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Abstract

A graph $G = (V, E)$ with p vertices and q edges is called edge-magic if there is a bijection $f : E \to \{1, 2, \ldots, q\}$ such that the induced mapping $f^+ : V \to \mathbb{Z}_p$ is a constant mapping, where $f^+(u) \equiv \sum_{v \in N(u)} f(uv) \pmod{p}$. The edge-magic index set of a graph G is the set of positive integer k such that the k-fold of G is edge-magic. In this paper, we find the edge-magic index set of the second power of a path.

Keywords: Edge-magic, edge-magic index, edge-magic index set, power of path.
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1 Introduction

In this paper, all graphs are finite connected multigraphs having no loop. All undefined symbols and concepts may be looked up from [1]. A graph $G = (V, E)$ is a (p, q)-graph if p and q are its order and size respectively, i.e., $|V| = p$ and $|E| = q$.

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A (p,q)-graph $G=(V,E)$ is called *edge-magic* if there exists a bijection

$$f : E \to \{1, 2, \ldots, q\}$$

such that the *induced mapping* $f^+ : V \to \mathbb{Z}_p$ is a constant mapping, where $f^+(u) \equiv \sum_{uv\in E} f(uv) \pmod p$ for $u \in V$. The mapping f is called an *edge-magic labeling* of G. This concept was introduced by Lee, Seah and Tan [3] in 1992. Some edge-magic graphs and some supermagic graphs were found in [2, 3, 5, 7, 9–11, 13, 14].

A necessary condition of a (p,q)-graph to be edge-magic is

$$q(q+1) \equiv 0 \pmod p. \tag{1.1}$$

But lots of graphs are not edge-magic even though they satisfy the necessary condition of edge-magicness (1.1), which we shall abbreviate as NCEM. For examples, cycles with more than 2 vertices; trees with more than 2 vertices, etc. However, all non edge-magic graphs can be embedded into some edge-magic graphs (see Theorem 1.1).

Let G be a graph and let k be a positive integer. $G[k]$ is a graph which is made up of k copies of G with the same set of vertices. We call $G[k]$ the *k-fold* of G.

Theorem 1.1 ([3]) *For any (p,q)-graph G, $G[2p]$ is edge-magic.*

Theorem 1.2 ([12]) *Let G be a (p,q)-graph. If p is odd, then $G[p]$ is edge-magic.*

Theorem 1.3 ([12]) *Suppose G is a (p,q)-graph and $G[k]$ is edge-magic. Then $G[k+2p]$ is edge-magic. Moreover, if p is odd, then $G[k+p]$ is edge-magic.*

Our problem is to find out all positive integers k such that $G[k]$ are edge-magic, i.e. to find the set $I_m(G) = \{k \mid G[k] \text{ is edge-magic}\}$. This set is called the *edge-magic index set* of G. We call $emi(G) = \min\{I_m(G)\}$ the *edge-magic index* of G. This concept was first appeared in [8]. Some edge-magic index sets of graphs were found [6, 8, 12].

Remark 1.1 Since p and kq satisfying the condition (1.1) implies p and $(k+lp)q$ satisfying the condition (1.1), by Theorems 1.1, 1.2 and 1.3 we can start at the case $1 \le k \le p$ or $1 \le k \le 2p$ according to $G[p]$ or $G[2p]$ is edge-magic, respectively. But sometimes a (p,q)-graph G (for example, a tree of order greater than 2) is not edge-magic even though it satisfies (1.1). In this case we must start at the case $2 \le k \le p+1$ or $2 \le k \le 2p+1$ according to $G[p]$ or $G[2p]$ is edge-magic, respectively.

In this paper we shall consider the edge-magic index set of the second power of path.

