Hall, and Hausman (1974).

der elliptical distributions. In addition, we illustrate the applicability of the additional moment matrices established in this paper by applying them to establish an AR-ARCH model under elliptical distribution.

The remainder of the paper is organised as follows: In Section 2, we discuss a general univariate conditional heteroskedastic time series model under an elliptical distribution and present its log-likelihood function. In Section 3, we establish our proposed moment matrices. We give an example to illustrate the applicability of our established moment matrices in an AR-ARCH model in Section 4. Section 5 rounds up the paper by providing several well-grounded observations.

## 2 The model and its log-likelihood function

In this section, we first discuss a general univariate conditional heteroskedastic time series model under an elliptical distribution, and thereafter present the form of its log-likelihood function.

We consider a general univariate conditional heteroskedastic time series model under an elliptical distribution. It is treated in a unified approach: the model needs only the specification of the conditional mean and conditional variance which is equivalent to specifying the scale structure. This model contains a class of GARCH models as a special case.

Firstly, we assume that the second conditional moment of the model is finite. Thereafter, we make specifications on the two equations involving both mean and scale parameters of the model. In this paper, we follow Liu and Heyde (2008) to consider the following model:

$$y_t = m_t + u_t, \quad t = 1, \dots, T, \quad (1)$$

where  $y_t$  is an observation of the time series,  $m_t = E(y_t | C_{t-1})$  is its corresponding conditional mean (in which  $C_{t-1}$  is the information set containing all available information at time  $t - 1$ ) and  $u_t$  is a disturbance term with conditional mean and conditional variance satisfying  $E(u_t | C_{t-1}) = 0$  and  $E(u_t^2 | C_{t-1}) = \sigma_t^2 > 0$ , respectively.

We further assume that  $u_t$  follows a conditional elliptical distribution (but, in general, it does not follow a conditional normal distribution), and assume that  $\sigma_t^2$  is linked to the scale parameter  $v_t$  such that  $\sigma_t^2 = av_t$  in which  $a$  is a constant. We also make the following assumptions on the third and fourth conditional moments, respectively, for the disturbance term,  $u_t$ :

$$E(u_t^3 | C_{t-1}) = s_t = 0 \quad \text{and} \quad E(u_t^4 | C_{t-1}) = k_t. \quad (2)$$

Let  $\theta = (\theta_1, \dots, \theta_p)'$  be a  $p \times 1$  vector of fundamental parameters. In this paper, we focus on developing the properties for  $m_t = m_t(\theta)$  and  $v_t = v_t(\theta)$ , both are scalar functions of  $\theta$ .

We turn to build up the log-likelihood function of the model being studied in our paper as follows:

$$L = L(y, \theta) = \sum_{t=1}^T L_t, \quad (3)$$

where

$$L_t = -\frac{1}{2} \log v_t + \log g(c_t), \text{ and} \quad (4)$$

$$c_t = u_t^2 v_t^{-1}. \quad (5)$$

$L_t$  is the conditional log-likelihood function associated with  $y_t$ ,  $u_t = y_t - m_t$  is the disturbance term, and  $g$  is the density generator for  $t = 1, \dots, T$ . The function forms of  $g$  contains members of the elliptical distribution family, see, for example, Liu (2004) for more information. Obviously,  $g$ ,  $L_t$ , and  $L$  are scalar functions of  $\theta$ .

Next, we let  $d$  be the differential operator such that both  $dm_t = dm_t(\theta)$  and  $dv_t = dv_t(\theta)$  stand for the differentials with respect to  $\theta$ , and  $\dot{g} = \dot{g}(c_t)$  stands for the derivative of  $g = g(c_t)$ . In addition, we define  $w = w(c_t) = g^{-1}\dot{g}$ . Thereby, taking the differential of  $L_t$  in (4) with respect to  $\theta$ , via  $m_t$  and  $v_t$ , we obtain  $dL_t$  to be a scalar function of  $\theta$  satisfying:

$$dL_t = -2wu_tv_t^{-1}dm_t - wu_t^2v_t^{-2}dv_t - \frac{1}{2}v_t^{-1}dv_t. \quad (6)$$

This leads to establish the score vector as follows, see Liu and Heyde (2008):

$$\begin{aligned} f(\theta) &= \sum_{t=1}^T \frac{\partial L_t}{\partial \theta} \\ &= -\sum_{t=1}^T 2wu_tv_t^{-1} \frac{\partial m_t}{\partial \theta} - \sum_{t=1}^T wu_t^2v_t^{-2} \frac{\partial v_t}{\partial \theta} - \sum_{t=1}^T \frac{1}{2}v_t^{-1} \frac{\partial v_t}{\partial \theta}. \end{aligned} \quad (7)$$

If we set  $f(\theta) = 0$  and if its solution could be solved explicitly, then one will obtain an analytical expression for the MLE's and it is not necessary to employ any algorithm in this situation. However, in most of the situations, it is impossible to obtain the explicit solution for the MLE's. In this situation, an algorithm has to be used to compute the MLE's. One may recommend to employ the BHHH, Newton-Raphson, and scoring methods for this purpose. In next section, we will derive the second differential of  $L$  with respect to  $\theta$  and then provide its analytical derivatives which will be needed in the iterations for the scoring method, to obtain the MLE's.

