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A NOTE ON CONVEX STOCHASTIC DOMINANCE

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ABSTRACT

In this paper, we extend Fishburn's convex stochastic dominance theorem to include any distribution function. This paper also considers risk takers as well as risk averters, and discusses third order stochastic dominance. We apply separation and representation theorems to obtain a concise alternative proof of the theorem. Our results are used to extend a theorem of Bawa et.al. on comparison between a convex combinations of several contiuous distributions and a single continuous distribution.

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1. INTRODUCTION

There are three major types of persons: risk averters, risk neutrals and risk seekers. Their corresponding utility functions are concave, linear and convex; all are increasing functions. A person may change from risk seeker to risk averter or vice verse. A very common type is risk seekers for small outcomes, risk averter for large outcomes. Fishburn (1974, 1980) and Bawa, et. al. (1985) discussed the stochastic dominance only for risk averters while Meyer (1977), Stoyan (1983) and Li and Wong (1998) discussed the stochastic dominance for both risk seekers and risk averters.

This paper extends results of the convex stochastic dominance theorem in Fishburn (1974) by including all distribution functions. It also addresses the situation for risk takers as well as risk averters and includes comments on third order stochastic dominance. Fishburn introduces three lemmas to prove the theorem. We apply separation and representation theorems to obtain a simpler proof of the theorem. Our results are used to extend a theorem of Bawa et.al. on comparison between a convex combination of several contiuous distributions and a single continuous distribution. In particular, our theorem covers the cases of arbitrary distribution functions of risk takers and risk adverters, and third order stochastic dominance.

The results in this paper can be easily extended to include higher order stochastic dominance: see, for example, Fishburn (1980), O'Brien (1984) and Mukherjee and Chatterjee (1992).

To aviod confusion, we call stochastic dominance for risk seekers descending stochastic dominance (DSD) and call stochastic dominance for risk averters ascending stochastic dominance (ASD). We remark that Stoyan (1983) used concave and convex ordings for risk averters and risk seekers, respectively.

Our note is organized as follows. Section 2 introduces notations and definitions. Section 3 extends and provides an alternative proof of the convex stochastic dominance theorem introduced in Fishburn (1974). We also extend the results in Bawa, et. al (1985) to compare a convex combination of continuous distributions with a continuous distribution.

2. DEFINITIONS AND NOTATIONS

Denote by $\mathbf{R}$ the set of real numbers and let $\overline{\mathbf{R}}$ be the set of extended real numbers. Suppose that $\Omega = [a, b]$ is a subset of $\overline{\mathbf{R}}$ in which a and b can be finite or infinite. Let $\mathbf{B}$ be the Borel σ-field of Ω and μ be a *probability measure* on $(\Omega, \mathbf{B})$, with $\mu(\Omega) = 1$. The *probability distribution function* F of the measure μ is defined as:

$$F(x) = \mu[a, x] \qquad \text{for all} \quad x \in \Omega\,. \tag{1}$$

We remark that in this paper we define F different from the traditional definition to include both ascending and descending stochastic dominance. By basic probability theory, for any random variable X and probability measure P, there exists a unique induced probability measure μ on $(\Omega, \mathbf{B})$ and a probability distribution function F such that F satisfies (1) and

$$\mu(B) = P(X^{-1}(B)) = P(X \in B) \qquad \text{for any} \quad B \in \mathbf{B}\,.$$

An integral written in the form of $\int_A f(t)\, d\,\mu(t)$ or $\int_A f(t)\, d\,F(t)$ is a Lebesgue integral for any integrable function $f(t)$. If the integral has the same value for any set A which is equal to $(c, d]$, $[c, d)$ or $[c, d]$, then we use the notation $\int_c^d f(t)\, d\,\mu(t)$ instead. In addition, if μ is a Borel measure with $\mu(c, d] = d - c$ for any $c < d$, then we write the integral as $\int_c^d f(t)\, dt$. The Lebesgue integral $\int_c^d f(t) dt$ is equal to the Riemann integral if f is bounded and continuous almost everywhere on $[c, d]$; see Theorem 1.7.1 in Ash (1972).

