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Abstract In this paper, we develop some stochastic dominance theorems for the location and scale family and linear combinations of random variables and for risk lovers as well as risk averters that extend results in Hadar and Russell (1971) and Tesfatsion (1976). The results are discussed and applied to decision-making.

Keywords: Ascending stochastic dominance, descending stochastic dominance, risk takers, risk averters, utility function

1 Introduction

There are three major types of persons: risk averters, risk neutrals and risk lovers. Their corresponding utility functions are concave, linear and convex; all are increasing functions. Many authors have studied the selection rules for risk averters. Markowitz (1952, 1970) and Tobin (1958, 1965) proposed the mean-variance selection rules for risk averters. Quirk and Saposnik (1962), Fishburn (1964, 1974), Hadar and Russell (1969, 1971), Hanoch and Levy (1969), Whitmore (1970), Rothschild and Stiglitz (1970, 1971), Tesfatsion (1976), Bawa (1975), and Bawa, et. al (1985) studied the stochastic dominance rules for risk averters. Meyer (1977) developed some results of second degree stochastic dominance with respect to a function. He discussed the stochastic dominance for risk lovers as well as risk averters.

In this paper we develop some stochastic dominance theorems for the location and scale family of random variables and linear combinations of random variables and for risk lovers as well as risk averters that extend results in Hadar and Russell (1971) and Tesfatsion (1976). We call stochastic dominance for risk lovers descending stochastic dominance (DSD). To avoid confusion, we call stochastic dominance for risk averters ascending stochastic dominance (ASD). We note that stochastic dominance for risk neutrals is a special case in the theory of stochastic dominance for risk averters or risk lovers. We also remark that Stoyan (1983) developed some results in ascending and descending stochastic dominances although he did not interpret the results in selecting rules for risk averters and risk lovers. Instead of using the terms ascending and descending stochastic dominances, he used concave and convex orderings.

We begin by introducing notation and definitions in Section 2. Section 3 discusses some basic properties for the stochastic dominance theory. Section 4 concerns the study of location and scale family of distributions and the properties of non-negative combinations of random variables for ASD and DSD. In Section 5, the stochastic dominance theories for risk lovers and risk averters are compared and applied to decision-making.

2. DEFINITIONS AND NOTATIONS

Denote by $\mathbf{R}$ the set of real numbers and let $\overline{\mathbf{R}}$ be the set of extended real numbers. Suppose that $\Omega = [a, b]$ is a subset of $\overline{\mathbf{R}}$ in which a and b can be finite or infinite. Let $\mathbf{B}$ be the Borel σ-field of Ω and μ be a *measure* on $(\Omega, \mathbf{B})$. The functions F and F^D of the measure μ are defined as:

$$F(x) = \mu[a, x] \quad \text{and} \quad F^D(x) = \mu[x, b] \qquad \text{for all} \quad x \in \Omega\,. \tag{1}$$

The function F is called a *probability distribution function* and μ is called a *probability measure* if $\mu(\Omega) = 1$. We remark that in this paper the definition of F which takes care of both ascending and descending stochastic dominance is different from the 'traditional' definition of F. By the basic probability theory, for any random variable X and for probability measure P, there exists a unique induced probability measure μ on $(\Omega, \mathbf{B})$ and the probability distribution function F such that F satisfies (1) and

$$\mu(B) = P(X^{-1}(B)) = P(X \in B) \qquad \text{for any} \quad B \in \mathbf{B}\,.$$

An integral written in the form of $\int_A f(t)\, d\,\mu(t)$ or $\int_A f(t)\, d\,F(t)$ is a Lebesgue-Stieltjes integral for any integrable function $f(t)$. If the integral has the same value for any set A which is equal to $(c, d]$, $[c, d)$ or $[c, d]$, then we use the notation $\int_c^d f(t)\, d\,\mu(t)$ instead. In addition, if μ is a Borel measure with $\mu(c, d] = d - c$ for any $c < d$, then we write the integral as $\int_c^d f(t)\, dt$. The Lebesgue-Stieltjes integral $\int_c^d f(t)\, dt$ is equal to the Riemann integral if f is bounded and continuous almost everywhere on $[c, d]$; see Theorem 1.7.1 in Ash (1972).

We consider random variables, denoted by $X, Y, \cdots$, defined on Ω. The probability distribution functions of X and Y are F and G respectively. The following notation will be used throughout this paper:

$$\mu_F = \mu_X = E(X) = \int_a^b x\, d\,F(x), \quad \mu_G = \mu_Y = E(Y) = \int_a^b x\, d\,G(x)\,;$$

$$
\begin{aligned}
&F_1^A(x) = F(x)\ , \quad G_1^A(x) = G(x)\ , \quad H_1^A(x) = F_1^A(x) - G_1^A(x)\ ; \qquad (2)\\
&F_1^D(x) = F^D(x)\ ; \quad G_1^D(x) = G^D(x)\ ; \quad H_1^D(x) = F_1^D(x) - G_1^D(x)\ ;\\
&M_n^A(x) = \int_a^x M_{n-1}^A(y)\, dy\ ,\ M_n^D(x) = \int_x^b M_{n-1}^D(y)\, dy\\
&\text{for} \quad n = 2, 3; \text{ and } M = F, G, \text{ or } H.
\end{aligned}
$$

Throughout this paper, all functions are assumed to be measureable, all random variables are assumed to satisfy:

$$F_1^A(a) = 0 \quad \text{and} \quad F_1^D(b) = 0. \qquad (3)$$

Condition (3) will hold for any random variable except a random variable with positive probability at the points negative infinity or positive infinity.