### 3 Moment matrices

In this paper, we consider four matrices of derivatives with respect to  $\theta$  as follows:

$$G = \sum_{t=1}^T \frac{\partial L_t}{\partial \theta} \frac{\partial L_t}{\partial \theta'},$$

$$F_1 = E(G \mid C_{t-1}), \quad (8)$$

$$H = \sum_{t=1}^T \frac{\partial^2 L_t}{\partial \theta \partial \theta'}, \quad \text{and}$$

$$F_2 = -E(H \mid C_{t-1}), \quad (9)$$

in which  $G = G(\theta)$ ,  $F_1 = F_1(\theta)$ ,  $H = H(\theta)$ , and  $F_2 = F_2(\theta)$  are  $p \times p$  matrices. In particular,  $H$  is the Hessian matrix, and  $F_1$  and  $F_2$  are conditional expectation matrices on  $C_{t-1}$  such that  $E(F_1) = E(G)$  and  $E(F_2) = E(H)$ . Thereby, we establish the following equations which are equivalent to (8) and (9), respectively:

$$(d\theta)' F_1 d\theta = (d\theta)' E(G \mid C_{t-1}) d\theta \quad \text{and}$$

$$(d\theta)' F_2 d\theta = -(d\theta)' E(H \mid C_{t-1}) d\theta. \quad (10)$$

Readers may refer to Magnus and Neudecker (1999) for the equivalence of the above equations and for the theory of the standard matrix differential calculus, and refer to Liu and Heyde (2008) for the detailed expressions of  $G$  and  $H$ .

$F_1$  and  $F_2$  can be used to implement the scoring method in the following iteration procedure; see, for example, Liu and Heyde (2008) and references therein:

$$\hat{\theta}_{i+1} = \hat{\theta}_i + F^{-1}(\hat{\theta}_i) f(\hat{\theta}_i), \quad (11)$$

where  $\hat{\theta}_i$  and  $\hat{\theta}_{i+1}$  are the estimates at the  $i^{\text{th}}$  and  $i + 1^{\text{th}}$  steps, respectively, of the numerical iteration,  $F$  is the moment matrix say to be  $F_1$  or  $F_2$ ,  $f$  is the score vector, and both  $F$  and  $f$  are evaluated at  $\hat{\theta}_i$ .

We are now ready to find the terms at the right-hand side of these equations, which will, then, be used to derive the expression of  $F_1$  and  $F_2$ . We will discuss their expressions in the following subsections.

#### 3.1 Matrix $F_1$

We first derive the expression of  $F_1$ . Multiplying  $dL_t$  in (7) by itself, we obtain

$$dL_t dL_t = 4w^2 u_t^2 v_t^{-2} dm_t dm_t$$

$$+ (2w^2 u_t^3 v_t^{-3} + w u_t v_t^{-2}) dm_t dv_t$$

$$+ (w^2 u_t^4 v_t^{-4} + w u_t^2 v_t^{-3} + \frac{1}{4} v_t^{-2}) dv_t dv_t. \quad (12)$$

Taking the expectation of  $dL_t dL_t$  in (12) conditional on  $C_{t-1}$ , and thereafter incorporating the results from (2), we have

$$E(dL_t dL_t | C_{t-1}) = 4w^2 a v_t^{-1} dm_t dm_t + (w^2 k_t v_t^{-4} + w a v_t^{-2} + \frac{1}{4} v_t^{-2}) dv_t dv_t. \quad (13)$$

From (10) and (13), we establish the expression of  $F_1$  as follows:

$$\begin{aligned} F_1 &= E \left( \sum_{t=1}^T \frac{\partial L_t}{\partial \theta} \frac{\partial L_t}{\partial \theta'} | C_{t-1} \right) \\ &= \sum_{t=1}^T 4w^2 a v_t^{-1} \left( \frac{\partial m_t}{\partial \theta'} \right)' \frac{\partial m_t}{\partial \theta'} \\ &\quad + \sum_{t=1}^T (w^2 k_t v_t^{-2} + w a + \frac{1}{4}) v_t^{-2} \left( \frac{\partial v_t}{\partial \theta'} \right)' \frac{\partial v_t}{\partial \theta'}. \end{aligned} \quad (14)$$

