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Preferences over Location-Scale Family

Summary. This paper extends the work on location-scale (LS) family with general n random seed sources. First, we clarify and generalize existing results in this multivariate setting. Some useful geometrical and topological properties of the location-scale expected utility functions are obtained. Second, we introduce and study some general non-expected utility functions defined over the LS family. Special care is taken in characterizing the shapes of the indifference curves induced by the location-scale expected utility functions and non-expected utility functions. Finally, efforts are also made to study several well-defined partial orders and dominance relations defined over the LS family. These include the first- and second-order stochastic dominances, the mean-variance rule, and a newly defined location-scale dominance.

Keywords and Phrases: location-scale family, inverse problem, non-expected utility function, stochastic dominance, location-scale dominance, mean-variance rule.

JEL classification Numbers: G11, C60, G10.

1 Introduction

After the pioneer work of Markowitz (1952), mean-variance efficient sets have been widely used in both economics and finance to analyze how people make their choices concerning risky investments. However, the literature largely reflected the use of quadratic utility functions in discussions and analyses and assumed normality in the distribution of an investment or its return (see, for example, Tobin 1958; Hanoch and Levy 1969; and Baron 1974). Meyer (1987), Sinn (1983, 1990), and Levy (1989) added to the literature by comparing the distributions that differ only by location and scale parameters while analyzing the class of expected utility functions with convexity or concavity restrictions. This paper extends their work on LS family with general n random seed sources. The extensions are carried out in two different directions. First, we allow for the possibility that the returns on the risky assets could be driven by more than one seed random variables (r.v.s), and we do not impose any distributional assumption on the seed r.v.s. Second, investors' preferences do not necessarily conform to von Neumann and Morgenstern's (1944) expected utility class.

The research has taken into considerations the perspectives of both economics and behavioral science regarding modern portfolio choice theory and asset pricing theory. On the one hand, the impact of multivariate seed variables on asset returns, in theory, provides more realistic and general framework for studying the randomness of asset returns (see, for example, Ross 1978). The returns on risky projects driven by a finite number of risky factors are not only a theoretical concept but are also commonly used in practice. For example, the relationship among the economic activities of the firm and the market returns on the debt and the equity of the firm are of interest to financial economists. Thus, there has been a renewed interest in the empirical relations between market return to equity and basic characteristics of the firm, such as the size, leverage, earnings yield, dividend-yield, book-to-market ratios, and leverage of the firm.[1] In addition, empirical

[1]The size and earnings yield anomalies were documented by Banz (1981), the book-to-market effect by Stattman (1980), the earnings-to-price ratios by Basu (1983), and the leverage by Bhandari (1988).

evidence is in favor of a multi-factor rather than single-factor asset pricing model (see, for example, Fama and French 1996).

On the other hand, there exists substantial experimental and empirical evidence in decision theory — all leading to the rejection of the expected utility functions in representing investors' behaviors in the presence of risk (see Machina 1982 and Epstein 1992, for surveys). This last set of observation leads us to consider general non-expected utility functions.

For the purpose of this paper, we shall focus on the class of betweenness utility functions axiomatized by Chew (1983) and Dekel (1986). The betweenness utility function is obtained by replacing the independence axiom of von Neumann and Morgenstern's expected utility representation with the so-called betweenness axiom. The betweenness axiom has been found to be well supported through experimental evidence, and provides predictions that are in line with Allais' (1953) paradox. The usefulness of the betweenness utility functions for resolving the well-known empirical puzzles in finance has been overwhelming; see Cochrane (2005) for good coverage of this and for extended reference.

The historical background prior to the work developed by Meyer (1987), Sinn (1983, 1990), and Levy (1989) on LS family is profound. To understand the importance of the LS family, we need to go back to, at least, Markowitz's (1952) classical mean-variance analysis and Tobin's (1958) mutual fund separation theorem. It is well-known that if investors rank risky portfolios through their mean and variance, Tobin's two-fund separation holds, and the separating portfolios will be located on Markowitz's efficient frontier. In the presence of a risk-free asset, investors would optimally hold a combination of the risk-free asset and a common risky portfolio. Tobin addressed an open question: how robust is the mutual-fund separation phenomenon for rational investors whose behaviors conform to some normality axioms such as those underlying von Neumann and Morgenstern's expected utility functions?

Seeking answers to this question has been an enduring task for academics in economics for more than forty years. The research on this subject can be roughly divided into two branches, each following its own school of thought. The first branch of research focuses on investors' behavior assumptions (see, for example, Cass and Stiglitz 1970, and more recently, Boyle and Ma 2005). The second branch aims at identifying the distributional assumptions on asset returns that are sufficient for mutual fund separation for expected-utility investors. This paper, along with those of Meyer (1987), Sinn (1983, 1990), and Levy (1989), falls into this second school.

The pioneering research that falls into this second branch is mainly represented by Ross (1978), Chamberlain (1983), Owen and Rabinovitch (1983), and Meyer (1987). Ross (1978) developed distributional conditions on asset returns to ensure that two-fund separation holds with the underlying separating portfolios common to all risk-averse expected-utility investors. Ross showed that two-fund separation holds if and only if asset returns are driven by two common factors with residual returns (to the factors) having zero mean conditional on the linear span formed by the factors. Ross's insight into two-fund separation allowed him to extend his analysis to some general observations on k-fund separation. Chamberlain (1983) and Owen and Rabinovitch (1983) showed that mean-variance preferences persist when asset returns are elliptically distributed.

Sinn (1983) and Meyer (1987) were among the first to explicitly study the expected utility functions defined over the LS family. Similar to Ross (1978), they obtained the LS family by restricting distributions to differ from the seed variable only by the location and scale parameters. This is done without restricting the random seed to be normally distributed or to be located within Chamberlain's elliptic class.

In fact, the seed variable may follow any distribution. Though the location-scale expected utility functions defined over the LS family are summarized through two parameters, the location-scale expected utility functions, in general, differ from the classical mean-variance criterion. This is because the underlying expected utility functions defined over Meyer's LS family can still be well-defined even when the seed random variable has no finite mean and variance. This is particularly true for bounded and continuous utility indexes. Sinn (1983) has also derived some properties for the indifference curves in relation to the linear distribution classes similar to the findings in Meyer (1987). Sinn (1990) further extended the work to find that decreasing (constant) absolute risk aversion implies that the slope of the indifference curve declines (stays constant) with an increase in μ, given a positive σ. However, if absolute risk aversion is non-increasing, the slope of the indifference curve will rise with an increase in σ, given μ. He also observed a change in the indifference curve's slope with an increase in σ under increasing absolute risk aversion. Levy (1989) extended Meyer's results to prove that the first- and second-degree stochastic dominance efficient sets are equal to the mean-variance efficient set under certain conditions. He also found an inequality relationship between the support of the seed random variable and the parameters of the linear functions of the seed random variables.

In light of the above established findings, this paper is best positioned as an extension of Sinn's (1983, 1990) and Meyer's (1987) research. Specifically, we extend and clarify the work by Meyer and others on the geometric and topological properties of the LS expected utility functions and non-expected utility functions defined over the LS family with n random seed variables. Our results also generalize Tobin's (1958) findings that the indifference curves are convex upward for risk-averters, and concave downward for risk-lovers, while keeping in mind that we are dealing with a wider n-dimensional LS family of distributions for general location-scale expected and non-expected utility functions.

As we mentioned before, our coverage of non-expected utility functions in this paper falls into the mathematical tractability of the betweenness utility class discussed by Chew (1983) and Dekel (1986).[2] We also use this utility class to discuss the usefulness of the model in resolving the Allais paradox, and the betweenness utility-based equilibrium asset pricing models (see Epstein and Zin 1989 and Ma 1998 for a general discussion).

Special efforts are also made to study several well-defined partial orders and dominance relations defined over the LS family. These include the first- and second-order stochastic dominances (FSD, SSD), the mean-variance (MV) rule, and a newly defined location-scale dominance (LSD). The linkage of the first and second orders to the corresponding utility classes has been well-documented in the literature. The "if and only if" relationships proved in this paper are somewhat stronger than those documented in

[2]The same observation on the tractability of the betweenness utility function holds true for the study of the corresponding investors' portfolio choice problem, along with the analytic derivations of some useful non-expected utilities-based equilibrium asset pricing models. These are not covered in this paper. Interested readers may refer to Ma (1993, 1998).

the existing literature.[3] First, the random variables are not assumed to have bounded supports. Second, we have restricted the utility functions to be continuously differentiable C^1 or to be twice continuously differentiable C^2, in which the discontinuous step functions are excluded from the class. With the step utility functions, the proofs for the sufficient part of the relationships are much simplified. This is at the expense of a statement that is weaker than what we need for this paper. Equipped with this result on the second-order stochastic dominance, we are able to establish a useful link between the newly defined location-scale dominance relation over the LS family and the SSD efficient set defined over the same family. This is summarized in Proposition 15 below.