We next define the first, second and third order ascending stochastic dominances which are applied to risk averters; and then define the first, second and third order descending stochastic dominances which are applied to risk lovers.

Definition 1. *Given two random variables X and Y with F and G as their respective probability distribution functions, X is at least as large as Y and F is at least as large as G in the sense of:*

a. *FASD, denoted by $X \succeq_1 Y$ or $F \succeq_1 G$, if and only if $F_1^A(x) \le G_1^A(x)$ for each x in $[a, b]$,*

b. *SASD, denoted by $X \succeq_2 Y$ or $F \succeq_2 G$, if and only if $F_2^A(x) \le G_2^A(x)$ for each x in $[a, b]$,*

c. *TASD, denoted by $X \succeq_3 Y$ or $F \succeq_3 G$, if and only if $F_3^A(x) \le G_3^A(x)$ for each x in $[a, b]$ and $\mu_F \ge \mu_G$,*

   where FASD, SASD and TASD stand for first, second and third order ascending stochastic dominance respectively.

If in addition there exists x in $[a, b]$ such that $F_i^A(x) < G_i^A(x)$ for $i = 1, 2$ and 3, we say that X is large than Y and F is large than G in the sense of SFASD, SSASD and

STASD, denoted by $X \succ_1 Y$ or $F \succ_1 G$, $X \succ_2 Y$ or $F \succ_2 G$, and $X \succ_3 Y$ or $F \succ_3 G$ respectively, where SFASD, SSASD, and STASD stand for strictly first, second and third order ascending stochastic dominance respectively.

Definition 2. *Given two random variables X and Y with F and G as their respective probability distribution functions, X is at least as large as Y and F is at least as large as G in the sense of:*

a. *FDSD, denoted by $X \succeq^1 Y$ or $F \succeq^1 G$, if and only if $F_1^D(x) \geq G_1^D(x)$ for each x in $[a, b]$,*

b. *SDSD, denoted by $X \succeq^2 Y$ or $F \succeq^2 G$, if and only if $F_2^D(x) \geq G_2^D(x)$ for each x in $[a, b]$,*

c. *TDSD, denoted by $X \succeq^3 Y$ or $F \succeq^3 G$, if and only if $F_3^D(x) \geq G_3^D(x)$ for each x in $[a, b]$ and $\mu_F \geq \mu_G$,*
where FDSD, SDSD, and TDSD stand for first, second and third order descending stochastic dominance respectively.

If in addition there exists x in $[a, b]$ such that $F_i^D(x) > G_i^D(x)$ for $i = 1, 2$ and 3, we say that X is large than Y and F is large than G in the sense of SFDSD, SSDSD, and STDSD, denoted by $X \succ^1 Y$ or $F \succ^1 G, X \succ^2 Y$ or $F \succ^2 G$, and $X \succ^3 Y$ or $F \succ^3 G$ respectively, where SFDSD, SSDSD, and STDSD stand for strictly first, second and third order descending stochastic dominance respectively.

We remark that if $F \succeq_i G$ or $F \succ_i G$, then $-H_j^A$ is a distribution function for any $j > i$, and there exists a unique measure μ such that $\mu[a, x] = -H_j^A(x)$ for any $x \in [a, b]$. Similarly, if $F \succeq^i G$ or $F \succ^i G$, then H_j^D is distribution function for any $j > i$. H_j^D and H_j^A are defined in (2).

Definition 3.

a. *For $n = 1, 2, 3, U_n^A, U_n^{SA}, U_n^D$ and U_n^{SD} are sets of utility functions u such that:*

$$U_n^A(U_n^{SA}) \quad = \quad \{u : (-1)^{i+1} u^{(i)} \geq (>) \, 0 \, , \, i = 1, \cdots, n\} \, ,$$

$$U_n^D(U_n^{SD}) \quad = \quad \{u : u^{(i)} \geq (>) \, 0 \,, \; i = 1, \cdots, n\}.$$

where $u^{(i)}$ is the i^{th} derivative of the utility function u.

b. *The extended sets of utility functions are defined as follows:*

$$\begin{aligned}
U_1^{EA}(U_1^{ESA}) &= \{u : u \text{ is (strictly) increasing }\}, \\
U_2^{EA}(U_2^{ESA}) &= \{u \text{ is increasing and (strictly) concave }\}, \\
U_2^{ED}(U_2^{ESD}) &= \{u \text{ is increasing and (strictly) convex }\}, \\
U_3^{EA}(U_3^{ESA}) &= \{u \in U_2^{EA} : u' \text{ is (strictly) convex }\}, \text{and} \\
U_3^{ED}(U_3^{ESD}) &= \{u \in U_2^{ED} : u' \text{ is (strictly) convex }\}.
\end{aligned}$$

Note that in Definition 3 'increasing' means 'nondecreasing' and 'decreasing' means 'nonincreasing'. We also remark that in Definition 3, $U_1^A = U_1^D$ and $U_1^{SA} = U_1^{SD}$. We will use two notation U_1^{ED} and U_1^{ESD} in this paper such that $U_1^{ED} \equiv U_1^{EA}$ and $U_1^{ESD} \equiv U_1^{ESA}$. It is known (e.g. see Theorem 11C in Roberts and Varberg 1973) that u in U_2^{EA}, U_2^{ESA}, U_2^{ED}, or U_2^{ESD} , and u' in U_3^{EA}, U_3^{ESA}, U_3^{ED} or U_3^{ESD} are differentiable almost everywhere and their derivatives are continuous almost everywhere.