### 3.2 Matrix $F_2$

We turn to derive the expression of  $F_2$ . In order to establish the Hessian matrix, we first develop the second differential of  $L$ . Let  $\dot{w}$  denote the derivative of  $w$ . Taking the differential of  $dL_t$  in (7) with respect to  $\theta$ , we get

$$\begin{aligned} d^2 L_t &= (4\dot{w} u_t^2 v_t^{-2} + 2w v_t^{-1}) dm_t dm_t \\ &\quad + (4\dot{w} u_t^3 v_t^{-3} + 4w u_t v_t^{-2}) dm_t dv_t \\ &\quad + (\dot{w} u_t^4 v_t^{-4} + 2w u_t^2 v_t^{-3} + \frac{1}{2} v_t^{-2}) dv_t dv_t. \end{aligned} \quad (15)$$

To derive the expression of  $F_2$ , we need to impose the assumptions on the conditional moments  $u_t^3$  and  $u_t^4$  as shown in (2). Similar to the procedure in finding  $F_1$ , taking the expectation of  $d^2 L_t$  in (15) conditional on  $C_{t-1}$ , and thereafter incorporating the results in (2), we have

$$\begin{aligned} E(d^2 L_t | C_{t-1}) &= E(4\dot{w} u_t^2 v_t^{-2} + 2w v_t^{-1}) dm_t dm_t \\ &\quad + E(4\dot{w} u_t^3 v_t^{-3} + 4w u_t v_t^{-2}) dm_t dv_t \\ &\quad + E(\dot{w} u_t^4 v_t^{-4} + 2w u_t^2 v_t^{-3} + \frac{1}{2} v_t^{-2}) dv_t dv_t \\ &= (d\theta)' (4\dot{w} a + 2w) v_t^{-1} \left( \frac{\partial m_t}{\partial \theta'} \right)' \frac{\partial m_t}{\partial \theta'} d\theta \\ &\quad + (d\theta)' (\dot{w} k_t v_t^{-2} + 2w a + \frac{1}{2}) v_t^{-2} \left( \frac{\partial v_t}{\partial \theta'} \right)' \frac{\partial v_t}{\partial \theta'} d\theta. \end{aligned} \quad (16)$$



Combining the results in (10) and (16), we establish  $F_2$  as follows:

$$\begin{aligned}
 F_2 &= -E \left( \sum_{t=1}^T \frac{\partial^2 L_t}{\partial \theta \partial \theta'} \mid C_{t-1} \right) \\
 &= - \sum_{t=1}^T (4\dot{w}a + 2w)v_t^{-1} \left( \frac{\partial m_t}{\partial \theta'} \right)' \frac{\partial m_t}{\partial \theta'} \\
 &\quad - \sum_{t=1}^T (\dot{w}k_t v_t^{-2} + 2wa + \frac{1}{2})v_t^{-2} \left( \frac{\partial v_t}{\partial \theta'} \right)' \frac{\partial v_t}{\partial \theta'}. \tag{17}
 \end{aligned}$$

Before we turn to discuss the applications of  $F_1$  and  $F_2$ , we summarize the above observations as follows.

**Theorem 1:** *For the model stated in (1) satisfying the assumptions stated in (2), the moment matrices  $F_1$  and  $F_2$  defined in (8) and (9), respectively, possess their explicit expressions as stated in (14) and (17), respectively.*

#### 4 An example

The theory of moment matrices developed in this paper could be applied in a variety of models. In this section, we illustrate the applicability of our proposed moment matrices by applying them to build up and estimate AR-ARCH models.

We first consider a special case of model (1) in which the parameter vector  $\theta$  is partitioned to be two vectors such that  $\theta = (\beta', \alpha')'$ , where  $\beta$  is the vector of parameters for  $m_t = m_t(\beta)$  and  $\alpha$  is the vector of parameters for  $v_t$  with  $v_t = v_t(\beta, \alpha)$ . Here,  $m_t$  and  $v_t$  have been discussed in model (1) such that  $y_t = m_t + u_t$  with scale  $v_t = \sigma_t^2/a$ .

For this model, Liu and Heyde (2008) have presented the expression of the score vector  $f$ . Based on the two additional moment matrices we developed in the previous section, we further express the functions  $F_1$  and  $F_2$  as follows: From (14), we get

$$F_1 = \begin{pmatrix} F_{1bb} & F_{1ba} \\ F'_{1ba} & F_{1aa} \end{pmatrix}, \tag{18}$$

where

$$\begin{aligned}
 F_{1bb}(\beta) &= \sum_{t=1}^T 4w^2 a v_t^{-1} \left( \frac{\partial m_t}{\partial \beta'} \right)' \frac{\partial m_t}{\partial \beta'} \\
 &\quad + \sum_{t=1}^T (w^2 k_t v_t^{-2} + wa + \frac{1}{4}) v_t^{-2} \left( \frac{\partial v_t}{\partial \beta'} \right)' \frac{\partial v_t}{\partial \beta'}, \\
 F_{1ba}(\beta, \alpha) &= \sum_{t=1}^T (w^2 k_t v_t^{-2} + wa + \frac{1}{4}) v_t^{-2} \left( \frac{\partial v_t}{\partial \beta'} \right)' \frac{\partial v_t}{\partial \alpha'}, \text{ and} \\
 F_{1aa}(\alpha) &= \sum_{t=1}^T (w^2 k_t v_t^{-2} + wa + \frac{1}{4}) v_t^{-2} \left( \frac{\partial v_t}{\partial \alpha'} \right)' \frac{\partial v_t}{\partial \alpha'}.
 \end{aligned}$$

