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Some results on k-edge-magic broken wheel graphs

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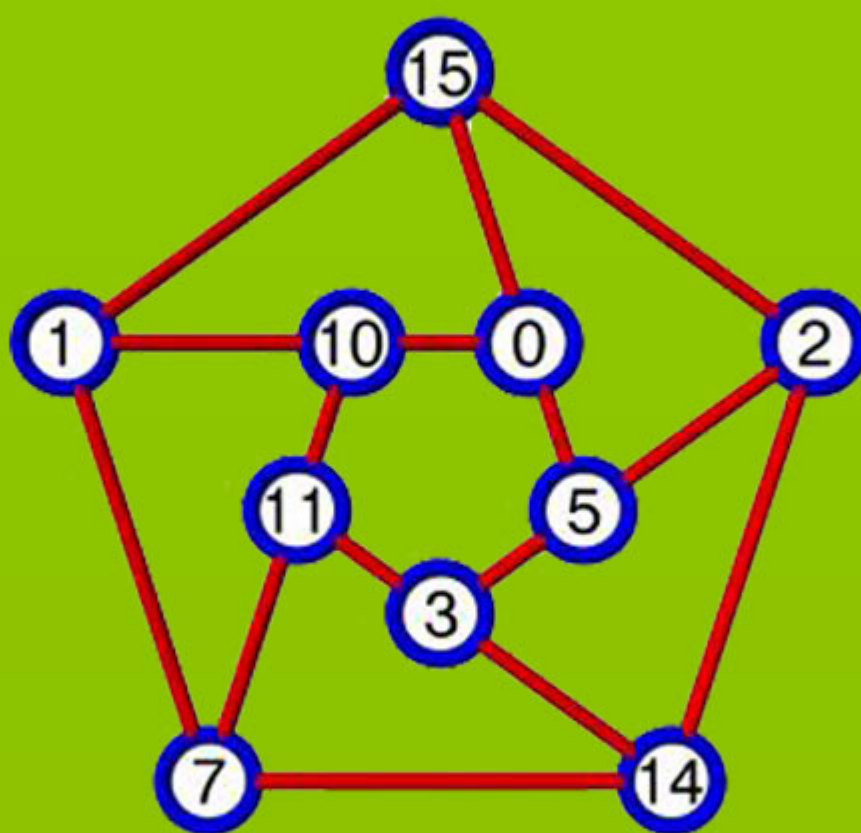
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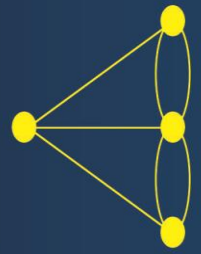
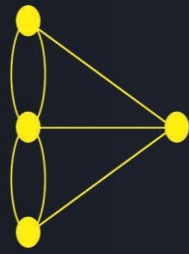
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SOME RESULTS ON k -EDGE-MAGIC BROKEN WHEEL GRAPHS

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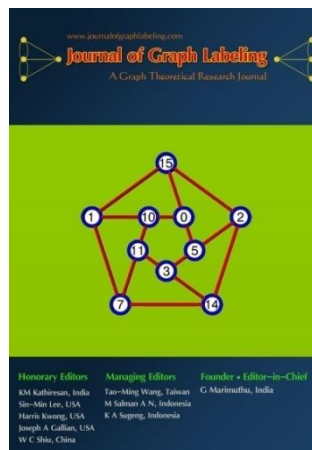
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Abstract

Let G be a (p, q) -graph in which the edges are labeled $k, k + 1, \dots, k + q - 1$, where $k \in \mathbb{Z}$. The vertex sum for a vertex v is the sum of the labels of the incident edges at v . If the vertex sums are constant modulo p , then G is said to be k -edge-magic. In this paper, we give necessary conditions for a family of regular broken wheel graphs to admit k -edge-magic labelings. Consequently, we show that some of these conditions are also sufficient.

1 Introduction

All undefined symbols and concepts may be looked up from [1]. Let $G = (V, E)$ be a (p, q) -graph, i.e., $|V| = p$ and $|E| = q$. Let $f : E \rightarrow \{k, k + 1, \dots, k + q - 1\}$ be a bijection for some $k \in \mathbb{Z}$. The induced mapping $f^+ : V \rightarrow \mathbb{Z}_r$ of f is defined by $f^+(u) = \sum_{uv \in E} f(uv)$

for $u \in V$, the sum is taken in \mathbb{Z}_r for some $r \geq 0$. Note that we fix $\mathbb{Z}_r = \{0, 1, \dots, r - 1\}$ for $r \geq 1$ and denote \mathbb{Z} by \mathbb{Z}_0 . If f^+ is a constant mapping, then G is called k -edge-magic over \mathbb{Z}_r . If $k = 1$, then G is simply called edge-magic over \mathbb{Z}_r , f an edge-magic labeling of G over \mathbb{Z}_r and the value of f^+ an edge-magic value of G over \mathbb{Z}_r . This concept was first introduced by Shiu and Lee [18] in 2002. Moreover, G being edge-magic over \mathbb{Z}_p or \mathbb{Z} is called edge-magic or supermagic, the labeling f is called an edge-magic labeling or supermagic labeling, respectively. These concepts were introduced by Lee, Seah and Tan [9] in 1992 and Stewart [22] in 1966, respectively. Note that edge-magic value is not unique in general.

The necessary condition (see [16]) for (p, q) -graph being k -edge-magic is

$$q(q + 2k - 1) \equiv 0 \pmod{p}. \quad (1.1)$$

Some edge-magic or supermagic graphs were found [5, 9–11, 14–23]. More about supermagic graphs can be found in [2, 4, 6, 7]. For regular graph, there is no different between k -edge-magic and edge-magic (see [16, 18, 19]).