The remainder of the paper is organized as follows: In Section 2, we clarify and extend the original work of Meyer (1987) and others on location-scale expected utility functions. An inverse problem associated with location-scale expected utility representation is studied in Section 3. In this section, we also introduce a family of location-scale non-expected utility functions defined over the n-dimensional LS family. Section 4 covers partial orders and dominance relationships defined over the LS family. Even though these partial orders and dominance relationships may not admit utility representations, their properties and implications for investors' choices can be readily studied. In this section, we also introduce the notion of a location-scale dominance relationship, in addition to the comparisons with those well-known dominance relationships in the literature. The latter includes the mean-variance rule and the first- and second-order stochastic dominances. Section 5 rounds up the paper by providing several well-grounded observations. Some technical proofs are provided in the Appendices.

2 Location-Scale Expected Utility

In this section we formulate and extend the work developed by Meyer (1987) and others on location-scale expected utility functions to a general n-dimensional setup. We also examine the shapes and other topological properties of indifference curves that are induced

[3]See Huang and Litzenberger (1988) and the extended reference there for more information.

by a location-scale expected utility function.

2.1 Preliminary

We assume that the returns of risky projects are driven by a finite number, say n, risky factors that are summarized by an $\mathbb{R}^n$-valued random vector $X = [X_1, \cdots, X_n]$; see, for example, Ross (1978) and Fama and French (1996). Let X_i be the i-th factor, and let X_{-i} be the vector of the factors excluding the i-th factor. For notational simplicity, we may write $X = [X_i, X_{-i}]$ for all i. We assume that $E[X_i \mid X_{-i}] = 0$ for all i. The random vector X satisfying these conditions is known to be a vector of *random seeds*. It is noted that the conditions for a zero conditional mean for the random seeds are satisfied when the random factors have zero mean and are independently distributed. So all observations and results derived later on in this paper are valid under the stronger assumption of independently distributed random factors.

For any given vector, X, of random seeds, we let

$$\mathcal{D} = \{\mu + \sigma \cdot X : \mu \in \mathbb{R}, \sigma \in \mathbb{R}^n_+\} \tag{1}$$

to denote the LS family induced by X. Here, $x \cdot y$ stands for the inner product defined on the Euclidean spaces, and $\mathbb{R}^n_+$ represents the non-negative cone of the Euclidian space. Later, we shall use $\mathbb{R}^n_{++}$ to represent the positive cone with all entries to be strictly positive. Elements in $\mathcal{D}$ can be interpreted as payoffs or returns associated with each of the risky projects. Here, all scaling factors σ_i in σ are restricted to be non-negative. We write $\sigma \leq \sigma'$ whenever $\sigma_i \leq \sigma'_i$ for all i.

Investors are thus assumed to express their preferences over all random payoffs in $\mathcal{D}$. Let $(\sigma, \mu) \to V(\sigma, \mu)$ be a location-scale utility function that represents investors' preference on $\mathcal{D}$. A location-scale utility function V is said to be located in Meyer's LS expected utility class if there exists a monotonic transformation of V, still denoted by V, and a well-defined utility index $u(\cdot)$ so that

$$V(\sigma,\mu) = \int_{\mathbb{R}^n} u(\mu + \sigma \cdot x)\, d\, F(x) \tag{2}$$

for all $(\sigma,\mu) \in \mathbb{R}^n_+ \times \mathbb{R}$.

Here, $F(\cdot)$ is the cumulative distribution function (c.d.f.) for the r.v. X. In this paper, unless otherwise specified, we shall assume that the utility index $u \in C^1(\mathbb{R})$ is monotonic increasing and continuously differentiable, and the c.d.f. $F(\cdot)$ satisfies Feller's property so that the LS expected utility function $V(\sigma,\mu)$ is well-defined and is continuously differentiable in (σ,μ).[4]

2.2 Monotonicity

Our first observation is that the monotonicity of the utility index $u(\cdot)$ implies and is implied by the monotonicity of the utility function $V(\sigma,\mu)$ with respect to the location variable μ. This was **Property 1** in Meyer (1987). Particularly, for any smooth utility index, u, with

$$V_\mu(\sigma,\mu) = \int_{\mathbb{R}^n} u'(\mu + \sigma \cdot x)\, d\, F(x),$$

Property 1 is stated as

$$V_\mu(\sigma,\mu) \geq 0 \Leftrightarrow u'(\cdot) \geq 0.$$

The marginal expected utility with respect to each of the scaling factors that is summarized by the n-dimensional gradient function $V_\sigma(\sigma,\mu) \equiv \left[\frac{\partial V(\sigma,\mu)}{\partial \sigma_i}\right]_{n\times 1}$ can be easily computed and be given by

$$V_\sigma(\sigma,\mu) = \left[\int_{\mathbb{R}^n} u'(\mu + \sigma \cdot x) x_i\, d\, F(x)\right]_{n\times 1}.$$

[4]A c.d.f. $F(\cdot)$ defined on $\mathbb{R}^n$ is said to satisfy Feller's property if, for all $g \in C(\mathbb{R}^n \times \mathbb{R}^m)$, the function $y \to \int_{\mathbb{R}^n} g(x,y)\, dF(x)$ defined on $\mathbb{R}^m$ is continuous. Feller's property is named after Feller (1968).

The marginal expected utility may take either + or - signs, depending on the curvature/convexity of the utility index $u(\cdot)$. With $u'(\cdot) \geq 0$, we can easily prove the validity of the following relationships respectively for risk averse, risk loving and risk neutral investors:

$$
\begin{aligned}
x \mapsto u(x) \text{ is concave} &\quad\Rightarrow\quad V_\sigma \leq 0; \\
x \mapsto u(x) \text{ is convex} &\quad\Rightarrow\quad V_\sigma \geq 0; \\
x \mapsto u(x) \text{ is linear} &\quad\Rightarrow\quad V_\sigma \equiv 0.
\end{aligned}
$$

This constitutes the "if" part of **Property 2** in Meyer's paper. We only need to prove the validity of the first relationship as follows and the rest can be obtained similarly: The concavity of the utility index implies that, for all $x = (x_i, x_{-i}) \in \mathbb{R}^n$, it must hold true that

$$u'(\mu + \sigma \cdot x)x_i \leq u'(\mu + \sigma_{-i} \cdot x_{-i})x_i$$

and that

$$
\begin{aligned}
& V_{\sigma_i}(\sigma, \mu) \\
= \; & E\left[u'(\mu + \sigma \cdot X)X_i\right] \\
\leq \; & E\left[u'(\mu + \sigma_{-i} \cdot X_{-i})X_i\right] \\
= \; & E\left[u'(\mu + \sigma_{-i} \cdot X_{-i})E\left[X_i \mid X_{-i}\right]\right] \\
= \; & 0
\end{aligned}
$$

since, by assumption, $E[X_i \mid X_{-i}] = 0$.

The converse to the above relationships are, in general, not valid (see, for example, Rothschild and Stiglitz 1970). But for distribution function $F(\cdot)$ to have a finite second moment and to satisfy Feller's property, the validity of the converse relationships can be proved under fairly general conditions. For example, if we assume that there exists an i such that X_i has its support located within a bounded open interval (a_i, b_i), and if the utility function is twice continuously differentiable, then we can readily prove the "only if' part of Property 2 as originally stated in Meyer (1987); that is,

$$\begin{aligned} V_\sigma \leq 0 &\Rightarrow u'' \leq 0; \\ V_\sigma \geq 0 &\Rightarrow u'' \geq 0; \\ V_\sigma = 0 &\Rightarrow u'' \equiv 0. \end{aligned}$$

Again, we only need to prove the validity of the first relationship as follows: Let $F_i(\cdot)$ be the marginal distribution function for X_i. Under Feller's condition, the marginal expected utility function $(\sigma, \mu) \to V_\sigma(\sigma, \mu) \leq 0$ is continuous. So, we may set $\sigma_{-i} = \emptyset$ for σ and for $V_{\sigma_i}(\sigma, \mu)$ so that, for all μ and $\sigma_i > 0$, we obtain

$$V_{\sigma_i}(\sigma_i, \mu) = \int_{a_i}^{b_i} u'(\mu + \sigma_i x) x dF_i(x) \leq 0.$$

Since, by assumption, $E[X_i] = \int_{a_i}^{b_i} x dF_i(x) = 0$, and since $u(\cdot)$ is continuously differentiable on $\mathbb{R}$, which have bounded first-order derivatives over (a_i, b_i), we have

$$\begin{aligned} \lim_{x \to a_i} u'(\mu + \sigma_i x) \int_{a_i}^{x} y dF_i(y) &= 0, \\ \lim_{x \to b_i} u'(\mu + \sigma_i x) \int_{a_i}^{x} y dF_i(y) &= 0. \end{aligned}$$

Applying integration by parts, we obtain

$$V_{\sigma_i}(\sigma_i, \mu) = -\sigma_i \int_{a_i}^{b_i} u''(\mu + \sigma_i x) \left(\int_{a_i}^{x} y dF_i(y) \right) dx.$$

This yields

$$\int_{a_i}^{b_i} u''(\mu + \sigma_i x) \left(\int_{a_i}^{x} y dF_i(y) \right) dx \geq 0, \forall \mu, \sigma_i > 0.$$

With $\int_{a_i}^{b_i} \int_{a_i}^{x} y dF_i(y) dx = -E[X_i^2] < 0$, by Feller's condition, we may set $\sigma_i \to 0_+$ to the above inequality to obtain $u''(x) \leq 0, \forall x \in \mathbb{R}$.