2 Some notation and concepts

Let S be a set. We use $S \times n$ to denote the *multiset* of n-copies of S. Note that S may be a multiset itself. From now on, the term "set" means "multiset". The set operations are considered as multiset operations. Let S be a set of qk elements. Let $\mathscr{P}$ be a partition of S such that each class of $\mathscr{P}$ contains k elements. We call $\mathscr{P}$ a (q,k)*-partition* of S. We let $[\boldsymbol{r}] = \{1,\ 2,\ \dots,\ r\}$ and let $[\mathbf{0}] = \varnothing$.

Suppose A is a set consisting of r integers. If the sum modulo p of elements of A is s, then A is called an $(s;r)$*-set* or simply an s*-set*. If $r = 1,\ 2$ or 3, it is called an s*-singleton*, an s*-doubleton* or an s*-tripleton* respectively. Let A and B be two sets. $A \equiv B \pmod p$ means two sets are the same after taking modulo p.

A mapping f is called a k*-fold edge-magic labeling* of a (p,q)-graph $G = (V,E)$ if there is a (q,k)-partition $\mathscr{P}$ of $[\boldsymbol{qk}]$ such that $f : E \to \mathscr{P}$ is a bijection and the induced mapping $f^+ : V \to \mathbb{Z}_p$ is a constant mapping, where

$$f^+(u) \equiv \sum_{uv \in E} \sum_{i \in f(uv)} i \pmod p.$$

Thus, finding an edge-magic labeling of $G[k]$ is equivalent to finding a k-fold edge-magic labeling of G.

3 Useful lemmas

For a fixed positive integer p, $F : E \to \mathbb{Z}_p$ is called a $\mathbb{Z}_p$*-magic labeling* of G if $F^+ : V \to \mathbb{Z}_p$ is a constant mapping. This concept was introduced in [12] by the authors. It is easy to prove the following theorem.

Theorem 3.1 ([12]) *Let $G = (V,E)$ be a (p,q)-graph. If $G[k]$ is edge-magic, then there is a $\mathbb{Z}_p$-magic labeling F of G such that $\sum_{e \in E} F(e) \equiv \frac{1}{2}kq(kq+1) \pmod p$.*

For $n \geq 3$, the *second power* of the path P_n, denoted by P_n^2, is the graph obtained from P_n by joining every pair of vertices of P_n whose distance is 2. Since $P_n^2[k]$ has $(2n-3)k$ edges, if it is edge-magic then $n|(2n-3)k[(2n-3)k+1]$, or equivalently,

$$n|3k(3k-1). \tag{3.1}$$

We shall show that the necessary condition (3.1) is also sufficient for $k \geq 2$ in the next two sections.

Lemma 3.2 *Suppose n is even and $x \in \mathbb{Z}_n$. There is a $\mathbb{Z}_n$-magic labeling F of P_n^2 such that $|F^{-1}(x)| = \frac{n}{2}$ and $|F^{-1}(0)| = \frac{3n}{2} - 3$.*

Proof: Let M be the perfect matching of P_n. If P_n is viewed as a subgraph of P_n^2, then M is also a perfect matching of P_n^2. Define $F : E(P_n^2) \to \mathbb{Z}_n$ by $F(e) = x$ if $e \in M$ and $F(e) = 0$ otherwise. Then $F^+(u) = x$ for all $u \in V(P_n^2)$. Hence F is a $\mathbb{Z}_n$-magic labeling. $\square$

Corollary 3.3 *Suppose n is even, k is positive and $x \in \mathbb{Z}_n$. If $[\boldsymbol{(2n-3)k}]$ has a $(2n-3,k)$-partition $\mathscr{P}$ consisting of $\frac{n}{2}$ x-sets for some x and $(\frac{3n}{2}-3)$ zero-sets, then $P_n^2[k]$ is edge-magic.*

Proof: Let M be the perfect matching of P_n^2 defined in the proof of Lemma 3.2. Define a bijection $f : E(P_n^2) \to \mathscr{P}$ by assigning each edge of M to a x-set of $\mathscr{P}$ and each edge not in M to a 0-set of $\mathscr{P}$, respectively. Then f is a k-fold edge-magic labeling of P_n^2. Hence $P_n^2[k]$ is edge-magic. $\square$