We consider random variables defined on Ω, denoted by $X, Y, \cdots$. The probability distribution functions of X and Y are F and G respectively. The following notations will be used throughout this paper:

$$\begin{aligned}
&\mu_F = \mu_X = E(X) = \int_a^b x \, d\, F(x), \quad \mu_G = \mu_Y = E(Y) = \int_a^b x \, d\, G(x) \,; \\
&F_1^A(x) = F(x) \,, \quad G_1^A(x) = G(x) \,, \quad H_1^A(x) = F_1^A(x) - G_1^A(x) \,; \\
&F_1^D(x) = F^D(x) \,; \quad G_1^D(x) = G^D(x) \,; \quad H_1^D(x) = F_1^D(x) - G_1^D(x) \,; \\
&M_n^A(x) = \int_a^x M_{n-1}^A(y) \, dy \,, \ M_n^D(x) = \int_x^b M_{n-1}^D(y) \, dy \quad n = 2, 3; \text{ and } M = F, G, \text{ or } H.
\end{aligned} \tag{2}$$

Throughout this paper, all functions are assumed to be measurable, all random variables are assumed to satisfy:

$$F_1^A(a) = 0 \quad \text{and} \quad F_1^D(b) = 0. \tag{3}$$

Condition (3) will hold for any random variable except a random variable with positive probability at negative infinity or positive infinity.

We next define first, second and third order ascending stochastic dominance for risk averters; and then define first, second and third order descending stochastic dominance for risk seekers.

Definition 1. *Given two random variables X and Y with F and G as their respective probability distribution functions, X is at least as large as Y and F is at least as large as G in the sense of:*

a. *FASD, denoted by $X \succeq_1 Y$ or $F \succeq_1 G$, if and only if $F_1^A(x) \le G_1^A(x)$ for each x in $[a, b]$,*

b. *SASD, denoted by $X \succeq_2 Y$ or $F \succeq_2 G$, if and only if $F_2^A(x) \le G_2^A(x)$ for each x in $[a, b]$, and*

c. *TASD, denoted by $X \succeq_3 Y$ or $F \succeq_3 G$, if and only if $F_3^A(x) \le G_3^A(x)$ for each x in $[a, b]$ and $\mu_F \ge \mu_G$,* *where FASD, SASD and TASD stand for first, second and third order ascending stochastic dominance respectively.*

If in addition there exists x in $[a, b]$ such that $F_i^A(x) < G_i^A(x)$ for $i = 1, 2$ and 3, we say that X is larger than Y and F is larger than G in the sense of SFASD, SSASD and STASD, denoted by $X \succ_1 Y$ or $F \succ_1 G$, $X \succ_2 Y$ or $F \succ_2 G$, and $X \succ_3 Y$ or $F \succ_3 G$ respectively, where SFASD, SSASD, and STASD stand for strictly first, second and third order ascending stochastic dominance respectively.

Definition 2. *Given two random variables X and Y with F and G as their respective probability distribution functions, X is at least as large as Y and F is at least as large as G in the sense of:*

a. *FDSD, denoted by $X \succeq^1 Y$ or $F \succeq^1 G$, if and only if $F_1^D(x) \geq G_1^D(x)$ for each x in $[a, b]$,*

b. *SDSD, denoted by $X \succeq^2 Y$ or $F \succeq^2 G$, if and only if $F_2^D(x) \geq G_2^D(x)$ for each x in $[a, b]$, and*

c. *TDSD, denoted by $X \succeq^3 Y$ or $F \succeq^3 G$, if and only if $F_3^D(x) \geq G_3^D(x)$ for each x in $[a, b]$ and $\mu_F \geq \mu_G$,*

where FDSD, SDSD, and TDSD stand for first, second and third order descending stochastic dominance respectively.

If in addition there exists x in $[a, b]$ such that $F_i^D(x) > G_i^D(x)$ for $i = 1, 2$ and 3, we say that X is larger than Y and F is larger than G in the sense of SFDSD, SSDSD, and STDSD, denoted by $X \succ^1 Y$ or $F \succ^1 G, X \succ^2 Y$ or $F \succ^2 G$, and $X \succ^3 Y$ or $F \succ^3 G$ respectively, where SFDSD, SSDSD, and STDSD stand for strictly first, second and third order descending stochastic dominance respectively.

We remark that if $F \succeq_i G$ or $F \succ_i G$, then $-H_j^A$ is a distribution function (not necessarily probability distribution function) for any $j > i$, and there exists a unique measure μ such that $\mu[a,x] = -H_j^A(x)$ for any $x \in [a,b]$. Similarly, if $F \geq^i G$ or $F \succ^i G$, then H_j^D is distribution function for any $j > i$. H_j^A and H_j^D are defined in (2).