The assumption on the existence of bounded support for the 'only if' part of Meyer's Property 2 can, in fact, be further relaxed. The arguments prevail if there exists a random source, X_i, with finite second moment so that, for all μ and $\sigma_i > 0$, the following limits exist:

$$\begin{array}{rl} \lim_{x\to\infty} x \int_{-\infty}^{x} y dF_i(y) & = 0 \\ \lim_{x\to\pm\infty} u'(\mu+\sigma_i x) \int_{-\infty}^{x} y dF_i(y) & = 0\,. \end{array} \tag{3}$$

The second condition is valid if the utility index $u(\cdot)$ has bounded first-order derivatives. The first condition is to ensure that the improper integral $\int_{-\infty}^{\infty}\int_{-\infty}^{x} y dF_i(y)\,dx$ is well-defined and takes a negative value. We have,

$$\begin{aligned} & \int_{-\infty}^{\infty}\int_{-\infty}^{x} y dF_i(y)\,dx \\ = & \lim_{b\to+\infty}\int_{-\infty}^{b}\int_{-\infty}^{x} y dF_i(y)\,dx \\ = & \lim_{b\to+\infty}\int_{-\infty}^{b}\int_{y}^{b} y dx dF_i(y) \\ = & \lim_{b\to+\infty} b\int_{-\infty}^{b} y dF_i(y) - \int_{-\infty}^{\infty} y^2 dF_i(y) \\ = & -E\left[X_i^2\right]. \end{aligned}$$

It is easy to verify that the condition $\lim_{x\to+\infty} x \int_{-\infty}^{x} y dF_i(y) = 0$ is satisfied when X_i is normally distributed with zero mean.

For future reference, we summarize the above observations on the monotonicity of the LS expected utility functions defined over the n-dimensional LS family. These are expressed formally in a proposition as follows:

Proposition 1 *Consider the expected utility function, $V(\sigma,\mu)$, on an n-dimensional LS family $\mathcal{D}$ as defined in (2). Letting $u \in C^1(\mathbb{R})$, we have*

(i) $u' \geq 0 \Leftrightarrow V_\mu \geq 0.$

(ii) *If $u' \geq 0$, then it must hold true that*

$$
\begin{aligned}
x \mapsto u(x) \text{ is concave } &\Rightarrow \quad V_\sigma \leq 0; \\
x \mapsto u(x) \text{ is convex } &\Rightarrow \quad V_\sigma \geq 0; \\
x \mapsto u(x) \text{ is linear } &\Rightarrow \quad V_\sigma \equiv 0.
\end{aligned}
$$

(iii) *If $u \in C^2(\mathbb{R})$ with $u' \geq 0$, and if there exists i so that condition (3) is satisfied, then it must hold true that*

$$
\begin{aligned}
V_\sigma \leq 0 &\Rightarrow u'' \leq 0; \\
V_\sigma \geq 0 &\Rightarrow u'' \geq 0; \\
V_\sigma = 0 &\Rightarrow u'' \equiv 0.
\end{aligned}
$$

2.3 Convexity

Now, let us prove the validity of the following statement. The statement is a modification of **Property 4** of Meyer's paper:

$$(\sigma,\mu) \to V(\sigma,\mu) \text{ is concave} \Leftrightarrow u(\cdot) \text{ is concave.}$$

For the 'if' part of the proof, let u be concave. For arbitrary (σ,μ) and (σ',μ') and $\alpha \in [0,1]$, let

$$(\sigma_\alpha,\mu_\alpha) \equiv \alpha(\sigma,\mu) + (1-\alpha)(\sigma',\mu').$$

We have: for all $x \in \mathbb{R}^n$, concavity of $u(\cdot)$ implies

$$\begin{aligned} & u(\mu_\alpha + \sigma_\alpha \cdot x) \\ = \; & u(\alpha(\mu + \sigma \cdot x) + (1-\alpha)(\mu' + \sigma' \cdot x)) \\ \geq \; & \alpha u(\mu + \sigma \cdot x) + (1-\alpha) u(\mu' + \sigma' \cdot x). \end{aligned}$$

This, in turn, implies

$$\begin{aligned} & V(\sigma_\alpha, \mu_\alpha) \\ = \; & \int_{\mathbb{R}^n} u(\mu_\alpha + \sigma_\alpha \cdot x)\, dF(x) \\ \geq \; & \alpha \int_{\mathbb{R}^n} u(\mu + \sigma \cdot x)\, dF(x) \\ & + (1-\alpha) \int_{\mathbb{R}^n} u(\mu' + \sigma' \cdot x)\, dF(x) \\ = \; & \alpha V(\sigma, \mu) + (1-\alpha) V(\sigma', \mu'). \end{aligned}$$

This is true for all (σ, μ) and (σ', μ') and for all $\alpha \in [0,1]$. This proves the concavity of $(\sigma, \mu) \to V(\sigma, \mu)$.

The 'only if' part of the statement is obvious: Setting $\sigma = \emptyset$. With $V(\emptyset, \cdot) \equiv u(\cdot)$, the concavity of $V(\emptyset, \cdot)$ is equivalent to the concavity of $u(\cdot)$.

Here, we intentionally drop the differentiability condition of the utility function. Meyer's original statement (Property 4) is obtained if we restrict $u \in C^2(\mathbb{R})$ to be twice continuously differentiable; that is, for all $u \in C^2(\mathbb{R})$,

$$(\sigma, \mu) \to V(\sigma, \mu) \text{ is concave} \Leftrightarrow u''(\cdot) \leq 0.$$

Examples can be easily constructed to show that the concavity of $(\sigma, \mu) \to V(\sigma, \mu)$ does not necessarily imply that $u(\cdot)$ is twice continuously differentiable. This is true even if $V(\sigma, \mu) \in C^\infty(\mathbb{R}^n_{++} \times \mathbb{R})$ is infinitely many times continuously differentiable.

2.4 Indifference Curves

We further explore the topological properties for the indifference curves induced by an LS expected utility function V. For an arbitrary constant a, let

$$C_a \equiv \{(\sigma, \mu) \in \mathbb{R}_+^n \times \mathbb{R} : V(\sigma, \mu) = a\} \tag{4}$$

be the indifference curve at utility level a. As a direct consequence of the 'if' part of Property 2 above, we can readily obtain the following observation with respect to the shapes of the indifference curves, which correspond to Property 3 in Meyer (1987): The indifference curve C_a is upward-sloping if u is concave and downward-sloping if u is convex. Moreover, by Property 4, concavity (convexity) of the utility index u implies concavity (convexity) of the utility function V. This, together with Property 3, results in the following stronger statement on the shapes of the indifference curves respectively for risk averse, risk loving and risk neutral investors:

Proposition 2 *Let $u \in C^1(\mathbb{R})$ be increasing and continuously differentiable. We have*

1. *The indifference curve C_a is convex upward if u is concave;*
2. *it is concave downward if u is convex; and*
3. *it is horizontal if u is a straight line.*

Proof. First, we characterize the monotonicity of the indifference curves. For all arbitrary $\sigma \geq \sigma'$, let $\mu = \mu(\sigma)$ and $\mu' = \mu(\sigma')$ be on the indifference curve so that $V(\sigma, \mu) = V(\sigma', \mu') = a$. Suppose u is concave (convex). This implies, by Proposition 1-(ii), $\sigma \to V(\sigma, \mu)$ is decreasing (increasing). So, we have $V(\sigma', \mu) \geq (\leq) V(\sigma, \mu) = a$. This, together with the monotonicity of $\mu \to V(\sigma, \mu)$ in Proposition 1-(i), yields $\mu \geq (\leq) \mu'$. That is, $\mu(\sigma) \geq (\leq) \mu(\sigma')$ whenever $\sigma \geq \sigma'$.