By a similar argument as the proof of Corollary 3.3 we have

Corollary 3.4 *If $[\boldsymbol{(2n-3)k}]$ has a $(2n-3,k)$-partition $\mathscr{P}$ consisting of all 0-sets, then $P_n^2[k]$ is edge-magic.*

From Corollaries 3.3 and 3.4, it suffices to find a $(2n-3,k)$-partition of $[\boldsymbol{(2n-3)k}]$ consisting of $(2n-3)$ zero-sets or $(\frac{3n}{2}-3)$ zero-sets and $\frac{n}{2}$ x-sets for some $x \in \mathbb{Z}_n$. Unfortunately, there may not exist such partition for some n (for example $n = 15$ and $k = 2$). Thus we have to find another way to label the graphs. From [4] we know that there is a $\mathbb{Z}_n$-magic labeling of P_n^2 for odd $n \geq 5$. Namely, the edges of P_n^2 are labeled by $(n-2)$ one's, $(n-3)$ two's and 2 four's described in Fig. 1. For the top edges, the first and the last edges are labeled by 2 and the others are labeled by 1. For the middle edges, the first and the last edges are labeled by 4, the second and the last second edges are labeled by 1, the others are labeled by 2. All the bottom edges are labeled by 1. Then the induced vertex labeling is a constant mapping with value 6.

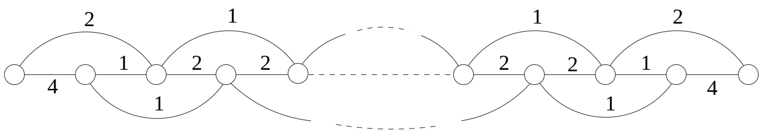

Figure 1: A $\mathbb{Z}_n$-magic labeling of P_n^2 with odd $n \geq 5$ by using 1, 2 and 4.

Thus we have

Corollary 3.5 *For odd n and $n|3k(3k-1)$ for some k, if there is a $(2n-3, k)$-partition of $[\boldsymbol{(2n-3)k}]$ consisting of $(n-2)$ one-sets, $(n-3)$ two-sets and 2 four-sets, then $P_n^2[k]$ is edge-magic.*

4 The cases for $1 \leq k \leq 3$

In this section, we only consider the cases for $1 \leq k \leq 3$. For a fixed k, there are finite possibilities of n satisfying (3.1). We list all possibilities of n for $1 \leq k \leq 3$ below.

k	$3k(3k-1)$	n
1	6	3, 6
2	30	3, 5, 6, 10, 15, 30
3	72	3, 4, 6, 8, 9, 12, 18, 24, 36, 72

In the following, we shall show that $P_n^2[k]$, where $1 \leq k \leq 3$ and $3k(3k-1) \equiv 0 \pmod{n}$, is edge-magic by either using a figure or listing a required partition of $[\boldsymbol{(2n-3)k}]$ which satisfies one of Corollaries 3.3, 3.4, and 3.5.

Case 1: $k = 1$. It is easy to see that $P_3^2 \cong C_3$, the 3-cycle, is not edge-magic. Fig. 2 shows that P_6^2 is edge-magic.

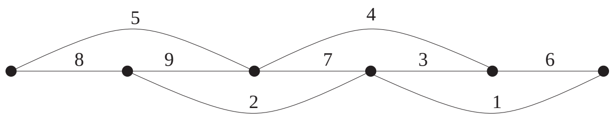

Figure 2: An edge-magic labeling of P_6^2.

Case 2: $k = 2$. We show a 2-fold edge-magic labeling for each of P_3^2, P_5^2 and P_6^2 below (see Fig. 3, 4 and 5).

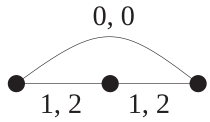

Figure 3: A 2-fold edge-magic labeling of P_3^2.