Definition 3.

a. *For $n = 1,2,3, U_n^A, U_n^{SA}, U_n^D$ and U_n^{SD} are sets of utility functions u such that:*

$$\begin{aligned} U_n^A(U_n^{SA}) &= \{u : (-1)^{i+1} u^{(i)} \geq (>)\, 0 \;,\; i = 1, \cdots, n\} \;, \\ U_n^D(U_n^{SD}) &= \{u : u^{(i)} \geq (>)\, 0 \;,\; i = 1, \cdots, n\}. \end{aligned}$$

where $u^{(i)}$ is the i^{th} derivative of the utility function u.

b. *The extended sets of utility functions are defined as follows:*

$$\begin{aligned} U_1^{EA}(U_1^{ESA}) &= \{u : u \textit{ is (strictly) increasing }\}, \\ U_2^{EA}(U_2^{ESA}) &= \{u \textit{ is increasing and (strictly) concave }\}, \\ U_2^{ED}(U_2^{ESD}) &= \{u \textit{ is increasing and (strictly) convex }\}, \\ U_3^{EA}(U_3^{ESA}) &= \{u \in U_2^{EA} : u' \textit{ is (strictly) convex }\}, \textit{and} \\ U_3^{ED}(U_3^{ESD}) &= \{u \in U_2^{ED} : u' \textit{ is (strictly) convex }\}. \end{aligned}$$

Note that in Definition 3 'increasing' means 'nondecreasing' and 'decreasing' means 'nonincreasing'. We also remark that in Definition 3, $U_1^A = U_1^D$ and $U_1^{SA} = U_1^{SD}$. We will use two notations U_1^{ED} and U_1^{ESD} in this paper such that $U_1^{ED} \equiv U_1^{EA}$ and $U_1^{ESD} \equiv U_1^{ESA}$. It is known (e.g. see Theorem 11C in Roberts and Varberg 1973) that u in U_2^{EA}, U_2^{ESA}, U_2^{ED}, or U_2^{ESD} , and u' in U_3^{EA}, U_3^{ESA}, U_3^{ED} or U_3^{ESD} are differentiable almost everywhere and their derivatives are continuous almost everywhere.

An individual chooses between F and G in accordance with a consistent set of preferences satisfying the Von Neumann-Morgenstern (1967) consistency properties. Accordingly, F is (strictly) preferred to G, or equivalently, X is (strictly) preferred to Y if

$$\Delta Eu \equiv u(F) - u(G) \equiv u(X) - u(Y) \geq 0 (> 0), \tag{4}$$

where $u(F) \equiv u(X) \equiv \int_a^b u(x)dF(x)$ and $u(G) \equiv u(Y) \equiv \int_a^b u(x)dG(x)$.

3. CONVEX STOCHASTIC DOMINANCE

Fishburn (1974) develops first and second order convex stochastic dominance theory of continuous distribution functions for risk averters. In this section we extend Fishburn's results by including all distribution functions, with application to risk seekers as well as for risk averters. We also cover the third order case. Denote the set of n-tuples of convex coefficients by:

$$\Lambda_n = \{(\lambda_1, \cdots, \lambda_n) : \lambda_i \geq 0 \text{ for } i = 1, \cdots, n, \text{ and } \sum_{i=1}^{n} \lambda_i = 1\} . \tag{5}$$

Theorem 1. *Let* $F_1, \cdots, F_n, G_1, \cdots, G_n$ *be distribution functions. For* $m = 1, 2,$ *or 3,*

a. *there exists* $(\lambda_1, \cdots, \lambda_n) \in \Lambda_n$ *such that*

$$\sum_{i=1}^{n} \lambda_i F_i \succeq_m (\succ_m) \sum_{i=1}^{n} \lambda_i G_i \tag{6}$$

if and only if for every utility function u *in* U *such that* $U_m^A \subseteq U \subseteq U_m^{EA}$ ($U_m^{SA} \subseteq U \subseteq U_m^{ESA}$) *there exists* $i \in \{1, \cdots, n\}$ *such that* F_i *is preferred to* G_i *for* u*;*

b. *there exists* $(\lambda_1, \cdots, \lambda_n) \in \Lambda_n$ *such that*

$$\sum_{i=1}^{n} \lambda_i F_i \geq^m (\succ^m) \sum_{i=1}^{n} \lambda_i G_i \tag{7}$$

if and only if for every utility function u *in* U *such that* $U_m^D \subseteq U \subseteq U_m^{ED}$ $(U_m^{SD} \subseteq U \subseteq U_m^{ESD})$ *there exists* $i \in \{1, \cdots, n\}$ *such that* F_i *is preferred to* G_i *for* u*.*