We further characterize the convexity of the indifference curve. For any arbitrary σ and σ' and for all $\alpha \in [0,1]$, let $\sigma_\alpha \equiv \alpha\sigma + (1-\alpha)\sigma'$, $\mu = \mu(\sigma), \mu' = \mu(\sigma')$ and

$\mu_\alpha = \mu(\sigma_\alpha)$, we have

$$V(\sigma,\mu) = V(\sigma',\mu') = V(\sigma_\alpha,\mu_\alpha).$$

Suppose u is concave (convex). This implies, by Property 4, $(\sigma,\mu) \to V(\sigma,\mu)$ is concave (convex). We have

$$\begin{aligned} & V(\sigma_\alpha, \alpha\mu + (1-\alpha)\mu') \\ \geq\ & (\leq)\, \alpha V(\sigma,\mu) + (1-\alpha) V(\sigma',\mu') \\ =\ & V(\sigma_\alpha,\mu_\alpha). \end{aligned}$$

The monotonicity of the utility function $V(\sigma_\alpha,\cdot)$ implies

$$\mu(\alpha\sigma + (1-\alpha)\sigma') \leq (\geq)\, \alpha\mu(\sigma) + (1-\alpha)\mu(\sigma').$$

The equality must hold when u is linear. $\square$

Note that the statements made in Proposition 2 about the shape and curvature of the indifference curves can be restated analytically in terms of the gradient and Hessian matrix of the indifference curve $\mu(\sigma), \sigma \in \mathbb{R}^n_+$. These, of course, require the standard regularity conditions on the utility function. For instance, by the implicit function theorem, the gradient vector $\mu_\sigma \equiv \left[\frac{\partial\mu}{\partial\sigma_j}\right]_{n\times 1}$ along the indifference curve is given by

$$\mu_\sigma = -\frac{V_\sigma(\sigma,\mu)}{V_\mu(\sigma,\mu)}, \forall (\sigma,\mu) \in C_a \tag{5}$$

which is non-negative (non-positive) when $u(\cdot)$ is concave (convex). We may further compute the Hessian matrix $\mu_{\sigma\sigma} \equiv \left[\frac{\partial^2\mu}{\partial\sigma_k\partial\sigma_j}\right]_{n\times n}$ for the $\mu(\cdot)$-function. This, of course, requires the utility index to be twice continuously differentiable. For all $(\sigma,\mu) \in C_a$, we have:

$$\mu_{\sigma\sigma} = -\frac{[\mu_\sigma, I_n]\, H(\sigma,\mu)\, [\mu_\sigma, I_n]^\intercal}{V_\mu(\sigma,\mu)} \tag{6}$$

in which $H(\sigma,\mu)$ is the $(n+1)\times(n+1)$ Hessian matrix for $V(\sigma,\mu)$, and I_n is the $n\times n$ unit matrix. From this expression, we see that concavity (convexity) of the utility index $u(\cdot)$ implies, by Property 4, negative (positive) semi-definiteness of the Hessian matrix $H(\sigma,\mu)$. With $V_\mu > 0$, the latter, in turn, implies $\mu_{\sigma\sigma}$ to be positive (negative) semi-definite.

In virtue of the above observations, we obtain the following analytic version of Proposition 2:

Corollary 3 *Let $u \in C^2(\mathbb{R})$ with $u' > 0$. Along the indifference curve $\mu(\sigma), \sigma \in \mathbb{R}^n_+$, it must hold true that*

$$u'' \le 0 \quad \Rightarrow \quad \mu_\sigma \ge 0, \mu_{\sigma\sigma} \ge 0;$$
$$u'' \ge 0 \quad \Rightarrow \quad \mu_\sigma \le 0,\ \mu_{\sigma\sigma} \le 0;$$
$$u'' \equiv 0 \quad \Rightarrow \quad \mu_\sigma = 0,\ \mu_{\sigma\sigma} \equiv 0.$$

3 Expected vs Non-Expected LS Utility Functions

This section introduces a class of LS utility functions that are not necessarily located in the expected utility class. To motivate our effort for considering a general class of non-expected utility functions, we raise and discuss in Section 3.1 the following so-called "inverse problem" with respect to Meyer's LS expected utility functions: for any arbitrarily given utility function $V(\sigma,\mu)$ defined over the LS family $\mathcal{D}$, which may satisfy all desirable topological properties (such as monotonicity and concavity), we wonder whether $V(\sigma,\mu)$ admits an expected utility representation or not.

Upon a negative answer to the inverse problem as illustrated below, we introduce, in Section 3.2, a class of non-expected utility functions over the LS family admitting all desirable properties that are possessed by the standard LS expected utility functions. In this paper we extend the betweenness utility functions (see, for example, Chew 1983 and Dekel 1986) to random variables belonging to Meyer's LS family.

3.1 An Inverse Problem

The inverse problem raised above can be formulated as the following mathematical problem:

Problem 4 *For any given utility function $V(\sigma,\mu) \in C(\mathbb{R}^n \times \mathbb{R})$ on the LS family $\mathcal{D}$, is there a utility index $u \in C(\mathbb{R})$ and a monotonic increasing function $\varphi \in C(\mathbb{R})$ such that*

$$\varphi(V(\sigma,\mu)) = \int_{\mathbb{R}^n} u(\mu + \sigma \cdot x)\, dF(x) \tag{7}$$

for all $(\sigma,\mu) \in \mathbb{R}^n_+ \times \mathbb{R}$?

Here, we take into account the ordinal property of the expected utility representation. It is well-known that for all arbitrary monotonic increasing functions $f(\cdot)$, $E[u(x)]$ and $f(E[u(x)])$ represent the same preference ordering. In light of the LS utility function, $V(\sigma,\mu) \in C(\mathbb{R}^n \times \mathbb{R})$ admits an expected utility representation if there exists a monotonic transformation of $V(\sigma,\mu)$ so that $\varphi(V(\sigma,\mu))$ admits an expected utility representation.

The following observation can be readily proved in working toward an answer to this inverse problem:

Proposition 5 *The inverse problem has a solution if and only if there exists a monotonic increasing function $\varphi \in C(\mathbb{R})$ such that*

$$V(\sigma,\mu) = \varphi^{-1}\left(\int_{\mathbb{R}^n} \varphi(V(\emptyset, \mu + \sigma \cdot x))\, dF(x)\right) \tag{8}$$

for all $(\sigma,\mu) \in \mathbb{R}^n_+ \times \mathbb{R}$; in particular, if a solution exists, the utility index is given by $u(x) = \varphi(V(\emptyset, x))$.

Proof. First, we prove the second part of the proposition. Suppose the inverse problem has a solution $\{u(\cdot), \varphi(\cdot)\}$. Setting $\sigma = 0$, we obtain $u(x) = \varphi(V(\emptyset, x)), x \in \mathbb{R}$. This, in turn, implies the validity of the first statement in establishing a necessary and sufficient condition for the existence of a solution to the inverse problem. $\square$

Assuming further that $(\sigma, \mu) \to V(\sigma, \mu)$ is continuously differentiable, from the above proposition, we can readily identify

$$V_\sigma(\emptyset, x) = \emptyset, \forall x \in \mathbb{R}, \tag{9}$$

as a necessary condition for the existence of a solution to the inverse problem. In fact, let $\varphi \in C^1(\mathbb{R})$ be a solution to equation (8). We may compute the utility gradient with respect to the scaling variables σ on both sides of equation (8), and set $\sigma \to \emptyset$ to obtain

$$\varphi'(V(\emptyset, \mu)) V_\sigma(\emptyset, \mu) = \emptyset, \forall \mu \in \mathbb{R}.$$

Since, by assumption, $\varphi' > 0$, we conclude that $V_\sigma(\emptyset, \mu) = \emptyset, \forall \mu \in \mathbb{R}$.

The necessary condition $V_\sigma(\emptyset, x) = \emptyset, \forall x \in \mathbb{R}$ for the existence of a solution to the inverse problem is, in general, too weak to constitute a sufficient condition for the existence of a solution. The following is an illustrative example.

Example 6 (non-existence) *Let $V(\sigma, \mu) = \mu - \sigma^2$, which obviously satisfies the necessary condition $V_\sigma(0, \mu) = 0$. The random seed X is a one-dimensional bivariate random variable taking values $\{-1, 1\}$ with equal probability of $\frac{1}{2}$. Equation (8) reduces to*

$$\frac{\varphi(\mu - \sigma) + \varphi(\mu + \sigma)}{2} = \varphi\left(\mu - \sigma^2\right), \forall (\sigma, \mu) \in \mathbb{R}_+ \times \mathbb{R}. \tag{10}$$

Suppose, to the contrary, that equation (10) has a monotonic solution $\varphi(\cdot)$. For all arbitrary $\mu \in \mathbb{R}$, setting $\sigma = 1$ to equation (10) to obtain $\varphi(\mu + 1) = \varphi(\mu - 1)$. This yields a constant function $\varphi(\cdot)$ on the interval $[\mu - 1, \mu + 1]$. Setting $\mu = \pm 2n, \cdots$ for all integer n, the constant function $\varphi(\cdot)$ extends to the entire real line $\mathbb{R} \equiv \cup_{n=-\infty}^{\infty}[2n - 1, 2n + 1]$. This enables us to conclude the non-existence of a monotonic solution to equation (10);

or equation (10) has only constant solutions.