From (17), we get

$$F_2 = \begin{pmatrix} F_{2bb} & F_{2ba} \\ F'_{2ba} & F_{2aa} \end{pmatrix}, \quad (19)$$

where

$$\begin{aligned}
 F_{2bb}(\beta) &= - \sum_{t=1}^T (4\dot{w}a + 2w) v_t^{-1} \left( \frac{\partial m_t}{\partial \beta'} \right)' \frac{\partial m_t}{\partial \beta'} \\
 &\quad - \sum_{t=1}^T (\dot{w}k_t v_t^{-2} + 2wa + \frac{1}{2}) v_t^{-2} \left( \frac{\partial v_t}{\partial \beta'} \right)' \frac{\partial v_t}{\partial \beta'}, \\
 F_{2ba}(\beta, \alpha) &= - \sum_{t=1}^T (\dot{w}k_t v_t^{-2} + 2wa + \frac{1}{2}) v_t^{-2} \left( \frac{\partial v_t}{\partial \beta'} \right)' \frac{\partial v_t}{\partial \alpha'}, \text{ and} \\
 F_{2aa}(\alpha) &= - \sum_{t=1}^T (\dot{w}k_t v_t^{-2} + 2wa + \frac{1}{2}) v_t^{-2} \left( \frac{\partial v_t}{\partial \alpha'} \right)' \frac{\partial v_t}{\partial \alpha'}.
 \end{aligned}$$

After expressing our proposed model (1) in a ‘bivariate’ format as shown in the above, we are now ready to estimate an AR(1)-ARCH(1) model as follows:

$$\begin{aligned}
 y_t &= m_t + u_t, \\
 m_t &= \beta_0 + \beta_1 y_{t-1}, \\
 v_t &= \alpha_0 + \alpha_1 u_{t-1}^2, \text{ and} \\
 \sigma_t^2 &= a v_t,
 \end{aligned} \quad (20)$$

where  $u_t$ ,  $t = 1, \dots, T$ , follows a conditional elliptical distribution. For the model in (20) with  $\beta = (\beta_0, \beta_1)'$  and  $\alpha = (\alpha_0, \alpha_1)'$ , we have

$$\begin{aligned} \frac{\partial m_t}{\partial \beta'} &= (1, y_{t-1}), \\ \frac{\partial m_t}{\partial \alpha'} &= (0, 0), \\ \frac{\partial v_t}{\partial \beta'} &= -2\alpha_1 u_{t-1} (1, y_{t-1}), \text{ and} \\ \frac{\partial v_t}{\partial \alpha'} &= (1, u_{t-1}^2). \end{aligned} \tag{21}$$

We note that when one makes further specification of the underlying distribution for the model, one can obtain  $w$ , the expressions in (21), and then obtain the exact expressions of  $F_1$  and  $F_2$ . These functions could then be used in the scoring method to implement the MLE.

## 5 Concluding remarks

In this paper, we extend the work by Liu and Heyde (2008) and others by establishing the additional moment matrices which could then be used in the scoring method to implement the MLE. Thereafter, we illustrate the applicability of the additional moment matrices to establish an AR-ARCH model under an elliptical distribution.

We note that the expressions of  $F_1$  and  $F_2$  are simpler than  $G$  and  $H$  given in Liu and Heyde (2008). These matrices can then be used to produce competitive or better results. When the normality assumption is imposed on our model settings, one could be able to obtain the explicit solutions for the MLE's in terms of  $G$ ,  $H$ ,  $F_1$ , and  $F_2$ , and thus the estimators could be easily estimated, see, for example, Engle (1982), Embrechts et al. (2002), and many others. However, in most of the situations, it is impossible to obtain the explicit solutions for the MLE's. In this situation, an algorithm has to be used to compute the MLE's. In this situation, academics and practitioners could apply the matrices developed in this paper to obtain better estimation. In addition, the moment matrices  $F_1$  and  $F_2$  could be used in hypothesis tests for ARCH models in econometrics and statistics. Thus, the moment matrices could help academics and practitioners to obtain better modeling and testing in many well-known portfolio optimization and other important economics and financial issues; see, for example, Fong et al. (2008) and Bai et al. (2009a, b).

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