An individual chooses between F and G in accordance with a consistent set of preferences satisfying the Von Neumann-Morgenstern (1944) consistency properties. Accordingly, F is (strictly) preferred to G, or equivalently, X is (strictly) preferred to Y if

$$\Delta Eu \equiv u(F) - u(G) \equiv u(X) - u(Y) \geq 0 (> 0), \tag{4}$$

where $u(F) \equiv u(X) \equiv \int_a^b u(x) dF(x)$ and $u(G) \equiv u(Y) \equiv \int_a^b u(x) dG(x)$.

3. BASIC PROPERTIES

In this section we present some lemmas which are useful for the extension of stochastic dominance theory to include any random variable with any distribution function defined on a finite or infinite interval. The lemmas also enable the stochastic dominance results to apply to utility functions without the differentiability constraints. We also state a basic theorem of stochastic dominance theory in this section.

Lemma 1. *Let μ be σ-finite measure defined on $([a,b], \mathbf{B})$ where $\mathbf{B}$ is a σ-field of $[a,b]$. Suppose $F(x) = \mu[a,x]$ and $F^D(x) = \mu[x,b]$ for all $x \in [a,b]$. We consider c and d with $a \le c < d \le b$. If $F^D(c), F(d)$ are finite, and if G is increasing and continuous on $[c,d]$, then there exists a measure ν with $\nu[c,x] = G(x) - G(c)$ such that*

$$\int_{(c,d]} G(x)\, d\,\mu(x) = F(d)G(d) - F(c)G(c) - \int_{(c,d]} F(t)\, d\,\nu(t) \tag{5}$$

$$\int_{(c,d)} G(x)\, d\,\mu(x) = F^D(c)G(c) - F^D(d)G(d) + \int_{(c,d)} F^D(t)\, d\,\nu(t)\,. \tag{6}$$

The proof of Lemma 1 is in the Appendix. We remark that if F is continuous on $[c,d]$, then the continuity requirement of G can be dropped and we will obtain results similar to (5) and (6). Where G is decreasing or differentiable, results similar to (5) and (6) are also obtained. Applying Theorem 3.2.3 in Rohatgi (1975) and Lemma 1, one can prove the following lemma:

Lemma 2. *If X and Y be random variables defined on Ω with finite means μ_X and μ_Y respectively, then*

$$\mu_X - \mu_Y = \int_\Omega [G(t) - F(t)]\, dt = \int_\Omega [F_1^D(t) - G_1^D(t)]\, dt\,.$$

Note that $E(X)$ is finite if and only if both $E[XI_{\{X>0\}}]$ and $E[XI_{\{X<0\}}]$ are finite in Lebesgue measure. We remark that the constraint of finite means in Lemma 2 can be further relaxed. The following theorem identifies conditions under which ascending stochastic dominance and descending stochastic dominance can be considered as dual problems of each other:

Lemma 3. *For any random variables X and Y, we have the following:*

a. $X \succeq_i (\succ_i) Y$ *if and only if* $-Y \succeq^i (\succ^i) - X$ *for* $i = 1, 2$ *or* 3.

b. $X \succeq_1 (\succ_1) Y$ *if and only if* $X \succeq^1 (\succ^1) Y$.

c. *If X and Y have the same mean which is finite, then*

$$X \succeq_2 (\succ_2) Y \quad \textit{if and only if} \quad Y \succeq^2 (\succ^2) X \, .$$

For most existing stochastic dominance results, it is not difficult to modify the proofs for the cases of continuous random variables to obtain the proofs for any general distribution function by using basic probability theory and Lemma 1. In addition, if the stochastic dominance results for continuous density functions are available, the following lemmas may be applied to extend the results to include any general probability distribution functions:

Lemma 4. *For any random variable X, there exists a sequence of random variables $\{X_n\}$ with finite supports and continuous density functions such that X_n converges to X in distribution. In addition if X is of finite mean, then $\{X_n\}$ can be uniformly integrable.*

We remark that $\{X_n\}$ in Lemma 4 can be constructed to be defined on $\mathbf{R}$ or on infinite intervals which are bounded from above or below.

Lemma 5. *Let X be a random variable, if $\{X_n\}$ is a sequence of random variables such than X_n converges to X in distribution, then*

$$F_{n,1}^A \to F_1^A \quad \textit{and} \quad F_{n,1}^D \to F_1^D \quad \textit{almost everywhere} \qquad \textit{as} \quad n \to \infty \, ,$$

in addition if X is of finite mean, then

$$F_{n,2}^A \to F_2^A \quad \textit{and} \quad F_{n,2}^D \to F_2^D \quad \textit{almost everywhere} \qquad \textit{as} \quad n \to \infty \, ,$$

where F_i^A and F_i^D are defined as in (2) for the probability distribution function F of X

and $F^A_{n,i}$ and $F^D_{n,i}$ are similarly defined for the probability distribution function F_n of X_n for $i = 1$ and 2.