So, in general, we might expect a negative answer to the inverse problem raised above; that is, it would not admit an LS expected utility representation for all (σ, μ)-preferences. The next section studies a class of non-expected utility functions defined over the LS family.

3.2 Location-Scale Non-Expected Utility

In light of the above example for a negative answer to the inverse problem for LS expected utility representation, we consider a general class of non-expected utility functions defined over the LS family. Although these utility functions may not necessarily admit some expected utility representations, the underlying behavior assumptions are well understood in decision theory and economics. The treatment below is based on the betweenness utility functions axiomatized by Chew (1983) and Dekel (1986), though much of the analysis can be readily extended to the broad class of Gateaux differentiable utility functions (see, for example, Machina 1982 and Ma 1993).

Definition 7 *A utility function U is said to be in the betweenness class if there exists a betweenness function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which is increasing in its first argument, and is decreasing in its second argument, and $H(x, x) \equiv 0$ for all $x \in \mathbb{R}$, such that, for all X, $U(X)$ is determined implicitly by setting $E[H(X, U(X))] = 0$. The corresponding LS betweenness utility function $V : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}$ on the LS family $\mathcal{D} \equiv \{\mu + \sigma \cdot X : \mu \in \mathbb{R}, \sigma \in \mathbb{R}^n_+\}$ induced by a r.v. X is, accordingly, defined by setting $V(\sigma, \mu) = U(\mu + \sigma \cdot X)$ as a unique solution to*

$$\int_{\mathbb{R}^n} H(\mu + \sigma \cdot x, V(\sigma, \mu))\, dF(x) = 0 \tag{11}$$

for all (σ, μ).

The betweenness utility function is known to be obtained by weakening the key independence axiom underlying the expected utility representation with the so-called betweenness axiom (Dekel 1986). The betweenness utility function is said to display risk aversion if, for all X, $U(X) \leq U(E[X])$, or, equivalently, $E[H(X, U(E[X]))] \leq 0$. It is well known that the betweenness utility function displays risk aversion if and only if the betweenness function is concave in its first argument (Epstein 1992).

The following result summarizes the properties of the LS betweenness utility function:

Proposition 8 *Let $H \in C^1(\mathbb{R} \times \mathbb{R})$ be a betweenness function. We have*

1. *$\mu \to V(\sigma, \mu)$ increasing; moreover,*
2. *if H is concave in its first argument, then $\sigma \to V(\sigma, \mu)$ must be monotonic decreasing, and $(\sigma, \mu) \to V(\sigma, \mu)$ must be quasi-concave; and*
3. *if H is jointly concave in both arguments, then $(\sigma, \mu) \to V(\sigma, \mu)$ must be concave in both arguments.*

Proof. The betweenness function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is, by definition, increasing in the first argument and decreasing in the second argument. For all arbitrary $\mu \geq \mu'$ and for all arbitrary $\sigma \geq \emptyset$, we have:

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} H(\mu' + \sigma \cdot x, V(\sigma, \mu'))\, dF(x) \\ &= \int_{\mathbb{R}^n} H(\mu + \sigma \cdot x, V(\sigma, \mu))\, dF(x) \\ &\geq \int_{\mathbb{R}^n} H(\mu' + \sigma \cdot x, V(\sigma, \mu))\, dF(x). \end{aligned}$$

This implies $V(\sigma, \mu) \geq V(\sigma, \mu')$ since $v \to \int_{\mathbb{R}^n} H(\mu' + \sigma \cdot x, v)\, dF(x)$ is decreasing. So, we conclude the monotonicity of $\mu \to V(\sigma, \mu)$.

Now, we assume further that H is concave in its first argument. For all arbitrary μ and $\sigma \geq \emptyset$, by the implicit functional theorem, we have, for all i,

$$V_{\sigma_i}(\sigma, \mu) = -\frac{\int_{\mathbb{R}^n} H_1(\mu + \sigma \cdot x, V(\sigma, \mu))\, x_i dF(x)}{\int_{\mathbb{R}^n} H_2(\mu + \sigma \cdot x, V(\sigma, \mu))\, dF(x)}.$$

The denominator is negative since H is decreasing in its second argument. The nominator also takes a negative sign because the concavity of $H(\cdot, v)$ implies

$$H_1(\mu + \sigma \cdot x, V(\sigma, \mu))\, x_i \leq H_1(\mu + \sigma_{-i} \cdot x_{-i}, V(\sigma, \mu))\, x_i$$

for all $x_i \in \mathbb{R}$. This, in turn, implies

$$\begin{aligned}
& \int_{\mathbb{R}^n} H_1(\mu + \sigma \cdot x, V(\sigma, \mu))\, x_i dF(x) \\
\leq \;& E[H_1(\mu + \sigma_{-i} \cdot X_{-i}, V(\sigma, \mu))\, E[X_i \mid X_{-i}]] \\
= \;& \emptyset
\end{aligned}$$

since $E[X_i \mid X_{-i}] = \emptyset$ by assumption. We thus conclude that $V_{\sigma_i}(\sigma, \mu) \leq \emptyset$ as desired.

We further verify the quasi-concavity of the utility function. Let (σ, μ) and (σ', μ') be such that

$$V(\sigma, \mu) = V(\sigma', \mu') = a.$$

For all arbitrary $\alpha \in [0, 1]$, let

$$(\sigma_\alpha, \mu_\alpha) \equiv \alpha(\sigma, \mu) + (1 - \alpha)(\sigma', \mu').$$

We want to show that $V(\sigma_\alpha, \mu_\alpha) \geq a$. For all v, the concavity of $H(\cdot, v)$ implies

$$\begin{aligned}
& H(\mu_\alpha + \sigma_\alpha \cdot x, v) \\
\geq \;& \alpha H(\mu + \sigma \cdot x, v) + (1 - \alpha) H(\mu' + \sigma' \cdot x, v)
\end{aligned}$$

for all x. In particular, setting $v = a$, we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n} H\left(\mu_\alpha + \sigma_\alpha \cdot x, a\right) dF\left(x\right) \\
\geq \; & \alpha \int_{\mathbb{R}^n} H\left(\mu + \sigma \cdot x, a\right) dF\left(x\right) \\
& + \left(1-\alpha\right) \int_{\mathbb{R}^n} H\left(\mu' + \sigma' \cdot x, a\right) dF\left(x\right) \\
= \; & \int_{\mathbb{R}^n} H\left(\mu_\alpha + \sigma_\alpha \cdot x, V\left(\sigma_\alpha, \mu_\alpha\right)\right) dF\left(x\right) \\
= \; & 0.
\end{aligned}$$

This implies $V\left(\sigma_\alpha, \mu_\alpha\right) \geq a$ since $v \to \int_{\mathbb{R}^n} H\left(\mu' + \sigma \cdot x, v\right) dF\left(x\right)$ is decreasing. The quasi-concavity of $\left(\sigma, \mu\right) \to V\left(\sigma, \mu\right)$ is thus proved.

We now turn to prove the concavity of $\left(\sigma, \mu\right) \to V\left(\sigma, \mu\right)$ under the additional joint concavity of the betweenness function $H\left(\cdot, \cdot\right)$. For arbitrary $\left(\sigma, \mu\right)$ and $\left(\sigma', \mu'\right)$ and for $\alpha \in \left[0, 1\right]$, we let

$$\begin{aligned}
\left(\sigma_\alpha, \mu_\alpha\right) & \equiv \alpha\left(\sigma, \mu\right) + \left(1-\alpha\right)\left(\sigma', \mu'\right), \\
V_\alpha & \equiv \alpha V\left(\sigma, \mu\right) + \left(1-\alpha\right) V\left(\sigma', \mu'\right) .
\end{aligned}$$

We have: for all $x \in \mathbb{R}^n$, concavity of $H\left(\cdot, \cdot\right)$ implies

$$\begin{aligned}
& H\left(\mu_\alpha + \sigma_\alpha \cdot x, V_\alpha\right) \\
\geq \; & \alpha H\left(\mu + \sigma \cdot x, V\left(\sigma, \mu\right)\right) \\
& + \left(1-\alpha\right) H\left(\mu' + \sigma' \cdot x, V\left(\sigma', \mu'\right)\right).
\end{aligned}$$