Let $[a, b]$ denote the set of integers from a to b . Let S and T be multisets of integers. $S \equiv T \pmod{r}$ means that two sets are equal after their elements are taken modulo r , where $r \geq 2$. From now on, the term “set” means multiset. Set operations are viewed as multiset operations.

A wheel graph (or wheel, for short) $W_p = C_{p-1} \vee K_1$ is the join graph of the cycle C_{p-1} and the complete graph K_1 , where $p \geq 4$. So W_p is a $(p, 2p - 2)$ -graph and hence the

necessary condition for W_p -graph being k -edge-magic is

$$4k \equiv 6 \pmod{p}. \quad (1.2)$$

Fukuchi [3] showed that wheel graph with p vertices is edge-magic if integer $p \geq 5$, and $p \not\equiv 0 \pmod{4}$. However, the term 'edge-magic' in [3] is different from edge-magic in this paper. Lee, Wong and Lo [12] studied on the $Q(a)$ -balance edge-magic graphs and provided some results, such as W_5 , W_7 , W_8 and W_9 are strong balance edge-magic. They also showed that all wheels are not edge-magic since (1.1) is not satisfied. By using (1.2) it is easy to obtain:

Proposition 1.1 ([13]). *If the wheel graph W_p is k -edge-magic, then we have the following three cases:*

1. $p = 4h + 1$ and $k \equiv 2h + 2 \pmod{p}$;
2. $p = 4h + 3$ and $k \equiv 2h + 3 \pmod{p}$;
3. $p = 4h + 2$ and $k \equiv h + 2$ or $3h + 3 \pmod{p}$.

Lee, Su and Wang [13] showed the converse for cases 1 and 2. They also showed that W_6 is k -edge-magic for all $k \equiv 0, 3 \pmod{6}$, as an example. They conjectured that the converse of case 3 holds. It is still open.

It is easy to obtain the following proposition.

Proposition 1.2. *Suppose G is a graph of order p and $k \in \mathbb{Z}$. Then G is k -edge-magic if and only if G is $(k + pt)$ -edge-magic for $t \in \mathbb{Z}$.*

So from now on we always assume that $1 \leq k \leq p$.

2 Broken wheel graphs

In this paper, we shall study the k -edge-magicness of some broken wheel graphs. Let us introduce some definitions and notation first.

Let $V(W_p) = \{c, u_1, \dots, u_{p-1}\}$, where $u_1u_2 \cdots u_{p-1}u_1$ is a $(p-1)$ -cycle and c is the *hub* (or the *center*) of the wheel, i.e., $\deg(c) = p-1 = \Delta(W_p)$. The edge cu_i , $1 \leq i \leq p-1$ is called a *spoke* of the wheel. Let $S = \{cu_i \mid 1 \leq i \leq p-1\}$ be the set of all spokes. Let $\emptyset \neq A \subset S$. The graph $W_p(A) = W_p - (S \setminus A)$ is called a *broken wheel graph* (or *broken wheel*, for short). We shall keep these notation throughout this paper.

Let w be a factor of $p-1$ with $w \geq 2$. Let $A_w = \{cu_{iw+1} \mid 0 \leq i \leq (p-1)/w-1\}$. The graph $RW_p(w) = W_p(A_w)$ is called a *regular broken wheel*. Note that $RW_p(w)$ is a

(p, q) -graph, where $q = (w + 1)(p - 1)/w$. Clearly $p \leq q < 2p$. The following figures show $W_9, RW_9(2), RW_9(4)$ and $RW_9(8)$.

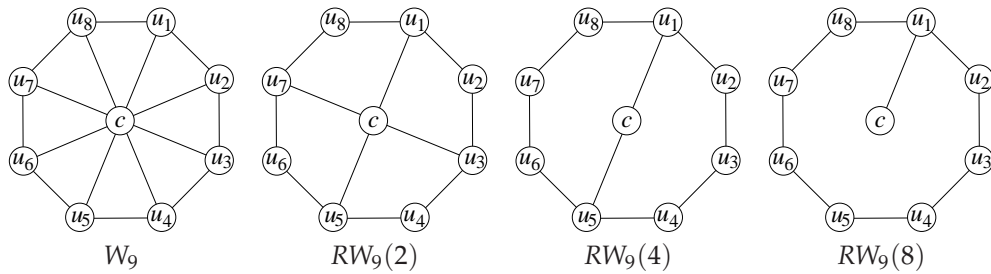


Figure 1

By pigeonhole principle, it is easy to show that

Lemma 2.1. *Let $p \geq 4$, $w|(p - 1)$ with $w \geq 2$ and $q = (w + 1)(p - 1)/w$. Let $L = [k, k + q - 1]$ for some k . Then every element in L is congruent modulo p to at most one other element in L . Moreover, $L \equiv \mathbb{Z}_p \cup R$, where $R \subset \mathbb{Z}_p$.*

Theorem 2.2. *If $RW_p(w)$ is k -edge-magic, then $w = 2$.*

Proof. Suppose $w \geq 5$ and f is a k -edge-magic labeling of $RW_p(w)$. Since $f^+(u_2) = f^+(u_3) = f^+(u_4) = f^+(u_5)$ in \mathbb{Z}_p , $f(u_1u_2) = f(u_3u_4) = f(u_5u_6)$ in \mathbb{Z}_p . But since the image of f is $L = [k, k + q - 1]$, where $q = (w + 1)(p - 1)/w$, by Lemma 2.1 we get a contradiction. So $2 \leq w \leq 4$.