Lemma 6. *Suppose X_n, Y_n, X and Y are random variables such that X_n converges to X in distribution and Y_n converges to Y in distribution. If X_n and Y_n are independent, then $X_n + Y_n$ converges to $X + Y$ in distribution.*

The proofs of Lemmas 3 to 6 are straightforward and we omit the proofs. The following theorem describes some basic relation between utility functions and distribution functions:

Theorem 7. *Let X and Y be random variables with probability distribution functions F and G respectively. Suppose u is a utility function. For $m = 1, 2$ and 3; we have the following:*

a. *$F \succeq_m (\succ_m) G$ if and only if $u(F) \geq (>) u(G)$ for any u in U such that $U^A_m \subseteq U \subseteq U^{EA}_m$ $(U^{SA}_m \subseteq U \subseteq U^{ESA}_m)$.*

b. *$F \succeq^m (\succ^m) G$ if and only if $u(F) \geq (>) u(G)$ for any u in U such that $U^D_m \subseteq U \subseteq U^{ED}_m$ $(U^{SD}_m \subseteq U \subseteq U^{ESD}_m)$.*

There are many papers studied the results similar to the results of the above theorem. For example, Hadar and Russell (1971) and Bawa (1975) proved the ascending stochastic dominance results for continuous density functions and continuously differentiable utility functions. Hanoch and Levy (1969) and Tesfatsion (1976) proved the first and second order ascending stochastic dominance for general distribution functions. Rothschild and Stiglitz (1970, 1971) studied the special case of distributions with equal means and have proposed a condition that is equivalent to the second order ascending stochastic dominance results. Meyer (1977) discussed second order stochastic dominance for risk lovers and risk averters. Stoyan (1983) proved the first and second order stochastic dominance results for risk lovers as well as risk averters. In this paper, we provide the proof of Theorem 7 in the Appendix.

It is known that if $\mu_F = \mu_G$, $F \succeq_2 G$ $(F \succ_2 G)$ and if their variances exist, then $\sigma^2_F \leq \sigma^2_G$ $(\sigma^2_F < \sigma^2_G)$. If $\mu_F = \mu_G$, $F \succeq^2 G$ $(F \succ^2 G)$ and if their variances exist, then

$\sigma_F^2 \geq \sigma_G^2$ ($\sigma_F^2 > \sigma_G^2$). These reflect the fact that risk averters prefer to invest in prospects or portfolios with smaller variances while risk lovers prefer larger variances.

4 STOCHASTIC DOMINANCE FOR RANDOM VARIABLES

In this section, we study the stochastic dominance for random variables, and non-negative combinations, or equivalently convex combinations, of random variables. Random variables $X, Y, \cdots$ can be regarded as the returns of individual prospects and convex combinations of random variables can be regarded as the returns of the portfolios of different prospects. Hence, stochastic dominance for the random variables can be applied to check the preferences of different prospects and the preferences of different portfolios.

We remark that for any pair of random variables X and Y, the statements $X \succeq_m Y$, and $F \succeq_m G$ are equivalent. But for $n > 1$, the statements $\sum_{i=1}^n \alpha_i X_i \succeq_m \sum_{i=1}^n \alpha_i Y_i$ and $\sum_{i=1}^n \alpha_i F_i \succeq_m \sum_{i=1}^n \alpha_i G_i$ are different because the distribution functions of $\sum_{i=1}^n \alpha_i X_i$ and $\sum_{i=1}^n \alpha_i Y_i$ are different from $\sum_{i=1}^n \alpha_i F_i$ and $\sum_{i=1}^n \alpha_i G_i$. Therefore, we cannot apply the convex stochastic dominance theorems obtained in Fishburn (1974) to the convex combinations of random variables.

First we study the stochastic dominance of random variables X and Y which are in the same location and scale family such that $Y = p + qX$. The location parameter p can be viewed as the random variable with degenerate distribution at p.

Hadar and Russell (1971) studied the relation among random variables in the same location and scale family in Theorem 4 of their paper. Tesfatsion [1976, Theorem 1'] improved their results by relaxing some constraints. We further relax the constraints and also include the risk lover case in the following theorem:

Theorem 8. *Let X be a random variable with range $[a,b]$ and finite mean μ_X. Define the random variable $Y = p + qX$ with mean μ_Y.*

a. *If $p + qy \geq y$ for all $y \in [a,b]$, then $Y \succeq_1 X$, equivalently $Y \succeq^1 X$.*

b. *If $0 \leq q < 1$ such that $p/(1-q) \geq \mu_X$, i.e., $\mu_Y \geq \mu_X$, then $Y \succeq_2 X$.*

c. *if $0 \leq q < 1$ such that $p/(1-q) \leq \mu_X$, i.e., $\mu_X \geq \mu_Y$, then $X \succeq^2 Y$.*

The proof of Theorem 8 is in the Appendix.