This, in turn, implies

$$\begin{aligned}
& \int_{\mathbb{R}^n} H\left(\mu_\alpha + \sigma_\alpha \cdot x, V_\alpha\right) dF\left(x\right) \\
\geq \;& \alpha \int_{\mathbb{R}^n} H\left(\mu + \sigma \cdot x, V\left(\sigma, \mu\right)\right) dF\left(x\right) \\
& + \left(1 - \alpha\right) \int_{\mathbb{R}^n} H\left(\mu' + \sigma' \cdot x, V\left(\sigma', \mu'\right)\right) dF\left(x\right) \\
= \;& 0;
\end{aligned}$$

or,

$$\begin{aligned}
& \int_{\mathbb{R}^n} H\left(\mu_\alpha + \sigma_\alpha \cdot x, V_\alpha\right) dF\left(x\right) \\
\geq \;& \int_{\mathbb{R}^n} H\left(\mu_\alpha + \sigma_\alpha \cdot x, V\left(\sigma_\alpha, \mu_\alpha\right)\right) dF\left(x\right).
\end{aligned}$$

Thus, we have $V_\alpha \leq V\left(\sigma_\alpha, \mu_\alpha\right)$ or

$$V\left(\sigma_\alpha, \mu_\alpha\right) \geq \alpha V\left(\sigma, \mu\right) + \left(1 - \alpha\right) V\left(\sigma', \mu'\right)$$

since $v \to \int_{\mathbb{R}^n} H\left(\mu' + \sigma \cdot x, v\right) dF\left(x\right)$ is decreasing. This proves the concavity of $\left(\sigma, \mu\right) \to V\left(\sigma, \mu\right)$. $\square$

Similar to LS expected utility functions, the monotonicity of a betweenness utility function with respect to μ and σ implies the monotonicity of the corresponding indifference curves; and the concavity of the utility function $\left(\sigma, \mu\right) \to V\left(\sigma, \mu\right)$ implies the quasi-concavity of the utility function, while the latter is equivalent to the convexity of the indifference curve C_a. Keeping in mind the equivalence between the concavity of $x \to H\left(x, v\right)$ and the risk aversion of the betweenness utility function, the relevance of the risk aversion and its implications for the shape of the indifference curve for this betweenness LS class can be readily established. Similar observations can be made when the betweenness utility functions display risk-loving or risk-neutrality, keeping in mind that the betweenness utility function displays risk-loving (risk-neutrality) if the betweenness function H is convex (linear) in its first argument. We may thus state without proof the following property:

Corollary 9 *Let $H \in C^1(\mathbb{R} \times \mathbb{R})$ be a betweenness function. We have*

1. *The indifference curve C_a is convex upward if the corresponding betweenness utility function displays risk aversion;*
2. *the indifference curve C_a is concave downward if the corresponding betweenness utility function displays risk-loving; and*
3. *the indifference curve C_a is horizontal if the corresponding betweenness utility function displays risk-neutrality.*

As an aside, the expected utility functions form a subclass to the class of betweenness utility functions. In fact, the standard expected utility function certainty equivalent induced by utility index $u(\cdot)$ is obtained by setting $H(x, y) = u(x) - u(y)$.

4 Dominance Relationships over the LS Family

Levy (1989) extended Meyer's results and proved that the first- and second-degree stochastic dominance efficient sets are equal to the mean-variance efficient set under certain conditions. To extend Levy's work, this section first develops several useful dominance relationships as partial orders defined over the LS family. These include the first- and second-order stochastic dominances,[5] in addition to a newly defined location-scale dominance (LSD) relationship defined over the LS family. These dominance relationships are known to admit no utility representations. Their properties over the LS family can be,

[5]Higher order stochastic dominances are not included in our discussion here, as it is not related to the newly introduced LSD in our paper. Interested readers are referred to Whitmore (1970), Stoyan (1983) and Li and Wong (1999) for treatments of higher-order stochastic dominances.

nevertheless, readily studied. We note that the LSD defined in our paper differs from the MV criterion used in the literature (see Markowitz 1952 or Tobin 1958), more information on which can be found in Definition 12 below.

The notions of first- and second-order stochastic dominances originated from Hadar and Russell (1969). For any pair of real-valued random variables Y and Y' with cumulative distribution functions to be respectively given by $F_Y(\cdot)$ and $F_{Y'}(\cdot)$, we say that Y dominates Y' by the first-order stochastic dominance (FSD) if $F_Y(y) \leq F_{Y'}(y)$ for all $y \in \mathbb{R}$; and that Y dominates Y' by the second-order stochastic dominance (SSD) if $\int_{-\infty}^{y} [F_Y(x) - F_{Y'}(x)]\, dx \leq 0$ for all $y \in \mathbb{R}$. We write $Y \succeq_1 Y'$ whenever Y dominates Y' by FSD, and $Y \succeq_2 Y'$ whenever Y dominates Y' by SSD. Moreover, we write $(Y, Y') \in D_{FSD}$ and $(Y, Y') \in D_{SSD}$ if the corresponding dominance relationships do not exist between the two random variables. D_{FSD} and D_{SSD} are respectively known as FSD- and SSD-efficient sets.

Under some fairly general conditions on the c.d.f.s of the underlying r.v.s, we shall show that $Y \succeq_1 Y'$ if and only if all expected utility investors with monotonic increasing utility functions ($u' \geq 0$) would prefer Y to Y'; and that $Y \succeq_2 Y'$ if and only if all expected utility investors with monotonic increasing and concave utility functions ($u' \geq 0, u'' \leq 0$) would prefer Y to Y'.

The proofs to the 'only if' or the necessary part of these statements are straightforward and are well-documented in the literature (See, for example, Hanoch and Levy 1969, and Hadar and Russell 1971). Huang and Litzenberger (1988) provided a proof for the 'if' part of the statements. They, nevertheless, restricted the utility functions to be continuous. For example, for the SSD, they showed that "if $u(Y) \geq u(Y')$ for all u that is continuous and concave, then $Y \succeq_2 Y'$."

For the purpose of this paper, we need some stronger results than those stated in Huang and Litzenberger (1988) and in other earlier work. First, we require utility functions to be monotonic increasing so that all investors prefer more to less. Second, we require the utility functions to be continuously differentiable. Formally, we may put these new results in the form of Propositions for future reference.

Proposition 10 *For all arbitrary r.v.s X and Y, we have*

$$X \succeq_1 Y \Leftrightarrow E\left[u\left(X\right)\right] \geq E\left[u\left(Y\right)\right] \tag{12}$$

for all bounded and increasing utility indices $u \in C^1\left(\mathbb{R}\right)$.

Proof. See Appendix 1.

To ensure that the SSD dominance relationships are well defined, we shall restrict the c.d.f.s to satisfy the following asymptotic and integrability conditions.

Asymptotic Condition: A c.d.f. $F\left(\cdot\right)$ is said to satisfy the asymptotic condition if

$$\begin{aligned} 1 - F\left(x\right) &= o\left(\frac{1}{x}\right) \text{ and} \\ F\left(x\right) &= o\left(\frac{1}{x}\right) \end{aligned} \tag{13}$$

as $x \to +\infty$ and $-\infty$ respectively.

Integrability Condition: A c.d.f. $F\left(\cdot\right)$ is said to satisfy the integrability conditions if the improper integrals

$$\begin{aligned} &\int_{-\infty}^{0} F\left(x\right) dx \geq 0 \text{ and} \\ &\int_{0}^{\infty} \left[1 - F\left(x\right)\right] dx \geq 0 \end{aligned} \tag{14}$$

exist and take finite values.

The integrability condition is to ensure that the SSD relationship is well-defined. The asymptotic condition is needed for the proof of Proposition 11 below. We have:

Proposition 11 *Suppose X and Y with c.d.f.s satisfy both the asymptotic and the integrability conditions (13) and (14). Then, it must hold true that*

$$X \succeq_2 Y \Leftrightarrow E\left[u\left(X\right)\right] \geq E\left[u\left(Y\right)\right] \tag{15}$$

for all increasing and concave utility indices $u \in C^2\left(\mathbb{R}\right)$ with bounded first-order derivatives.

Proof. See Appendix 2.

In contrast to what covered in the existing literature, we do not assume the r.v.s to be bounded. These are replaced with some asymptotic conditions with respect to the c.d.f.s, along with some boundedness assumptions on the utility function or the marginal utility function. In fact, both conditions (13) and (14) are virtually satisfied when the underlying r.v.s have bounded support $[A, B]$. For bounded random variables, we may drop the boundedness assumptions imposed on the utility indexes. So, as a corollary to the above proposition, we may readily obtain a stronger statement on SSD for bounded r.v.s.; that is,

"for X and Y with bounded support $[A, B]$, $X \succeq_2 Y$ if and only if $E\left[u\left(X\right)\right] \geq E\left[u\left(Y\right)\right]$ for all $u \in C^2\left(\mathbb{R}\right)$ with $u' \geq 0$ and $u'' \leq 0$."

4.1 Location-Scale Dominance

To extend Levy's work on stochastic dominance efficient sets and the mean-variance efficient set, we introduce the following LS dominance relationship defined over Meyer's LS family.