Suppose $w = 4$. Consider the subpath $u_{1+4i}u_{2+4i}u_{3+4i}u_{4+4i}u_{5+4i}$ of the cycle $C = u_1u_2 \cdots u_{p-1}u_1$, $0 \leq i \leq (p - 5)/4$ (here $u_p = u_1$). By a similar argument above we obtain that $f(u_{1+4i}u_{2+4i}) = f(u_{3+4i}u_{4+4i})$ and $f(u_{2+4i}u_{3+4i}) = f(u_{4+4i}u_{5+4i})$ in \mathbb{Z}_p . Since f is a bijection, by Lemma 2.1 $L \equiv Q \cup Q \pmod{p}$ for some $Q \subseteq \mathbb{Z}_p$. But it is impossible by Lemma 2.1 again.

Suppose $w = 3$. Consider the subpath $u_{1+3i}u_{2+3i}u_{3+3i}u_{4+3i}$ of the cycle C , $0 \leq i \leq (p - 4)/3$. Similarly, we have $f(u_{1+3i}u_{2+3i}) = f(u_{3+3i}u_{4+3i})$ in \mathbb{Z}_p . Then

$$\{f(u_{1+3i}u_{2+3i}), f(u_{3+3i}u_{4+3i}) \mid 0 \leq i \leq (p - 4)/3\} \equiv Q \cup Q \pmod{p},$$

where $Q \subseteq \mathbb{Z}_p$ with $|Q| = (p - 1)/3$. By viewing Q as a subset of \mathbb{Z} , let r be the largest integer in Q . Then there is an edge e such that $f(e) = r + p$. Since $|Q| = (p - 1)/3$, $r + p \geq (k + (p - 1)/3 - 1) + p = k + q$ which is impossible. \square

By the above theorem, we only focus on $w = 2$. Thus if $RW_p(2)$ is k -edge-magic, then $p = 2n + 1$ for some $n \geq 2$. In some articles, for example [8], $RW_{2n+1}(2)$ is also called a gear graph and denoted by G_n . For simplicity, we shall use this notation for the rest of this paper.

3 Property of edge-magic gear graphs

Suppose $f : E(G_n) \rightarrow L = [k, k + 3n - 1]$ is a k -edge-magic labeling of G_n . We let $L \equiv \mathbb{Z}_{2n+1} \cup R \pmod{2n+1}$ for some $R \subset \mathbb{Z}_{2n+1}$. Note that $|R| = n - 1$. For convenience, we let $u_0 = u_{2n}$ and $u_{2n+1} = u_1$. We shall keep these notation throughout this paper.

Theorem 3.1. *For $n \geq 2$, if f is a k -edge-magic labeling of G_n , then $f^+ = 0$.*

Proof. Suppose $f^+ = s$ for some $s \in \mathbb{Z}_{2n+1}$. Then we have

$$f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) \equiv s \pmod{2n+1}, \quad 1 \leq i \leq n; \quad (3.1)$$

$$f(u_{2i-2}u_{2i-1}) + f(u_{2i-1}u_{2i}) + f(cu_{2i-1}) \equiv s \pmod{2n+1}, \quad 1 \leq i \leq n; \quad (3.2)$$

$$\sum_{i=1}^n f(cu_{2i-1}) \equiv s \pmod{2n+1}. \quad (3.3)$$

Subtracting (3.2) by (3.1) we have

$$f(u_{2i-2}u_{2i-1}) + f(cu_{2i-1}) - f(u_{2i}u_{2i+1}) \equiv 0 \pmod{2n+1}.$$

Hence

$$\sum_{i=1}^n f(u_{2i-2}u_{2i-1}) + \sum_{i=1}^n f(cu_{2i-1}) - \sum_{i=1}^n f(u_{2i}u_{2i+1}) \equiv 0 \pmod{2n+1}.$$

So we have $\sum_{i=1}^n f(cu_{2i-1}) \equiv 0 \pmod{2n+1}$.

Hence from (3.3) we have $s \equiv 0 \pmod{2n+1}$. □

Following we show the necessary condition for $G_{(p-1)/2} = RW_p(2)$ being k -edge-magic.

Lemma 3.2. *Suppose G_n is k -edge-magic for $n \geq 2$, where $1 \leq k \leq 2n + 1$. We have*

1. if $n = 6h$, then $k = 9h + 2$;
2. if $n = 6h + 1$, then $k = 3h + 2, 7h + 3$ or $11h + 4$;
3. if $n = 6h + 2$, then $k = 9h + 5$;
4. if $n = 6h + 3$, then $k = 3h + 3$;
5. if $n = 6h + 4$, then $k = h + 2, 5h + 5$ or $9h + 8$;
6. if $n = 6h + 5$, then $k = 3h + 4$.

Proof. From (1.1) we have

$$\frac{3}{2}(p-1) \left(\frac{3}{2}(p-1) + 2k - 1 \right) \equiv 0 \pmod{p}. \tag{3.4}$$

1. When $n = 6h$, from (3.4), we have

$$(18h)(18h + 2k - 1) \equiv 0 \pmod{12h + 1}.$$

Equivalently,

$$(6h - 1)(6h + 2k - 2) \equiv 0 \pmod{12h + 1}.$$

This implies that

$$(-3)(3h + k - 1) \equiv 0 \pmod{12h + 1}.$$

Thus

$$k \equiv 9h + 2 \pmod{12h + 1}.$$

2. When $n = 6h + 1$, from (3.4), we have

$$(18h + 3)(18h + 2k + 2) \equiv 0 \pmod{12h + 3}$$

which implies that

$$3k \equiv -3h + 3 \pmod{12h + 3}.$$

Thus

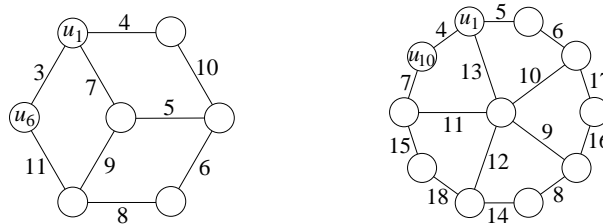
$$k \equiv -h + 1 \equiv 3h + 2 \pmod{4h + 1}.$$

So $k = 3h + 2, 7h + 3$ or $11h + 4$.