Proof: We only prove the second order descending stochastic dominance case. The proofs for other cases can be obtained similarly. We let f_i and g_i to be the probability density functions of F_i and G_i respectively. Suppose (7) is satisfied for second order descending stochastic dominance. Then for any $u \in U_2^{ED}$,

$$\sum_{j=1}^{n} \lambda_j \int_a^R u(t)\,(f_j(t) - g_j(t))\,dt \geq 0.$$

So there exists $i \in \{1, \cdots, n\}$ such that

$$\int_a^R u(t)\,(f_i(t) - g_i(t))\,dt \geq 0.$$

To prove the converse, let $\psi_i : [a, b] \to \mathbf{R}$ be defined by

$$\psi_i(R) = \int_R^b \int_y^b (f_i(t) - g_i(t))\,dt\,dy \quad (i = 1, \cdots, n)\,.$$

Suppose there is no $(\lambda_1, \cdots, \lambda_n) \in \Lambda_n$ such that

$$\sum_{j=1}^{n} \lambda_j\, \psi_j(R) \geq 0 \quad \text{for all } R \in [a, b].$$

We show that there exists u in U_2^D such that

$$\int_a^R u(t)\,(f_i(t) - g_i(t))\,dt < 0$$

for all $i = 1, \cdots, n$ as follows. Let $C[a, b]$ be the linear space of all real-valued measurable functions defined on $[a, b]$. Let A be the convex hull of the set

$$\{\psi_j : j = 1, \cdots, n\}.$$

Let $B \subset C[a, b]$ be the collection of all the functions Φ defined by

$$\Phi(R) = \int_R^b \int_y^b \phi(t)\, dt\, dy$$

for some $\phi \in C[a, b]$ that satisty $\Phi(R) \geq 0$ for all $R \in [a, b]$. Then the asserted condition implies $A \cap B = \Omega$. Since A is compact and convex and B is closed and convex, by a separation theorem (see Limaye [1981, p.54]) we can find a linear functional L on $c[a, b]$ and $\gamma_1, \gamma_2 \in \mathbf{R}$ such that

$$L(\Phi) > \gamma_2 > \gamma_1 > L(H)$$

for every $\Phi \in B$ and $H \in A$. Since $C[a, b]$ is a Hilbert space, by the Riesz Representation Theorem (see Limaye [1981, p.216]) we can find $\nu \in C[a, b]$ such that

$$\int_a^R \nu(R)\, \Phi(R)\, dR > \gamma_2 > \gamma_1 > \int_a^R \nu(R)\, \psi(R)\, dR.$$

Clearly we must have $\nu(R) \geq 0$ for all $R \in [a, b]$; otherwise, we can find $\Phi \in B$ such that $\int_a^R \nu(R)\, \Phi(R)\, dR < \gamma_2$ no matter how small γ_2 is. Moreover, $0 \in B$ implies $0 > \gamma_2$. So

$$0 > \int_a^R \nu(R)\, \psi(R)\, dR \quad \text{for all } \psi \in A\,.$$

Let $u(R) = \int_a^b \int_a^x \nu(t)\, dt\, dx$. Then for $i = 1, \cdots, n$,

$$\int_a^R u(t)\,(f_i(t) - g_i(t))\, dt \;=\; \int_a^R \int_a^x \int_a^y \nu(x)\,(f_i(t) - g_i(t))\, dx\, dy\, dt$$

$$
\begin{aligned}
&= \int_a^R \int_x^b \int_y^b \nu(x)\,(f_i(t) - g_i(t))\,dt\,dy\,dx \\
&= \int_a^R \nu(x)\,\psi_i(x)\,dx < 0\,.
\end{aligned}
$$

So u is the required function. ||

Theorem 1 generalizes Fishburn's results, and our use of separation and representation theorems allows a more concise proof.

In Theorems 4 to 6 of Bawa et. al (1985), the authors applied Theorem 2 of Fishburn (1974) to compare a convex combination of several contiuous distributions and a single continuous distribution for risk averters. Applying Theorem 1 in this section, we extend their results as follows:

Corollary 1. *Let $F_1, \cdots, F_n, F_{n+1}$ be distribution functions. For $m = 1, 2$ and 3,*

a. *there exists $(\lambda_1, \cdots, \lambda_n) \in \Lambda_n$ such that*

$$\sum_{i=1}^{n} \lambda_i F_i \succeq_m (\succ_m)\, F_{n+1}$$

if and only if for every utility function u in U_m^A (U_m^{SA}) there exists $i \in \{1, \cdots, n\}$ such that F_i is preferred to F_{n+1} for u.

b. *there exists $(\lambda_1, \cdots, \lambda_n) \in \Lambda_n$ such that*

$$\sum_{i=1}^{n} \lambda_i F_i \succeq^m (\succ^m)\, F_{n+1}$$

if and only if for every utility function u in U_m^D (U_m^{SD}) there exists $i \in \{1, \cdots, n\}$ such that F_i is preferred to F_{n+1} for u.