Definition 12 *Let X be an R^n-valued r.v. with zero mean and conditional mean $E\left[X_i \mid X_{-i}\right] = 0$ for all i. Let D be an LS family generated from X. For all $Y = \mu + \sigma \cdot X$ and $Y' = \mu' + \sigma' \cdot X$, we say that Y dominates Y' according to the LS rule if $\mu \geq \mu'$*

and $\sigma \leq \sigma'$. We write $Y \succeq_{LS} Y'$ whenever Y dominates Y' according to the LS rule. Otherwise, we write $(Y, Y') \in D_{LSD}$ if Y and Y' do not dominate each other in the sense of LSD. The set D_{LSD} is referred to as the LS-efficient set.

For $n = 1$, when the random seed X has zero mean and a finite second moment, the LS rule defined on D is equivalent to Markowitz's (1952) MV rule defined over the family. The equivalence breaks down when X does not have a finite second moment, for which the variance of X does not exist; yet, the LS expected utility functions are still well-defined for all bounded continuous utility indexes.

For random payoffs belonging to the high dimensional ($n > 1$) LS family, the equivalence between the LS-rule and the MV criterion breaks down even when the seeds r.v. X have finite second moments. In fact, with

$$\sigma [Y] = (\sigma^{\intercal} \Sigma_X \sigma)^{1/2} \text{ and}$$
$$\sigma [Y'] = ((\sigma')^{\intercal} \Sigma_X \sigma')^{1/2}$$

where Σ_X is the positive variance-covariance matrix for the vector, X, of random seeds, we have: $\sigma \geq \sigma'$ implies but is not implied by $\sigma [Y] \geq \sigma [Y']$. Accordingly, for LS expected utility functions, monotonicity in σ does not necessarily imply monotonicity in $\sigma [Y]$. The following is an illustrative example of this last observation.

Example 13 *Let*

$$\Omega = \{(i, j) : i \in \{-1, 0, 1\}, j \in \{-1, 0, 1\}\}$$

be a state space that contains 9 elements with equal probabilities $p_{ij} = \frac{1}{9}$. Let X_1 and X_2 be two random seed variables on Ω, which are defined, respectively, by setting

$$X_1 (i, j) = i \text{ and } X_2 (i, j) = j \text{ for all } (i, j) \in \Omega.$$

We have $E[X_1 \mid X_2] = E[X_2 \mid X_1] = 0$ *and* $\sigma[X_1] = \sigma[X_2] = \sqrt{\frac{2}{3}}$. *Consider the following LS random variables*

$$Y = 300 + 90(2X_1 + X_2), Z = 299 + 202X_1.$$

We have

$$\begin{aligned} E[Y] &= 300, \sigma[Y] = 90\sqrt{\frac{10}{3}}; \\ E[Z] &= 299, \sigma[Z] = 202\sqrt{\frac{2}{3}}. \end{aligned}$$

Evidently, Y *dominates* Z *according to the MV rule. Now, consider the LS expected utility function* V *resulting from the log-utility index* $u(x) = \ln x$*; that is,* $V(\sigma_1, \sigma_2, \mu) = E[\ln(\mu + \sigma_1 X_1 + \sigma_2 X_2)]$. *We have*

$$E[u(Y)] \doteq 5.45 < 5.50 \doteq E[u(Z)];$$

that is, although Y *dominates* Z *according to the MV rule, but we have* $E[u(Z)] > E[u(Y)]$. *We also note that, by Proposition 1, the utility function* V *defined over* D *must display monotonicity with respect to LSD; that is, for all* Y *and* $Y' \in D$*, it holds true that*

$$E[u(Y)] \geq E[u(Y')] \text{ whenever } Y \succeq_{LS} Y'.$$

More generally, as a direct consequence of Proposition 1, we can readily state without proof the following observation on LSD defined over an LS family:

Proposition 14 *For* $n = 1$*, let* Y *and* Y' *belong to the same LS family* D *generated from the seed r.v.* X*. Suppose* X *has a (zero mean) finite second moment. We have:* Y

dominates Y' according to the MV rule if and only if $Y \succeq_{LS} Y'$. Moreover, for $n > 1$, for all Y and Y' belonging to the same LS family D, it holds true that

$$Y \succeq_{LS} Y' \Rightarrow E\left[u\left(Y\right)\right] \geq E\left[u\left(Y'\right)\right]$$

for all increasing and concave utility indexes $u \in C^1\left(\mathbb{R}\right)$.

4.2 FSD, SSD and LSD

The relationships among the three forms of dominance relationships, namely, FSD, SSD, and LSD, defined over an n-dimensional Meyer LS family can be readily studied. The following proposition summarizes our findings on these.

Proposition 15 *Let $\mathcal{D}$ be an LS family induced by an n-dimensional vector, X, of seed r.v.s with bounded supports. We have:*

1. *$D_{SSD} \subset D_{FSD}$;*
2. *$D_{SSD} \subset D_{LSD}$; and*
3. *(a) $D_{LSD} - D_{FSD} \neq \emptyset$ and*

   (b) $D_{FSD} - D_{LSD} \neq \emptyset$.

Proof. By definition, we have $D_{SSD} \subseteq D_{FSD}$. To show $D_{SSD} \subset D_{FSD}$, we set $Y = \sigma X$, $0 < \sigma < 1$, where X is with zero mean $E(X) = 0$. Obviously, we have $Y \succ_2 X$ but X and Y do not dominate each other in the sense of FSD. Hence, $(X, Y) \in D_{FSD}$ but $(X, Y) \notin D_{SSD}$.

To prove the validity of Part 2 of the proposition, we let $Y = \mu + \sigma \cdot X$ and $Y' = \mu' + \sigma' \cdot X$. Assume that $\mu \geq \mu'$ and $\sigma \leq \sigma'$ so that $Y \succeq_{LS} Y'$. By Proposition 14, we conclude that

$$V\left(\sigma, \mu\right) = E\left[u\left(Y\right)\right] \geq E\left[u\left(Y'\right)\right] = V\left(\sigma', \mu'\right)$$

for all increasing and concave utility indexes $u \in C^1(\mathbb{R})$. This implies that $Y \succeq_2 Y'$. Therefore, we have $Y \succeq_{LS} Y' \Rightarrow Y \succeq_2 Y'$; or, equivalently, $D_{SSD} \subseteq D_{LSD}$.

The following example shows that D_{SSD} is a proper subset of D_{LSD}. Let X have its support given by $[A, B] = [-1, 1]$. Let $Y = \mu + \sigma X$ and $Y' = \mu' + \sigma' X$ with $\sigma > \sigma' > 0$ and $\mu = \mu' + \sigma - \sigma'$. By definition, we have: $(Y, Y') \in D_{LSD}$ and

$$Y - Y' = (\sigma - \sigma')(1 + X) \geq 0.$$

This implies $Y \succ_2 Y'$ and $(Y, Y') \notin D_{SSD}$. Hence, D_{SSD} is a proper subset of D_{LSD}.

In fact, in the above example we have $Y \succ_1 Y'$; that is $(Y, Y') \notin D_{FSD}$. This confirms the validity of (3a) of the proposition.

One can also easily postulate the first example to show (3b). For any $\sigma \in (0, 1)$, we have $(X, \sigma X) \in D_{FSD}$ and $\sigma X \succ_{LS} X$. $\square$

So, we see that both notions of FSD and LSD relations are stronger than that of SSD. Part 3 of Proposition 15 suggests that there is no specific logical relationship between FSD and LSD. The LSD neither implies nor is implied by the FSD.

5 Conclusion

This paper extends the work of Meyer (1987) and others by studying the expected and non-expected utility functions defined over the multivariate LS family. These are in addition to several useful dominance relationships, including FSD, SSD, and LSD dominances, defined over the family. For the special case when the random seeds are jointly normally distributed, or when they fall into the broader class of n-dimensional elliptic distributions (Chamberlain 1983, and Owen and Rabinovitch 1983), the LS utility defined in this paper reduces to the MV utility function with the n scaling factors collapsing into a single scaling coefficient, which is the standard deviation of the random wealth.

The mean-standard deviation (μ, σ) and mean-variance (μ, σ^2) approach have received great attention in recent years (Lajeri and Nielsen, 2000; Eichner and Wagener, 2003; Eichner, 2005) after the seminal work by Markowitz (1952). On the other hand, researchers have started to apply SD in many different areas, for example, financial theory (Fong, Wong and Lean, 2005), international trade (Broll, Wahl and Wong, 2006), game theory (Schulteis, Perea and Peters, 2007), duality theory (Martínez-Legaz and Quah, 2007) and business planning (Wong, 2007). Though both MV and SD approaches have been widely used, so far, it is not common for practitioners to examine their relationships in their analyses. Since this paper establishes some relationships between MV and SD, we recommend that practitioners apply the approach introduced in this paper and examine the SD and MV relationships in their studies.