3. The proofs of the remaining cases are similar. We leave to readers.

□

Example 3.1. Following are two k -edge-magic labelings of the graphs G_3 and G_5 for a suitable k , respectively.



3-edge-magic labeling for G_3 . 4-edge-magic labeling for G_5 .

Figure 2

For convenience, we will represent the labeling f of the gear graph G_n by the following list:

| | | | | | | |
|------------------|-------------|-------------|-------------|---------|---------------------|----------------|
| $[f(u_{2n}u_1)]$ | $f(u_1u_2)$ | $f(u_2u_3)$ | $f(u_3u_4)$ | \dots | $f(u_{2n-1}u_{2n})$ | $f(u_{2n}u_1)$ |
| . | $f(cu_1)$ | . | $f(cu_3)$ | \dots | $f(cu_{2n-1})$ | . |

So for the 4-edge-magic labeling of G_5 above, we represent it as

| | | | | | | | | | | |
|-----|----|---|----|----|---|----|----|----|----|---|
| [4] | 5 | 6 | 17 | 16 | 8 | 14 | 18 | 15 | 7 | 4 |
| . | 13 | . | 10 | . | 9 | . | 12 | . | 11 | . |

After taking modulo 11, the set of edge labels $L = [4, 18] \equiv \mathbb{Z}_{11} \cup [4, 7] \pmod{11}$. So we will represent the list above as

| | | | | | | | | | | |
|-----|---|---|----|---|---|---|---|---|---|---|
| [4] | 5 | 6 | 6 | 5 | 8 | 3 | 7 | 4 | 7 | 4 |
| . | 2 | . | 10 | . | 9 | . | 1 | . | 0 | . |

Lemma 3.3. *Suppose $f : E(G_n) \rightarrow \mathbb{Z}_{2n+1} \cup R$ is a k -edge-magic labeling. Suppose $0 \notin R$. Then $f(cu_{2i+1}) = 0$ for some i .*

Proof. Suppose an edge e is incident with a vertex v of degree 2 and $f(e) = 0$. Since $f^+ = 0$, another edge incident with v must be labeled by 0. Thus, $f(cu_{2i+1}) = 0$ for some i . □

Lemma 3.4. *Suppose $f : E(G_n) \rightarrow \mathbb{Z}_{2n+1} \cup R$ is a k -edge-magic labeling. Suppose $0 \in R$. Then either $f(cu_{2i+1}) = 0 = f(cu_{2j+1})$ for some distinct i, j or $f(u_{2i-1}u_{2i}) = f(u_{2i}u_{2i+1}) = 0$ for some i .*

Proof. By the proof of Lemma 3.3, if an edge e in the outer cycle of G_n is labeled by 0, then the edge with a common vertex of degree 2 with e is also labeled by 0. This completes the proof since there are exactly two zeros in the list of edge labels. □

For some cases, the necessary condition showed in Lemma 3.2 are not sufficient. Let us show you some examples. All arithmetics about the labels are taken in \mathbb{Z}_{2n+1} .

Example 3.2. Let f be a k -edge-magic labeling of G_2 . According to the notation in Lemma 3.2, $h = 0$ and $k = 5$. Then $L \equiv \mathbb{Z}_5 \cup \{0\} \pmod{5}$. Up to isomorphic and by Lemma 3.4 we may assume that $f(cu_1) = f(cu_3) = 0$. It is easy to see that f does not exist.

Example 3.3. Let f be a k -edge-magic labeling of G_4 . According to the notation in Lemma 3.2, $h = 0$ and $k = 2, 5$ or 8 .

Suppose $k = 2$. Then $L \equiv \mathbb{Z}_9 \cup \{2, 3, 4\}$. By Lemma 3.3, without loss of generality we may assume $f(cu_1) = 0$. Suppose $f(u_1u_2) = a \neq 0$. Then $f(u_2u_3) \equiv f(u_8u_1) \equiv -a$ and $f(u_7u_8) \equiv a$. But there is no such a in L .

By a similar proof as above, we can show that there is no 5-edge-magic labeling of G_4 .

Suppose $k = 8$. Then $L \equiv \mathbb{Z}_9 \cup \{8, 0, 1\} = \mathbb{Z}_9 \cup R$. By Lemma 3.4, without loss of generality, either $f(cu_1) = 0 = f(cu_{2i-1})$ with $i = 2, 3$ or $f(u_1u_2) = f(u_2u_3) = 0$. If $f(cu_1) = 0 = f(cu_3)$, then $f(u_1u_2) = f(u_3u_4) = f(u_7u_8)$ which is impossible. If $f(cu_1) = 0 = f(cu_5)$, then $f(u_1u_2) = f(u_3u_4)$, $f(u_8u_1) = f(u_2u_3)$, $f(u_3u_4) = f(u_5u_6)$ and $f(u_4u_5) = f(u_6u_7)$. It is impossible since $|R| = 3$. Suppose $f(u_1u_2) = f(u_2u_3) = 0$. Let $f(u_3u_4) = a$, and $f(u_7u_8) = b$. Then $f(u_4u_5) = -a$, $f(u_8u_1) = -b$, $f(cu_1) = b$ and $f(cu_3) = -a$. So $\{-a, b\} \equiv \{1, 8\}$. That means $a - b = 0$ or equivalent to $a = b$. It is impossible by Lemma 2.1. So there is no 8-edge-magic labeling of G_4 .