Hadar and Russell [1971, Theorem 4] and Tesfatsion [1976, Theorem 1'] studied the results for (a) and (b) of the above theorem. Both papers impose the constraints in (a) such that $p \geq 0$, $q \geq 1$ and X is nonnegative. In this paper, we release all these constraints. Hadar and Russell [1971, Theorem 4] impose the constraints in (b) such that $p > 0$, $0 < q < 1$ and X is nonnegative. Tesfatsion [1976, Theorem 1'] releases the constraint on p and reduce the constraints for q to be $0 \leq q \leq 1$. In this paper, we further release the constraint imposed on X. We further improve the results to include the situation for descending stochastic dominance.

Hadar and Russell (1971) studied the invariance property of the stochastic dominance and obtained the following theorem for continuous distributed random variables in Theorem 5 of their paper:

Theorem 9. *Let X and Y denote two random variables with distribution functions F and G respectively, and assume that random variable W is independent of both X and Y. Let the distribution functions of the random variables $aX + bW$ and $aY + bW$ be denoted by $\hat{F}$ and $\hat{G}$, respectively, where $a > 0$, and $b \geq 0$. Then the following statements are true:*

a. *If G is larger than F in the sense of FASD, then $\hat{G}$ is larger than $\hat{F}$ in the sense of FASD.*

b. *If G is larger than F in the sense of SASD, then $\hat{G}$ is larger than $\hat{F}$ in the sense of SASD.*

Tesfatsion [1976, Theorem 2'] extended the results to include random variables with any distribution functions and release the nonnegative constraint imposed on b. However, this still requires that W is independent of both X and Y. We relax this constraint and compare two sets of independent variables and improve the results to include the situation for descending stochastic dominance as shown in the following theorem:

Theorem 10. *Let $\{X_1, \cdots, X_m\}$ and $\{Y_1, \cdots, Y_m\}$ be two sets of independent variables. For $n = 1, 2$ and 3; we have:*

a. *$X_i \succeq_n (\succ_n) Y_i$ for $i = 1, \cdots, m$ if and only if $\sum_{i=1}^m \alpha_i X_i \succeq_n (\succ_n) \sum_{i=1}^m \alpha_i Y_i$ for any $\alpha_i \geq 0, i = 1, \cdots, m$; and*

b. *$X_i \succeq^n (\succ^n) Y_i$ for $i = 1, \cdots, m$ if and only if $\sum_{i=1}^m \alpha_i X_i \succeq^n (\succ^n) \sum_{i=1}^m \alpha_i Y_i$ for any $\alpha_i \geq 0, i = 1, \cdots, m$.*

The proof of Theorem 10 is in the Appendix. The following corollary is obtained by applying Theorem 10:

Corollary 11. *Let X, Y be random variables and $k \in \mathbf{R}$ (the set of real number). For $n = 1, 2$ and 3,*

a. *if $X \succeq_n (\succ_n) Y$ then $X + k \succeq_n (\succ_n) Y + k$; and*

b. *if $X \succeq^n (\succ^n) Y$ then $X + k \succeq^n (\succ^n) Y + k$.*

In Theorems 8 and 9 of Hadar and Russell (1971), it is proved that if X_1 and X_2 are two independent and identically distributed non-negative random variables with continuous distributed functions, then

$$\frac{1}{2}(X_1 + X_2) \succeq_2 \lambda_1 X_1 + \lambda_2 X_2 \succeq_2 X_1 \text{ for any } (\lambda_1, \lambda_2) \in \Lambda_2 .$$

Tesfatsion (1976) extended the results by dropping the non-negative constraint on the random variables and the continuous requirement on the distribution functions. We remark that an alternative proof of this extension is simply to apply Lemmas 4 to 6 and

Corollary 11 in this paper to Theorems 8 and 9 in Hadar and Russell (1971). Then the results can be obtained immediately. In addition, one can easily extend the results to n random variables as shown in the following theorem:

Theorem 12. *Let $n \geq 2$. If $X_1, \cdots, X_n$ are independent and identically distributed, then*

$$\text{a.} \qquad \frac{1}{n}\sum_{i=1}^{n} X_i \succeq_2 \sum_{i=1}^{n} \lambda_i X_i \succeq_2 X_i \qquad \textit{for any} \quad (\lambda_1, \cdots, \lambda_n) \in \Lambda_n \ , \qquad \textit{and}$$

$$\text{b.} \qquad X_i \succeq^2 \sum_{i=1}^{n} \lambda_i X_i \succeq^2 \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \textit{for any} \quad (\lambda_1, \cdots, \lambda_n) \in \Lambda_n \ ,$$

where $\Lambda_n = \{(\lambda_1, \cdots, \lambda_n) : \lambda_i \geq 0 \text{ for } : i = 1, \cdots, n, \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}$. The proof of Theorem 12 is in the Appendix.

5. PREFERENCES OF RISK AVERTERS AND RISK LOVERS

In this section, we study the preferences of risk averters and risk lovers in an investment or gamble. We also study their preferences in a portfolio or any non-negative combination of investments or gambles. We call a person a second order ascending stochastic dominance (SASD) risk averter if his/her utility function belongs to U_2^{EA}, and a second order descending stochastic dominance (SDSD) risk lover if his/her utility function belongs to U_2^{ED}.