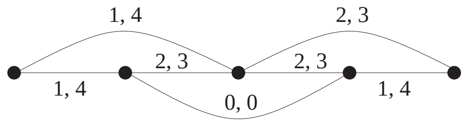

Figure 4: A 2-fold edge-magic labeling of P_5^2.

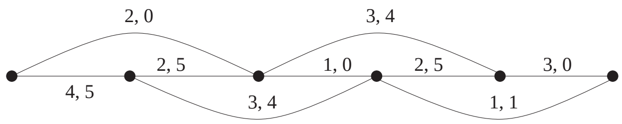

Figure 5: A 2-fold edge-magic labeling of P_6^2.

We shall list a $(2n-3, 2)$-partition of $[\mathbf{2(2n-3)}]$ which satisfies either Corollaries 3.3 or 3.5 below, where $n = 10$, 15, and 30.

When $n = 10$. $[\mathbf{34}] \equiv ([\mathbf{10}] \times 3) \cup [\mathbf{4}] \pmod{10}$. $[\mathbf{10}] \times 2$ and $[\mathbf{10}] \cup [\mathbf{4}]$ are grouped as follows:

1	2	3	4	5	6	7	8	9	10
9	8	7	6	5	4	3	2	1	10

	1	2	3	4	5	6	7
	2	1	10	9	8	4	3
Sum	3	3	3	3	3	0	0

Thus, we have five 3-doubletons and twelve 0-doubletons. This partition of $[\mathbf{34}]$ satisfies Corollary 3.3.

When $n = 15$. $[\mathbf{54}] \equiv ([\mathbf{15}] \times 3) \cup [\mathbf{9}] \pmod{15}$. A required partition of $[\mathbf{54}]$ satisfying Corollary 3.5 is listed below:

Pair	3 13	4 12	5 11	6 10	7 9	8 8	15 2	1 1	3 14	4 13	5 12	6 11	15 4	2 2
Sum	.... 1						.... 2						. . 4 . .	
Quantity	1	1	2	3	4	2	2	2	3	2	2	1	1	1

When $n = 30$. $[\mathbf{114}] \equiv ([\mathbf{30}] \times 3) \cup [\mathbf{24}] \pmod{30}$. We group up two copies of $[\mathbf{30}]$. Each $[\mathbf{30}]$ is grouped as follows:

	1 29	2 28	3 30	4 26	5 25	6 27	7 23	8 22	9 24	10 20	11 19	12 21	13 17	14 16	15 18
Sum	0	0	3	0	0	3	0	0	3	0	0	3	0	0	3

We rearrange $[\mathbf{30}] \cup [\mathbf{24}]$ as follows:

	1 2	2 1	3 30	4 29	5 28	6 24	7 23	$\cdots$ $\cdots$	26 4	27 3
Sum	3	3	3	3	3	0	0	$\cdots$	0	0

This partition of $[\mathbf{114}]$ satisfies Corollary 3.3.

Case 3: $k = 3$. By Theorem 1.2, $P_3^2[3]$ is edge-magic. Fig. 6 shows a 3-fold edge-magic labeling of P_4^2.

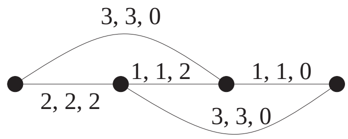

Figure 6: A 3-fold edge-magic labeling of P_4^2.

When $n = 6$. $[\mathbf{27}] \equiv ([\mathbf{6}] \times 4) \cup [\mathbf{3}] \pmod{6}$ can be grouped into nine 0-tripletons as follows:

1	2	3	4	5	1	2	3	1
4	5	1	2	3	5	4	3	2
1	5	2	6	4	6	6	6	3

When $n = 9$. $[\mathbf{45}] \equiv [\mathbf{9}] \times 5 \pmod{9}$. Each $[\mathbf{9}]$ can be grouped into three 0-tripletons: $\{1, 3, 5\}$, $\{2, 7, 9\}$ and $\{4, 6, 8\}$. Thus $[\mathbf{45}]$ has a partition consisting of fifteen 0-tripletons.