Theorem 3.5. For $h \geq 0$, G_{6h+3} is $(3h+3)$ -edge-magic.

Proof. For $h = 0$, it is shown in Example 3.1. So we assume that $h \geq 1$. Again the arithmetics about the labels are taken in \mathbb{Z}_{12h+7} . So we omit to write $(\text{mod } 12h+7)$.

Case 1. Suppose $h = 2m$ for $m \geq 1$. Then $L = [6m+3, 42m+11]$. An edge-labeling f is defined by

$$f(u_{4t+1}u_{4t+2}) = \begin{cases} 9m+3+t, & \text{if } 0 \leq t \leq 3m; \\ 39m+11-t, & \text{if } 3m+1 \leq t \leq 6m+1, \end{cases} \quad (3.5)$$

$$f(u_{4t+2}u_{4t+3}) = \begin{cases} 15m+4-t, & \text{if } 0 \leq t \leq 3m; \\ 33m+10+t, & \text{if } 3m+1 \leq t \leq 6m+1, \end{cases} \quad (3.6)$$

$$f(u_{4t+3}u_{4t+4}) = \begin{cases} 9m+2-t, & \text{if } 0 \leq t \leq 3m-1; \\ 27m+9+t, & \text{if } 3m \leq t \leq 6m, \end{cases} \quad (3.7)$$

$$f(u_{4t+4}u_{4t+5}) = \begin{cases} 15m+5+t, & \text{if } 0 \leq t \leq 3m; \\ 45m+12-t, & \text{if } 3m+1 \leq t \leq 6m, \end{cases} \quad (3.8)$$

$$f(cu_j) = \begin{cases} 24m+7-2i, & \text{if } j = 4i+1, 1 \leq i \leq 3m; \\ 12m+4+2i, & \text{if } j = 4i+1, 3m+1 \leq i \leq 6m+1; \\ 24m+8+2i, & \text{if } j = 4i+3, 0 \leq i \leq 3m; \\ 36m+9-2i, & \text{if } j = 4i+3, 3m+1 \leq i \leq 6m+1. \end{cases} \quad (3.9)$$

We can check that the image of f is

$[9m+3, 12m+3] \cup [33m+10, 36m+10]$ from (3.5);
 $[12m+4, 15m+4] \cup [36m+11, 39m+11]$ from (3.6);
 $[6m+3, 9m+2] \cup [30m+9, 33m+9]$ from (3.7);
 $[15m+5, 18m+5] \cup [39m+12, 42m+11]$ from (3.8); and
 $[18m+6, 30m+8]$ from (3.9).

So f is a bijection.

Now we are going to check that $f^+ = 0$.

$$\begin{aligned} f^+(u_{4t+4}) &= f(u_{4t+3}u_{4t+4}) + f(u_{4t+4}u_{4t+5}) \\ &= \left\{ \begin{array}{ll} (9m+2-t) + (15m+5+t) = 24m+7, & \text{if } 0 \leq t \leq 3m-1 \\ (27m+9+3m) + (15m+5+3m) = 48m+14, & \text{if } t = 3m \\ (27m+9+t) + (45m+12-t) = 72m+21, & \text{if } 3m+1 \leq t \leq 6m \end{array} \right\} \equiv 0. \end{aligned}$$

Similarly, we can verify that $f^+(u_{4t+2}) = 0$ for $0 \leq t \leq 6m+1$.

$$\begin{aligned} f^+(u_1) &= f^+(u_{24m+7}) = f(u_{24m+6}u_1) + f(cu_{24m+7}) + f(u_1u_2) \\ &= (33m+10+6m+1) + (36m+9-12m-2) + (9m+3+0) \\ &= 72m+21 \equiv 0. \end{aligned}$$

$$\begin{aligned}
 f^+(u_{4t+3}) &= f(u_{4t+2}u_{4t+3}) + f(cu_{4t+3}) + f(u_{4t+3}u_{4t+4}) \\
 &= \begin{cases} (15m + 4 - t) + (24m + 8 + 2t) + (9m + 2 - t), & \text{if } 0 \leq t \leq 3m - 1 \\ (15m + 4 - 3m) + (24m + 8 + 6m) + (27m + 9 + 3m), & \text{if } t = 3m \\ (33m + 10 + t) + (36m + 9 - 2t) + (27m + 9 + t), & \text{if } 3m + 1 \leq t \leq 6m \end{cases} \\
 &= \begin{cases} 48m + 14 \\ 72m + 21 \\ 96m + 28 \end{cases} \equiv 0.
 \end{aligned}$$

Similarly, we can verify that $f^+(u_{4t+1}) = 0$ for $0 \leq t \leq 6m + 1$. Finally, since the order of any gear graph is odd and from (1.1), the sum of labels is 0. Thus $f^+(c) = 0$.