Tesfatsion [1976, Theorem 1'] extended the results in Hadar and Russell [1971, Theorem 4]. From the theorem, he claimed that the decision maker is confronted with the choice of transforming his current portfolio containing a random prospect into a diversified portfolio containing a sure prospect and a specified amount of the original random prospect. He also claimed that part (ii) of his theorem gives a necessary and sufficient condition for the second degree stochastic dominance of one portfolio over the other, assuming the diversified portfolio contains a positive "percentage" of the random respect.

By Theorem 8 in our paper, we further include the following information for risk averters or risk lovers in a single investment or gamble:

Property 13.

a. *Let X and Y be the returns of two investments or gambles. If X has the same distribution form as Y but has a higher mean, then all risk averters and risk lovers will prefer X.*

b. *For an investment or gamble with the mean of return less than or equal to zero, the highest preference of SASD risk averters is not to invest or gamble.*

c. *For an investment or gamble with the mean of return which is greater than or equal to zero , SDSD risk lovers will prefer to invest or gamble as much as possible.*

d. *Let X be the return of an investment or gamble with zero return, and $Y = qX$ with $0 \leq q < 1$, then SASD risk averters will prefer Y while SDSD risk lovers will prefer X.*

Hadar and Russell (1971) have pointed out that a diversified portfolio can be larger in the sense of SASD than a specialized portfolio only if its constituent prospects have equal means. They also derived several useful results in the portfolio diversification for risk averters in the case that all prospects are of the same mean. Applying Theorem 12, we can extend Theorem 9 in Hadar and Russell (1971) for the portfolio of n independent and identically distributed prospects to the following property:

Property 14. *For the portfolio of n independent and identically distributed prospects with $n \geq 2$, SASD risk averters will prefer the equal weight portfolio whereas SDSD risk lovers will prefer a single prospect.*

Finally, we remark that all other theorems in this paper can be applied to make inferences about the preferences of the risk averters and risk lovers. For example in the sufficient part of Theorem 10, we can infer that if a risk averter prefers prospect X_i to prospect Y_i for each i, then he will prefer a portfolio formed by the convex combination of X_i rather than the corresponding portfolio of Y_i.

6. CONCLUDING REMARKS

In this paper we establish some stochastic dominance theorems for risk lovers as well as risk averters, and apply the results to investment decision-making. We first proved basic properties which are helpful in generalizing existing stochastic dominance results, and then illustrated the techniques if generalization by proving some theorems.

Our development excluded only random variables with positive probability at the points of negative infinity or positive infinity. While it would not have been difficult to include such random variables in the theory,they seem of little practical interest.

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APPENDIX

Proof of Lemma 1. For the proof of (5) in the case in which G is increasing, we let

$$\chi(t,x) = \begin{cases} 1 & c < t < x \\ 0 & x \le t \le d \end{cases} .$$

Since G is continuous and increasing on $[c, d]$, there exists a measure ν such that

$$G(x) = G(c) + \nu(c,x) = G(c) + \int_{(c,d]} \chi(t,x) d\nu(t) .$$

By Fubini's Theorem and Corollary 2.6.5 in Ash (1972), we have

$$\begin{aligned} \int_{(c,d]} \int_{(c,d]} \chi(t,x)\, d\nu(t)\, d\mu(x) &= \int_{(c,d]} \left[\int_{(c,d]} \chi(t,x) d\mu(x) \right] d\nu(t) \\ &= \int_{(c,d]} \left[\int_{(t,d]} d\mu(x) \right] d\nu(t) \\ &= \int_{(c,d]} [F(d) - F(t)]\, d\nu(t) . \end{aligned}$$

Hence,

$$\begin{aligned} \int_{(c,d]} G(x) d\mu(x) &= \int_{(c,d]} [G(c) + \int_{(c,d]} \chi(t,x) d\nu(t)] d\mu(x) \\ &= F(d)G(d) - F(c)G(c) - \int_{(c,d]} F(t) d\nu(t) . \end{aligned}$$

The proof for (6) can be obtained similarly. ||

Proof of Theorem 7.

We only prove the necessary condition for both Theorem 7. The sufficient condition can be proved by contradiction. Huang and Litzenberger (1988) and others have proved the sufficient condition of SD for risk averters. One could easily modify their proofs to obtain the proof of sufficient conditions in Theorem 7 of our paper.

Proof of Part (a):

$$
\begin{aligned}
\Delta Eu &\equiv u(F) - u(G) \equiv \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) \\
&= [F(x) - G(x)]u(x)\big|_a^b - \int_a^b [F(x) - G(x)]u^{(1)}(x)\,dx \\
&= \int_a^b [G(x) - F(x)]u^{(1)}(x)\,dx \qquad (7) \\
&= \int_a^b [G_1^A(x) - F_1^A(x)]u^{(1)}(x)\,dx \qquad (8) \\
&= \int_a^b u^{(1)}(x)\,d\,[G_2^A(x) - F_2^A(x)] \\
&= [G_2^A(x) - F_2^A(x)]u^{(1)}(x)\Big|_a^b - \int_a^b [G_2^A(x) - F_2^A(x)]u^{(2)}(x)\,dx \\
&= A_1 + \int_a^b [F_2^A(x) - G_2^A(x)]u^{(2)}(x)\,dx \qquad (9) \\
&= A_1 + \int_a^b u^{(2)}(x)\,d\,[F_3^A(x) - G_3^A(x)] \\
&= A_1 + [F_3^A(x) - G_3^A(x)]u^{(2)}(x)\Big|_a^b - \int_a^b [F_3^A(x) - G_3^A(x)]u^{(3)}(x)\,dx \\
&= A_1 + A_2 + \int_a^b [G_3^A(x) - F_3^A(x)]u^{(3)}(x)\,dx \qquad (10)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= [G_2^A(b) - F_2^A(b)]u^{(1)}(b) \quad \text{and} \\
A_2 &= [F_3^A(b) - G_3^A(b)]u^{(2)}(b)\ . \qquad (11)
\end{aligned}
$$