The sufficient part of Theorem 1 allows us to draw conclusions about preferences of risk seekers that supplement those specified by Theorem 1. For example, if we assume that u is in U_2^{SD}, and if it is not true that ($F_1 \succ^2 G_1$ or $G_1 \succ^2 F_1$), and not true that ($F_2 \succ^2 G_2$ or $G_2 \succ^2 F_2$) and if $\lambda_1 F_1 + (1 - \lambda_1) F_2 \succ^2 \lambda_1 G_1 + (1 - \lambda_1) G_2$ for some λ_1 strictly between 0 and 1, then we know that either F_1 is preferred to G_1 or F_2 is preferred to G_2 for u.

Similarly we can also draw some conclusions about the preferences of distribution functions by using the necessary part of Theorem 1. For example, assuming $u \in U_2^{SD}$, if $\sum \lambda_i F_i \succ^2 \sum \lambda_i G_i$ is false for every $\lambda \in \Lambda_n$, then there exists $u \in U_2^{SD}$ such that $u(G_i) \geq u(F_i)$ for all i, so that it is not possible to conclude that F_i is preferred to G_i for some i .

Corollary 1 is a special case of Theorem 1 in which all G_i are identical. This corollary can be used to compare a convex combination of distributions with a distribution for risk averters and risk seekers.

4. CONCLUDING REMARKS

Our development excluded only random variables with positive probability at the points of negative infinity or positive infinity. While it would not have been difficult to include such random variables in the theory,they seem to be of little practical interest.

Li and Wong (1998) establish some stochastic dominance theorems for risk seekers as well as risk averters, and apply the results to investment decision-making. One may use the findings in Li and Wong (1998) and the findings in our paper to study the behavior of risk averters and risk seekers in stock market or any other investment decision-making, see for example Tobin (1958), Markowitz (1970), Thompson and Wong (1991, 1996), Bian and Wong (1997) and Wong and Chew (1998) for reference.

BIBLIOGRAPHY

Ash, R.B. (1972). Real analysis and probability, Academic Press, New York.

Bawa, V.S., Bodurtha. JR.J.N., Rao, M.R. and Suri, H.L. (1985). 'On determination of stochastic dominance optimal sets'. *Journal of Finance,* XL, 2, 417-431.

Bian, G. and Wong,W.K. (1997). 'An Alternative Approach to Estimate Regression Coefficients,' The Journal of Applied Statistical Science, Vol 6, No. 1, 21-44.

Fishburn, P.C. (1974). 'Convex stochastic dominance with continuous distribution functions'. *Journal of Economic Theory,* 7, 143-158.

Fishburn, P.C. (1980). 'Stochastic dominance and moments of distributions'. *Mathematics of Operations Research,* 5, 94-100.

Limaye, B.V. (1981). Functional Analysis, John Wiley, New York.

Li, C.K. and Wong, W.K. (1998). 'A Note on Stochastic Dominance for Risk Averters and Risk Takers', *RAIRO Recherche Opérationnelle,* (forthcoming).

Markowitz, H. (1970). Portfolio Selection: Efficient Diversification of Investments, John Wiley, New York.

Meyer, J. (1977). 'Second degree stochastic dominance with respect to a function'. *International Economic Review,* 18, 476-487.

Mukherjee, S.P. and Chatterjee, A. (1992). 'Stochastic dominance of higher orders and its implications'. *Communications in Statistics – Theory and Methods,* 21, 1977-1986.

O'Brien, G.L. (1984). 'Stochastic dominance and moments inequalities". *Mathematics of Operations Research,* 9, 475-477.

Roberts, A.W. and Varberg, D.E. (1973). Convex Functions, John Wiley, New York.

Stoyan, D. (1983). Comparison Methods for Queues and Other Stochastic Models, John Wiley, New York.

Thompson H.E. and Wong, W.K. (1991). 'On the unavoidability of "scientific" judgement in estimating the cost of capital,' Managerial and Decision Economics, 12, 27-42.

Thompson H.E. and Wong, W.K. (1996). 'Revisiting "Dividend Yield Plus Growth" and Its Applicability,' Engineering Economist, Winter, Vol 41, No. 2, 123-147.

Tobin, J. (1958). 'Liquidity preference as behavior towards risk'. *Review of Economic Studies,* 25, 65-86.

Von Neumann, J. and Morgenstern, O. (1967). Theory of Games and Economic Behavior, John Wiley, New York.

Wong, W.K. and Chew, B.K. (1998). 'Can P/E ratio and bond yield be used to beat stock markets?' *Multinational Finance Journal,* (forthcoming).