Case 2. Suppose $h = 2m + 1$ for $m \geq 0$. We only define the labeling f for this case. To show f^+ being an edge-magic labeling is similar to Case 1. It is left to readers.

$$\begin{aligned}
 f(u_{4t+1}u_{4t+2}) &= \begin{cases} 9m + 7 - t, & \text{if } 0 \leq t \leq 3m + 1; \\ 27m + 22 + t, & \text{if } 3m + 2 \leq t \leq 6m + 4, \end{cases} \\
 f(u_{4t+2}u_{4t+3}) &= \begin{cases} 15m + 12 + t, & \text{if } 0 \leq t \leq 3m + 2; \\ 45m + 35 - t, & \text{if } 3m + 3 \leq t \leq 6m + 4, \end{cases} \\
 f(u_{4t+3}u_{4t+4}) &= \begin{cases} 9m + 8 + t, & \text{if } 0 \leq t \leq 3m + 1; \\ 39m + 30 - t, & \text{if } 3m + 2 \leq t \leq 6m + 3, \end{cases} \\
 f(u_{4t+4}u_{4t+5}) &= \begin{cases} 15m + 11 - t, & \text{if } 0 \leq t \leq 3m + 1; \\ 33m + 27 + t, & \text{if } 3m + 2 \leq t \leq 6m + 3, \end{cases} \\
 f(cu_j) &= \begin{cases} 24m + 19 + 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m + 2; \\ 36m + 28 - 2i, & \text{if } j = 4i + 1, 3m + 3 \leq i \leq 6m + 4; \\ 24m + 18 - 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m + 1; \\ 12m + 11 + 2i, & \text{if } j = 4i + 3, 3m + 2 \leq i \leq 6m + 3. \end{cases}
 \end{aligned}$$

This completes the proof. □

Example 3.4. According to the proof above, we have a 9-edge-magic labeling for G_{15}

| | | | | | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|-----|
| [50] | 12 | 19 | 11 | 20 | 13 | 18 | 10 | 21 | 14 | 17 | 9 | 22 | 15 | 16 | 39 | --> |
| . | 31 | . | 32 | . | 29 | . | 34 | . | 27 | . | 36 | . | 25 | . | 38 | |
| | | | | | | | | | | | | | | | | |
| --> | 23 | 46 | 47 | 40 | 53 | 45 | 48 | 41 | 52 | 44 | 49 | 42 | 51 | 43 | 50 | |
| | . | 24 | . | 37 | . | 26 | . | 35 | . | 28 | . | 33 | . | 30 | . | |

and a 6-edge-magic labeling of G_9

| | | | | | | | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| [31] | 7 | 12 | 8 | 11 | 6 | 13 | 9 | 10 | 24 | 14 | 28 | 29 | 25 | 32 | 27 | 30 | 26 | 31 |
| . | 19 | . | 18 | . | 21 | . | 16 | . | 23 | . | 15 | . | 22 | . | 17 | . | 20 | . |

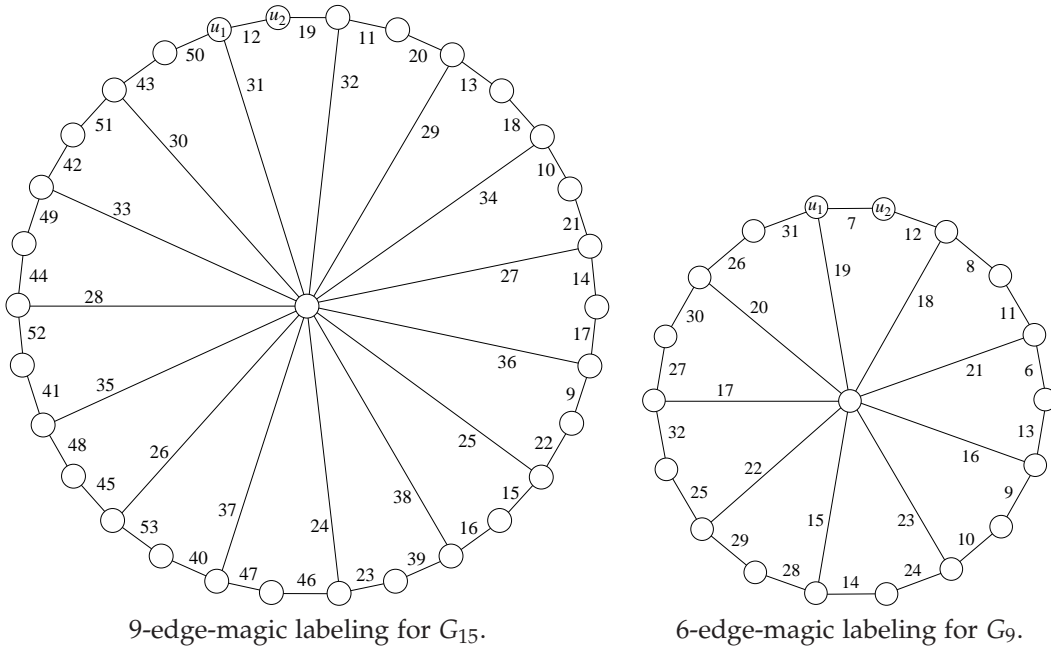


Figure 3

The proofs of the following theorems are similar to that of Theorem 3.5, so we only provide corresponding labelings and omit the proofs.

Theorem 3.6. For $h \geq 0$, G_{6h+5} is $(3h + 4)$ -edge-magic.

Proof. Suppose $h = 2m$ for $m \geq 0$. Define f by

$$\begin{aligned}
 f(u_{4t+1}u_{4t+2}) &= \begin{cases} 9m + 4 - t, & \text{if } 0 \leq t \leq 3m; \\ 27m + 13 + t, & \text{if } 3m + 1 \leq t \leq 6m + 2, \end{cases} \\
 f(u_{4t+2}u_{4t+3}) &= \begin{cases} 15m + 7 + t, & \text{if } 0 \leq t \leq 3m + 1; \\ 45m + 20 - t, & \text{if } 3m + 2 \leq t \leq 6m + 2, \end{cases} \\
 f(u_{4t+3}u_{4t+4}) &= \begin{cases} 9m + 5 + t, & \text{if } 0 \leq t \leq 3m; \\ 39m + 17 - t, & \text{if } 3m + 1 \leq t \leq 6m + 1, \end{cases} \\
 f(u_{4t+4}u_{4t+5}) &= \begin{cases} 15m + 6 - t, & \text{if } 0 \leq t \leq 3m; \\ 33m + 16 + t, & \text{if } 3m + 1 \leq t \leq 6m + 1, \end{cases} \\
 f(cu_j) &= \begin{cases} 24m + 11 + 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m + 1; \\ 36m + 16 - 2i, & \text{if } j = 4i + 1, 3m + 2 \leq i \leq 6m + 2; \\ 24m + 10 - 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m; \\ 12m + 7 + 2i, & \text{if } j = 4i + 3, 3m + 1 \leq i \leq 6m + 1. \end{cases}
 \end{aligned}$$