Since it is sensible to assume that asset returns are driven by a large and arbitrary number of random seed factors, and since the random seeds may admit arbitrary distributional representations falling out of Chamberlain's elliptic class or Meyer's one-dimensional LS family, the message in this research concerning MV utility representation is viewed as a negative one. Having said that, we must clarify that our research findings do not diminish the great importance of Markowitz's path-breaking work on MV analysis and his in-depth original insight into portfolio selection and risk diversification. In fact, according to a recent study by Boyle and Ma (2005), the relevance of Markowitz's MV efficient frontier for portfolio choices goes beyond the MV preferences and normally or general elliptically distributed asset returns. They showed that all investors who display risk aversion in the sense of the mean-preserving-spread (MPS) must optimally choose to invest along Markowitz's efficient frontier. And we must stress that this observation is made without imposing any distributional assumption on asset returns (except the existence of finite first two moments) and without explicitly specifying an arbitrary number of random seed factors in driving the asset returns. The MPS partial order may not admit a utility representation; particularly, it may not admit an MV utility representation.

Our coverage of the non-expected utility functions and partial orders is not exhaustive. The analysis can be further extended to a more general class of Gateaux differentiable non-expected utility functions examined by Machina (1982) and Ma (1993), and the rank-dependent utility functions of Quiggin (1982) and Yaari (1984, 1987) to be narrated within the general non-expected utility framework. Also, in addition to the partial orders studied in this paper, several other partial orders attracted our attention. These are (a) Boyle and Ma's (2005) mean-preserving-spread (MPS) dominance relationships, and (b) the Markowitz stochastic dominance and prospect stochastic dominance developed by Levy and Wiener (1998), Levy and Levy (2002, 2004), and Wong and Chan (2008). Their significance should clearly be scrutinized carefully in a separate paper.

Finally, we note that the main barrier to using the approach in this paper to extend the prospect theory (Rieger and Wang, 2006; Neilson, 2006) is that prospect theory is a well-known paradigm challenging the expected utility theory (Kahneman and Tversky, 1979). The allegation is that the prospect theory invalidates the expected utility theory as being "subjectively distorted probabilities" (Levy and Wiener 1998). To circumvent this problem, one could incorporate a Bayesian approach (Matsumura, Tsui and Wong, 1990) and distribution-free statistics (Wong and Miller, 1990) into the subjective probability (Klibanoff and Ozdenoren, 2007) or subjective uncertainty (Machina, 2004) to estimate the subjectively distorted probabilities. Prospect theory will satisfy the Bayesian expected utility maximization. Thus, the problem that the prospect theory violates the expected utility theory could be circumvented.

6 Appendices

Appendix 1. Proof of Proposition 10.

The necessary part of the proof is standard and is thus omitted. To prove the sufficiency, suppose $E\left[u\left(X\right)\right] \geq E\left[u\left(Y\right)\right]$ for all bounded and increasing index functions $u \in C^1\left(\mathbb{R}\right)$, particularly for those belonging to $C^\infty\left(\mathbb{R}\right)$. For any arbitrary $x \in \mathbb{R}$, consider the sequence of bounded, increasing and smooth utility functions $\{u_n\} \subset C^\infty\left(\mathbb{R}\right)$ defined by setting

$$u_n\left(y\right) = \frac{1}{2}\left[1 + \frac{y-x}{\sqrt{\left(y-x\right)^2 + n^{-1}}}\right], \forall y \in \mathbb{R}$$

for all $n = 0, 1, \cdots$. We have: $\lim_{n\to\infty} u_n\left(y\right) = 0$ for $y < x$, $\lim_{n\to\infty} u_n\left(y\right) = 1$ for $y > x$, and $u_n\left(x\right) = \frac{1}{2}$ for all n. We have:

$$\begin{aligned} 0 &\leq E\left[u_n\left(X\right)\right] - E\left[u_n\left(Y\right)\right] \\ &= \int_{-\infty}^{x} u_n\left(y\right) d\left[F_X\left(y\right) - F_Y\left(y\right)\right] \\ &\quad + \int_{x}^{\infty} u_n\left(y\right) d\left[F_X\left(y\right) - F_Y\left(y\right)\right]. \end{aligned}$$

Setting $n \to \infty$, by Monotonic Convergence Theorem (Billingsley, Theorem 16.2), we have

$$\int_x^\infty d\left[F_X\left(x\right) - F_Y\left(x\right)\right] = F_Y\left(x\right) - F_X\left(x\right) \geq 0.$$

Appendix 2. Proof of Proposition 11.

For the sufficiency, suppose $X \succeq_2 Y$. For all $u \in C^2(\mathbb{R})$ with bounded first-order derivative u' and with negative second-order derivatives $u''(\cdot) \leq 0$, we obtain

$$\begin{aligned}
0 &\leq \int_{-\infty}^{\infty} u''(x) \left(\int_{-\infty}^{x} [F_X(y) - F_Y(y)]\, dy \right) dx \\
&= \int_{-\infty}^{\infty} \int_{y}^{\infty} u''(x) [F_X(y) - F_Y(y)]\, dxdy \\
&= \lim_{x \to +\infty} \int_{-\infty}^{x} [u'(x) - u'(y)] [F_X(y) - F_Y(y)]\, dy \\
&= \lim_{x \to +\infty} u'(x) \int_{-\infty}^{x} [F_X(y) - F_Y(y)]\, dy \\
&\quad + \lim_{x \to +\infty} u(x) [F_X(x) - F_Y(x)] \\
&\quad - \lim_{x \to -\infty} u(x) [F_X(x) - F_Y(x)] \\
&\quad + E[u(X)] - E[u(Y)].
\end{aligned}$$

By assumption, $u(\cdot)$ has bounded first-order derivatives. This implies the utility function $u(x)$ as $x \to \pm\infty$ is of order $O(x)$. This, together with the asymptotic conditions (13), implies

$$\lim_{x \to \pm\infty} u(x) [F_X(x) - F_Y(x)] = 0.$$

With these, the above inequality reduces to

$$\begin{aligned}
& E[u(X)] - E[u(Y)] \\
\geq & -\lim_{x \to \infty} u'(x) \int_{-\infty}^{x} [F_X(y) - F_Y(y)]\, dy \\
\geq & \ 0
\end{aligned}$$

which takes positive value since $u' \geq 0$ and since, by assumption, $\int_{-\infty}^{x} [F_X(y) - F_Y(y)]\, dy \leq 0$ for all x.

For the necessary part of Proposition 11, suppose $E[u(X)] \geq E[u(Y)]$ for all u with bounded first-order derivatives $u' \geq 0$ and with $u'' \leq 0$. We have,

$$\begin{aligned} 0 &\leq E\left[u\left(X\right)\right] - E\left[u\left(Y\right)\right] \\ &= -\int_{-\infty}^{\infty} u'\left(y\right)\left[F_X\left(y\right) - F_Y\left(y\right)\right] dy \end{aligned}$$

or,

$$\int_{-\infty}^{\infty} u'\left(y\right)\left[F_X\left(y\right) - F_Y\left(y\right)\right] dy \leq 0. \tag{16}$$

This inequality holds true for all increasing and concave smooth utility functions with bounded first-order derivatives. Now, for any arbitrary $x \in \mathbb{R}$, we consider the following sequence of utility functions $\{u_n\}_{n=1}^{\infty}$ in $C^{\infty}\left(\mathbb{R}\right)$ that are defined by

$$u_n\left(y\right) = \frac{y + x - \sqrt{\left(y - x\right)^2 + n^{-1}}}{2}, \forall y \in \mathbb{R}$$

for all positive integers n. For each n, we have

$$u_n'\left(y\right) = \frac{1}{2}\left[1 - \frac{y - x}{\sqrt{\left(y - x\right)^2 + n^{-1}}}\right] \in \left(0, 1\right);$$

that is, the utility functions are increasing and concave with its first-order derivatives to be strictly bounded within $(0, 1)$. Setting $n \to \infty$, we have $\lim_{n\to\infty} u_n'\left(y\right) = 1$ for $y < x$, $\lim_{n\to\infty} u_n'\left(y\right) = 0$ for $y > x$, and $u_n'\left(x\right) = \frac{1}{2}$ at $y = x$.

In light of inequality (16), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} u_n'\left(y\right)\left[F_X\left(y\right) - F_Y\left(y\right)\right] dy \\ = &\int_{-\infty}^{x} u_n'\left(y\right)\left[F_X\left(y\right) - F_Y\left(y\right)\right] dy \\ &+ \int_{-\infty}^{x} u_n'\left(y\right)\left[F_X\left(y\right) - F_Y\left(y\right)\right] dy \\ \leq &\ 0 \text{ for all } n = 1, 2, \cdots. \end{aligned}$$

Again, by the Monotonic Convergence Theorem (Billingsley, Theorem 16.2), we obtain

$$\int_{-\infty}^{x} \left[F_X\left(y\right) - F_Y\left(y\right)\right] dy \leq 0.$$

This holds for all arbitrary $x \in \mathbb{R}$. We, therefore, conclude that $X \succeq_2 Y$.

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