When $n = 18$. $[\mathbf{99}] \equiv [\mathbf{18}] \times 5 \cup [\mathbf{9}] \pmod{18}$. $[\mathbf{18}] \times 3 \setminus \{9, 18, 18\}$ is grouped as follows:

1	2	3	4	5	6	7	8	10	11	12	13	14	15	16	17	18
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
16	14	12	10	8	6	4	2	17	15	13	11	9	7	5	3	1

$[\mathbf{18}] \times 2 \cup [\mathbf{9}] \cup \{9, 18, 18\}$ is grouped as follows:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	17
1	2	3	4	5	6	7	8	9	15	16	17	18	9	18	18
16	14	12	10	8	6	4	2	18	11	9	7	5	13	3	1

All tripletons are 0-sets.

All the above cases satisfy the condition of Corollary 3.4.

When $n = 8$. $[\mathbf{39}] \equiv ([\mathbf{8}] \times 4) \cup [\mathbf{7}] \pmod 8$ can be grouped into nine 0-tripletons and four 1-tripletons as follows:

	1	2	3	4	5	6	7	8	1	1	3	3	4
	1	3	5	7	1	3	5	7	2	2	6	6	4
	6	4	8	6	2	8	4	2	5	5	7	7	8
Sum	0	1	0	1	0	1	0	1	0	0	0	0	0

When $n = 12$. $[\mathbf{63}] \equiv [\mathbf{12}] \times 5 \cup [\mathbf{3}] \pmod{12}$. Each $[\mathbf{12}]$ can be grouped into three 0-tripletons and one 6-tripleton: $\{1, 3, 8\}$, $\{4, 9, 11\}$, $\{5, 7, 12\}$ and $\{2, 6, 10\}$. Combining with $[\mathbf{3}]$ we have six 6-tripletons and twenty-five 0-tripletons.

When $n = 24$. $[\mathbf{135}] \equiv [\mathbf{24}] \times 5 \cup [\mathbf{15}] \pmod{24}$. $[\mathbf{24}] \times 3 \setminus \{12, 24, 24\}$ is grouped into twenty-three 0-tripletons as follows:

1	2	$\cdots$	10	11	13	14	$\cdots$	23	24
1	2	$\cdots$	10	11	12	13	$\cdots$	22	23
22	20	$\cdots$	4	2	23	21	$\cdots$	3	1

$[\mathbf{24}] \times 2 \cup [\mathbf{15}] \cup \{12, 24, 24\}$ is grouped as follows:

	1	2	$\cdots$	11	12	13	14	15	16	17	18	2	12	13	24
	1	2	$\cdots$	11	12	19	20	21	22	23	24	4	14	15	24
	23	21	$\cdots$	3	1	16	14	12	10	8	6	18	22	20	24
Sum	1	1	$\cdots$	1	1	0	0	0	0	0	0	0	0	0	0

Thus we have thirty-three 0-tripletons and twelve 1-tripletons.

When $n = 36$. $[\mathbf{207}] \equiv [\mathbf{36}] \times 5 \cup [\mathbf{27}] \pmod{36}$. Each $[\mathbf{36}]$ is grouped into nine 0-tripletons and three 6-tripletons as follows:

	1	2	3	4	5	6	8	17	19	7	15	21
	9	10	11	12	13	14	31	23	25	35	29	27
	26	24	22	20	18	16	33	32	28	36	34	30
Sum	0	0	0	0	0	0	0	0	0	6	6	6

$[\mathbf{27}]$ is grouped as six 0-tripletons and three 6-tripletons as follows:

	1	2	3	4	6	22	5	7	12
	8	9	10	11	13	24	18	15	14
	27	25	23	21	17	26	19	20	16
Sum	0	0	0	0	0	0	6	6	6

Thus we have fifty-one 0-tripletons and eighteen 6-tripletons.