If $F \succeq_1^A G$ then $F_1^A(x) \le G_1^A(x)$ for all x. If $u \in U_1^A$ then $u^{(1)} \ge 0$. Hence, from (8), we have $\Delta Eu = u(F) - u(G) \ge 0$ if $F \succeq_1^A G$ and $u \in U_1^A$.

If $F \succeq_2^A G$, then $F_2^A(x) \le G_2^A(x)$ for all x. If $u \in U_2^A$ then $u^{(1)} \ge 0$ and $u^{(2)}(x) \le 0$ for x. From (11), $A_1 \ge 0$, and hence from (9), $\Delta Eu = u(F) - u(G) \ge 0$.

If $F \succeq_3^A G$, then $F_3^A(x) \le G_3^A(x)$ for all x. If $u \in U_3^A$ then $u^{(1)} \ge 0$, $u^{(2)}(x) \le 0$ and $u^{(3)} \ge 0$. From (11), we have $A_2 > 0$. As we have shown $A_1 > 0$. Hence, from (10), $\Delta Eu = u(F) - u(G) \ge 0$.

Proof of Part (b):

$$
\begin{aligned}
\Delta Eu &\equiv u(F)-u(G)\equiv \int_a^b u(x)dF(x)-\int_a^b u(x)dG(x)\\
&= \int_a^b [G(x)-F(x)]u^{(1)}(x)\,dx \qquad \text{from (7)}\\
&= \int_a^b [F_1^D(x)-G_1^D(x)]u^{(1)}(x)\,dx \qquad (12)\\
&= -\int_a^b u^{(1)}(x)\,d\,[F_2^D(x)-G_2^D(x)]\\
&= -\,[F_2^D(x)-G_2^D(x)]u^{(1)}(x)\Big|_a^b+\int_a^b [F_2^D(x)-G_2^D(x)]u^{(2)}(x)\,dx\\
&= B_1+\int_a^b [F_2^D(x)-G_2^D(x)]u^{(2)}(x)\,dx \qquad (13)\\
&= B_1-\int_a^b u^{(2)}(x)\,d\,[F_3^D(x)-G_3^D(x)]\\
&= B_1-[F_3^D(x)-G_3^D(x)]u^{(2)}(x)\Big|_a^b+\int_a^b [F_3^D(x)-G_3^D(x)]u^{(3)}(x)\,dx\\
&= B_1+B_2+\int_a^b [F_3^D(x)-G_3^D(x)]u^{(3)}(x)\,dx \qquad (14)
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= [F_2^D(a)-G_2^D(a)]u^{(1)}(a) \quad \text{and}\\
B_2 &= [F_3^D(a)-G_3^D(a)]u^{(2)}(a)\;. \qquad (15)
\end{aligned}
$$

If $F \succeq_1^D G$ then $F_1^D(x) \geq G_1^D(x)$ for all x. If $u \in U_1^D$ then $u^{(1)} \geq 0$. Hence, from (12), we have $\Delta Eu = u(F) - u(G) \geq 0$

If $F \succeq_2^D G$, then $F_2^D(x) \geq G_2^D(x)$ for all x. If in addition, $u \in U_2^D$ then $u^{(1)} \geq 0$ and $u^{(2)}(x) \geq 0$ for all x. From (15), $B_1 \geq 0$, and hence from (13), $\Delta Eu = u(F) - u(G) \geq 0$.

If $F \succeq_3^D G$, then $F_3^D(x) \geq G_3^D(x)$ for all x. If in addition, $u \in U_3^R$ then $u^{(1)} \geq 0$, $u^{(2)}(x) \geq 0$ and $u^{(3)} \geq 0$ for all x. From (15), we have $B_2 > 0$. As we have shown $B_1 > 0$. Hence, from (14), $\Delta Eu = u(F) - u(G) \geq 0$.

||

Proof of Theorem 8. For part (a),

$$P(Y \leq y) \leq P(Y \leq p + qy) = P(p + qX \leq p + qy) = P(X \leq y)$$

Hence, $Y \succeq_1 X$. Apply Lemma 3b, we have $Y \succeq^1 X$. Refer to Tesfatsion (1976) for the proof of part (b). For part (c), we let $Y' = -X$, $X' = -Y$, and $p' = -p$, apply Lemma 3(a) and part (b) of this theorem, then delete all the $'$, we get the result.