Suppose $h = 2m + 1$ for $m \geq 0$. Define f by

$$\begin{aligned}
 f(u_{4t+1}u_{4t+2}) &= \begin{cases} 9m + 9 + t, & \text{if } 0 \leq t \leq 3m + 2; \\ 39m + 37 - t, & \text{if } 3m + 3 \leq t \leq 6m + 5, \end{cases} \\
 f(u_{4t+2}u_{4t+3}) &= \begin{cases} 15m + 14 - t, & \text{if } 0 \leq t \leq 3m + 2; \\ 33m + 32 + t, & \text{if } 3m + 3 \leq t \leq 6m + 5, \end{cases} \\
 f(u_{4t+3}u_{4t+4}) &= \begin{cases} 9m + 8 - t, & \text{if } 0 \leq t \leq 3m + 1; \\ 27m + 27 + t, & \text{if } 3m + 2 \leq t \leq 6m + 4, \end{cases} \\
 f(u_{4t+4}u_{4t+5}) &= \begin{cases} 15m + 15 + t, & \text{if } 0 \leq t \leq 3m + 2; \\ 45m + 42 - t, & \text{if } 3m + 3 \leq t \leq 6m + 4, \end{cases} \\
 f(cu_j) &= \begin{cases} 24m + 23 - 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m + 2; \\ 12m + 12 + 2i, & \text{if } j = 4i + 1, 3m + 3 \leq i \leq 6m + 5; \\ 24m + 24 + 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m + 2; \\ 36m + 33 - 2i, & \text{if } j = 4i + 3, 3m + 3 \leq i \leq 6m + 4. \end{cases}
 \end{aligned}$$

□

Example 3.5. According to the construction above, we have a 4-edge-magic labeling for G_5

| | | | | | | | | | | |
|------|----|---|----|---|----|---|----|----|----|----|
| [18] | 4 | 7 | 5 | 6 | 14 | 8 | 16 | 17 | 15 | 18 |
| . | 11 | . | 10 | . | 13 | . | 9 | . | 12 | . |

and a 7-edge-magic labeling for G_{11}

| | | | | | | | | | | | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| [37] | 9 | 14 | 8 | 15 | 10 | 13 | 7 | 16 | 11 | 12 | 29 | 17 | 34 | 35 | 30 | 39 | 33 | 36 | 31 | 38 | 32 | 37 |
| . | 23 | . | 24 | . | 21 | . | 26 | . | 19 | . | 28 | . | 18 | . | 27 | . | 20 | . | 25 | . | 22 | . |

Theorem 3.7. For $h \geq 1$, G_{6h+1} is $(3h + 2)$ -edge-magic.

Proof. Suppose $h = 2m$ for $m \geq 1$. Define f by

$$\begin{aligned}
 f(u_{4t+1}u_{4t+2}) &= \begin{cases} 9m + 1 - t, & \text{if } 0 \leq t \leq 3m - 1; \\ 27m + 4 + t, & \text{if } 3m \leq t \leq 6m, \end{cases} \\
 f(u_{4t+2}u_{4t+3}) &= \begin{cases} 15m + 2 + t, & \text{if } 0 \leq t \leq 3m; \\ 45m + 5 - t, & \text{if } 3m + 1 \leq t \leq 6m, \end{cases} \\
 f(u_{4t+3}u_{4t+4}) &= \begin{cases} 9m + 2 + t, & \text{if } 0 \leq t \leq 3m - 1; \\ 39m + 4 - t, & \text{if } 3m \leq t \leq 6m - 1, \end{cases} \\
 f(u_{4t+4}u_{4t+5}) &= \begin{cases} 15m + 1 - t, & \text{if } 0 \leq t \leq 3m - 1; \\ 33m + 5 + t, & \text{if } 3m \leq t \leq 6m - 1, \end{cases}
 \end{aligned}$$

$$f(cu_j) = \begin{cases} 24m + 3 + 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m; \\ 36m + 4 - 2i, & \text{if } j = 4i + 1, 3m + 1 \leq i \leq 6m; \\ 24m + 2 - 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m - 1; \\ 12m + 3 + 2i, & \text{if } j = 4i + 3, 3m \leq i \leq 6m - 1. \end{cases}$$