When $n = 72$. $[\mathbf{423}] \equiv [\mathbf{72}] \times 5 \cup [\mathbf{63}] \pmod{72}$. $[\mathbf{72}] \times 3 \setminus \{36, 72, 72\}$ is grouped into seventy-one 0-tripletons as follows:

1	2	⋯	34	35	37	38	⋯	71	72
1	2	⋯	34	35	36	37	⋯	70	71
70	68	⋯	4	2	71	69	⋯	3	1

$[\mathbf{72}] \times 2 \cup [\mathbf{63}] \cup \{36, 72, 72\}$ is grouped into thirty-six 1-tripletons and eighteen 0-tripletons as follows:

	1	2	⋯	35	36	37	38	⋯	53	54
	1	2	⋯	35	36	55	56	⋯	71	72
	71	69	⋯	3	1	52	50	⋯	20	18
Sum	1	1	⋯	1	1	0	0	⋯	0	0

Thus we have a partition of $[\mathbf{423}]$.

The above cases satisfy the condition of Corollary 3.3.

Thus we have

Theorem 4.1 *P_6^2 is edge-magic, but P_3^2 is not.*

Theorem 4.2 *If $n | 3k(3k-1)$ and $2 \le k \le 3$, then $P_n^2[k]$ is edge-magic.*

5 Main results for $4 \le k \le n+1$

In this section, we shall consider $4 \le k \le n+1$ satisfying $n|3k(3k-1)$. We shall prove three useful lemmas below. From now on, the *sum* of integers will mean 'their sum modulo n' and the symbol $\equiv$ will denote 'congruence modulo n'. The integer x is said to be *complementary* with $n - x$.

Divide $(2n-3)k$ by n we have $(2n-3)k = (2k-b)n + r$ for some b and $0 \le r < n$. So we have $\frac{12}{n} \le b < 1 + \frac{3k}{n} \le 4 + \frac{3}{n} \le 5$. Since $3k = bn - r > (b-1)n$, if $k \le n$, then $1 \le b \le 3$. When $k = n+1$. Since $3k(3k-1) \equiv n \pmod{2n}$, we have $n = 6$. This is the only case for $b = 4$.

Lemma 5.1 *Let n be an even integer. Then $[\boldsymbol{(2n-3)n}]$ can be partitioned into $(2n-3)$ sets of order n. $(\frac{3n}{2} - 3)$ of them are 0-sets and $\frac{n}{2}$ of them are 1-sets.*

Proof: Note that $[\boldsymbol{n}]$ is an $\frac{n}{2}$-set. We choose a complete pass $[\boldsymbol{n}]$ and perform the following *rectifying process* by arranging it as follows:

$(n+2)/2$	$(n+4)/2$	$(n+6)/2$	⋯	$n-2$	$n-1$	n
$n/2$	$(n-2)/2$	$(n-4)/2$	⋯	3	2	1

These $\frac{n}{2}$ sets are all 1-doubletons. Since n is even, each of the other complete passes may be grouped into $\frac{n-1}{2}$ 0-doubletons, one 0-singleton and one $\frac{n}{2}$-singleton.

-	$(n+2)/2$	$(n+4)/2$	$\cdots$	$n-2$	$n-1$	n
$n/2$	$(n-2)/2$	$(n-4)/2$	$\cdots$	2	1	-

Now we have $\frac{n}{2}$ one-doubletons, a number of 0-doubletons, a number of 0-singletons and a number of $\frac{n}{2}$-singletons. Since $[\boldsymbol{(2n-3)n}] \equiv [\boldsymbol{n}] \times (2n-3)$ is an $\frac{n}{2}$-set, the number of $\frac{n}{2}$-singletons and that of 0-singletons are even. Thus we can pair up the $\frac{n}{2}$-singletons to and the 0-singletons to produce 0-doubletons, respectively. Therefore, we can form $\frac{n}{2}$ one-sets and $\frac{3n}{2} - 3$ zero-sets, all of which are of order n. $\square$