Proof of Theorem 10. The proofs for the necessary parts of the theorem are obvious. For the sufficient part in part (a), it suffices to prove the following two lemmas:

Lemma A: *X and Y are random variables. For $n = 1, 2$ and 3, and for $\alpha > 0, X \succeq_n (\succ_n) Y$ implies $\alpha X \succeq_n (\succ_n) \alpha Y$.*

Lemma B: *Suppose X_1, X_2, Y_1 and Y_2 are random variables such that X_1 and X_2 are independent, and Y_1 and Y_2 are independent. For $n = 1, 2$ or 3, if $X_i \succeq_n (\succ_n) Y_i$ for $i = 1$ and 2, then $X_1 + X_2 \succeq_n (\succ_n) Y_1 + Y_2$.*

The proof of Lemma A is obvious. For Lemma B, we only prove the case for the second order ascending stochastic dominance. The proofs for other cases can be obtained similarly. We suppose without loss of generality that X_1, X_2, Y_1 and Y_2 are defined on [a,b]. Let $X = X_1 + X_2$ and $Y = Y_1 + Y_2$. Let the probability distribution functions of X, X_1, X_2, Y, Y_1 and Y_2 be F, F_1, F_2, G, G_1 and G_2 respectively. We define $H^A_{i,n}$, $F^A_{i,n}$, and $G^A_{i,n}$ in terms of F_i and G_i for $i = 1, 2$ and for $n = 1, 2$ in the same manner of (2).

Since X_1 and X_2 are independent and Y_1 and Y_2 are independent, by Theorem 6.1.1 in Chung (1975), we have

$$F^A_1(x) = \int_a^R F^A_{1,1}(x - t)\, d\, F^A_{2,1}(t) \quad \text{and} \quad G^A_1(x) = \int_a^R G^A_{1,1}(x - t)\, d\, G^A_{2,1}(t)\, .$$

Hence,

$$H^A_2(y) \;\; = \;\; \int_{2a}^{y} F^A_1(x) - G^A_1(x)\, dx$$

$$= \int_{2a}^{y}\int_{a}^{R} F_{1,1}^{A}(x-t)\, d\, F_{2,1}^{A}(t)\, dx - \int_{2a}^{y}\int_{a}^{R} G_{1,1}^{A}(x-t)\, d\, G_{2,1}^{A}(t)\, dx\,.$$

by Fubini's Theorem and Corollary 2.6.5 in Ash (1972), we have

$$\begin{aligned} H_2^A(y) &= \int_a^R \int_{2a}^{y} F_{1,1}^A(x-t)\, dx\, d\, F_{2,1}^A(t) - \int_a^R \int_{2a}^{y} G_{1,1}^A(x-t)\, dx\, d\, G_{2,1}^A(t) \\ &= \int_a^R F_{1,2}^A(z-t)\, d\, F_{2,1}^A(t) - \int_a^R G_{1,2}^A(z-t)\, d\, G_{2,1}^A(t) \\ &\le \int_a^R G_{1,2}^A(y-t)\, d\, [F_{2,1}^A - G_{2,1}^A](t) \quad \text{since} \quad X_1 \succeq_2 Y_1\,. \end{aligned}$$

Applying Lemma 1 twice, we have

$$H_2^A(y) \le G_{1,1}^A(y-b) H_{2,2}^A(b) + \int_{y-b}^{y-a} H_{2,2}^A(y-s) dG_{1,1}^A(s) \le 0$$

as $H_{2,2}^A \le 0$ and $G_{1,1}^A$ is the probability distribution function.

Hence,

$$X_1 + X_2 \succeq_2 Y_1 + Y_2.$$

For the proof of part (b), the results hold by applying Lemma 3 and part (a) of this theorem. ||

Proof of Theorem 12. We prove by induction on n. The result is true if $n = 2$. Suppose the result is true up to $(n-1)$ independent variables with $n > 3$. We consider the case with n variables $X_1, ..., X_n$. Let $(\lambda_1, ..., \lambda_n) \in \Lambda_n$.

For part (a), to prove the second inequality, construct the new variable $Y = (\lambda_2 X_2 + ... + \lambda_n X_n)/(1-\lambda_1)$. Then $\lambda_1 X_1 + (1-\lambda_1) Y \ge_2 X_1, Y$; and also $Y \ge_2 X_i$ for $i = 2, ..., n$, by induction assumption. The result follows.

To prove the first inequality, let λ_i and λ_j be the maximum and minimum among λ_k's. If $\lambda_i > \lambda_j$, we replace both λ_i and λ_j by their average $\lambda = (\lambda_i + \lambda_j)/2$. Then $(X_i + X_j)/2 \ge_2 (\lambda_i X_i + \lambda_j X_j)/2\lambda$ by the 2-variable result, and hence $\lambda X_i + \lambda X_j \ge_2 \lambda_i X_i + \lambda_j X_j$. Adding the other $\lambda_k X_k$'s on both sides will clearly preserve $\ge_2$ by Theorem 10. As a result, whenever λ_i are not all equal, one can find a convex combination of $X_1, ..., X_n$ with larger value under the ordering $\geq_2$. Hence the maximum value must occur at the combination with equal $\lambda_i, i.e., \lambda_i = 1/n$ for all i.

One can prove (b) by similar arguments.

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