Suppose $h = 2m + 1$ for $m \geq 0$. Define f by

$$\begin{aligned} f(u_{4t+1}u_{4t+2}) &= \begin{cases} 9m + 6 + t, & \text{if } 0 \leq t \leq 3m + 1; \\ 39m + 24 - t, & \text{if } 3m + 2 \leq t \leq 6m + 3, \end{cases} \\ f(u_{4t+2}u_{4t+3}) &= \begin{cases} 15m + 9 - t, & \text{if } 0 \leq t \leq 3m + 1; \\ 33m + 21 + t, & \text{if } 3m + 2 \leq t \leq 6m + 3, \end{cases} \\ f(u_{4t+3}u_{4t+4}) &= \begin{cases} 9m + 5 - t, & \text{if } 0 \leq t \leq 3m; \\ 27m + 18 + t, & \text{if } 3m + 1 \leq t \leq 6m + 2, \end{cases} \\ f(u_{4t+4}u_{4t+5}) &= \begin{cases} 15m + 10 + t, & \text{if } 0 \leq t \leq 3m + 1; \\ 45m + 27 - t, & \text{if } 3m + 2 \leq t \leq 6m + 2, \end{cases} \\ f(cu_j) &= \begin{cases} 24m + 15 - 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m + 1; \\ 12m + 8 + 2i, & \text{if } j = 4i + 1, 3m + 2 \leq i \leq 6m + 3; \\ 24m + 16 + 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m + 1; \\ 36m + 21 - 2i, & \text{if } j = 4i + 3, 3m + 2 \leq i \leq 6m + 2. \end{cases} \end{aligned}$$

□

Example 3.6. According to the construction above, we have an 8-edge-magic labeling for G_{13}

| | | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| [44] | 10 | 17 | 11 | 16 | 9 | 18 | 12 | 15 | 8 | 19 | 13 | 14 | 34 |
| . | 27 | . | 26 | . | 29 | . | 24 | . | 31 | . | 22 | . | 33 |

 \dashrightarrow

| | | | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 20 | 40 | 41 | 35 | 46 | 39 | 42 | 36 | 45 | 38 | 43 | 37 | 44 |
| . | 21 | . | 32 | . | 23 | . | 30 | . | 25 | . | 28 | . |

and an 8-edge-magic labeling for G_7

| | | | | | | | | | | | | | | |
|------|----|---|----|----|----|---|----|----|----|----|----|----|----|----|
| [24] | 6 | 9 | 5 | 10 | 7 | 8 | 19 | 11 | 22 | 23 | 20 | 25 | 21 | 24 |
| . | 15 | . | 16 | . | 13 | . | 18 | . | 12 | . | 8 | . | 14 | . |

We also find some ad hoc examples. We have six 11-edge-magic labelings for G_6 . They are

| | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|
| [27] | 12 | 14 | 20 | 19 | 11 | 15 | 24 | 28 | 16 | 23 | 25 | 27 |
| . | 13 | . | 18 | . | 22 | . | 26 | . | 21 | . | 17 | . |

| | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|
| [25] | 14 | 12 | 17 | 22 | 11 | 15 | 24 | 28 | 21 | 18 | 27 | 25 |
| . | 13 | . | 23 | . | 19 | . | 26 | . | 16 | . | 20 | . |

| | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|
| [14] | 25 | 27 | 17 | 22 | 20 | 19 | 15 | 11 | 28 | 24 | 12 | 14 |
| . | 26 | . | 21 | . | 23 | . | 18 | . | 13 | . | 16 | . |

| | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|
| [27] | 12 | 14 | 15 | 24 | 28 | 11 | 20 | 19 | 17 | 22 | 25 | 27 |
| . | 13 | . | 23 | . | 26 | . | 21 | . | 16 | . | 18 | . |

| | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|
| [14] | 13 | 26 | 11 | 15 | 18 | 21 | 17 | 22 | 23 | 16 | 12 | 14 |
| . | 25 | . | 28 | . | 19 | . | 27 | . | 20 | . | 24 | . |

| | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|
| [14] | 13 | 26 | 11 | 15 | 17 | 22 | 16 | 23 | 18 | 21 | 12 | 14 |
| . | 25 | . | 28 | . | 20 | . | 27 | . | 24 | . | 19 | . |

We obtain two 14-edge-magic labelings for G_8 . They are

| | | | | | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| [37] | 14 | 20 | 35 | 16 | 18 | 33 | 27 | 24 | 15 | 19 | 23 | 28 | 36 | 32 | 31 | 37 |
| . | 17 | . | 30 | . | 34 | . | 25 | . | 29 | . | 26 | . | 21 | . | 22 | . |

| | | | | | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| [20] | 17 | 34 | 18 | 16 | 32 | 36 | 23 | 28 | 30 | 21 | 22 | 29 | 15 | 19 | 14 | 20 |
| . | 31 | . | 33 | . | 37 | . | 26 | . | 27 | . | 25 | . | 24 | . | 35 | . |

We get one 23-edge-magic labeling for G_{14} :

| | | | | | | | | | | | | | | |
|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| [60] | 27 | 31 | 34 | 24 | 26 | 32 | 35 | 23 | 25 | 33 | 41 | 46 | 30 | 28 |
| . | 29 | . | 51 | . | 37 | . | 49 | . | 39 | . | 42 | . | 40 | . |

-->

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| | | | | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 59 | 57 | 43 | 44 | 54 | 62 | 64 | 52 | 55 | 61 | 63 | 53 | 56 | 60 |
| 58 | .. | 45 | . | 47 | . | 48 | . | 38 | . | 50 | . | 36 | . |

From those examples, we have not discovered any regulation to obtain edge-magic labelings for other unsolved cases. We summarize those unsolved problems below:

Problem 3.1. Find a $(9h + 2)$ -edge-magic labeling for G_{6h} , $h \geq 1$.

Problem 3.2. Find a $(7h + 3)$ -edge-magic labeling and a $(11h + 4)$ -edge-magic labeling for G_{6h+1} , $h \geq 1$.

Problem 3.3. Find a $(9h + 5)$ -edge-magic labeling for G_{6h+2} , $h \geq 1$.

Problem 3.4. Find a $(h + 2)$ -edge-magic labeling, a $(5h + 5)$ -edge-magic labeling, and a $(9h + 8)$ -edge-magic labeling for G_{6h+4} , $h \geq 1$.

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