Corollary 5.2 *For $n \geq 3$, $P_n^2[n]$ is edge-magic.*

Proof: It follows from Lemma 5.1 and Corollary 3.3. $\square$

Lemma 5.3 *$P_6^2[7]$ is edge-magic.*

Proof: By Corollary 5.2, $P_6^2[6]$ is edge-magic. Since P_6^2 and $P_6^2[6]$ are edge-magic, by Theorem 1.3 we have the lemma. $\square$

So we just only need to deal with $4 \leq k \leq n-1$.

Lemma 5.4 *Let n be an odd integer greater than 1 and k an integer, where $4 \leq k \leq n-1$ and $n|3k(3k-1)$. Then $[\boldsymbol{(2n-3)k}]$ has a $(2n-3, k)$-partition consisting of $(2n-3)$ 0-sets.*

Proof: Write $[\boldsymbol{(2n-3)k}] \equiv \{[\boldsymbol{n}] \times (2k-b)\} \cup [\boldsymbol{r}]$, where $0 \leq r = bn - 3k < n$, $1 \leq b \leq 3$. Since n is odd, each $[\boldsymbol{n}]$ may be grouped into $\frac{n-1}{2}$ 0-doubletons and one 0-singleton as follows:

$(n+1)/2$	$(n+3)/2$	$\cdots$	$n-2$	$n-1$	n
$(n-1)/2$	$(n-3)/2$	$\cdots$	2	1	$-$

Now we shall deal with the incomplete pass $[\boldsymbol{r}]$. Note that since n is odd and $n|3k(3k-1)$, $[\boldsymbol{(2n-3)k}]$ and $[\boldsymbol{n}]$ are 0-sets, hence so is $[\boldsymbol{r}]$.

When $2 \leq b \leq 3$. We have $3k = bn - r > (b-1)n \geq n > r$. When $b = 1$. If $r = n - 3k \leq n/2$, then $3k > n/2 \geq r$. If $r = n - 3k > n/2$ then the set $\{n-r,\ n-r+1,\ \ldots,\ r-1,\ r\}$ may be grouped into 0-doubletons, so we have to deal only with the incomplete pass $[\boldsymbol{n-r-1}]$, where $3k > n-r-1$. All of the above may reduced to the case of handling the incomplete pass $[\boldsymbol{r}]$, where $r < 3k$ and $[\boldsymbol{r}]$ is a 0-set.

If $r > k$, then we shall perform the following *reduction process*.

Reduction process: Let $S \subset [\boldsymbol{r}]$ be an x_1-set of order $k-1$. If $x_1 \equiv 0$, then take the singleton $\{n\}$ from one pass of $[\boldsymbol{n}]$ to form a 0-set $S \cup \{n\}$ of order k. If $x_1 \not\equiv 0$, then we may take the 0-doubleton $\{x_1, n-x_1\}$ from one pass of $[\boldsymbol{n}]$ to form the 0-set $S \cup \{n-x_1\}$ and the residual set $N = ([\boldsymbol{r}] \setminus S) \cup \{x_1\}$. Clearly N is a 0-set.

This reduction process transfers $[\boldsymbol{r}]$ into a set with order reduced by $k-2$ plus a 0-set of order k. We will continue this process until the order of the residual set is k or less. Since $r \le 3k-1 = 4(k-2)+(7-k)$ and $k \ge 4$, we obtain a residual 0-set with order of at most k after performing this process for at most 4 times. This process works since we have $2k-b \ge 5$ complete passes $[\boldsymbol{n}]$. So far we have up to four 0-sets of order k, one residual 0-set of order k or less, a number of 0-doubletons and a number of 0-singletons. It remains to combine these into 0-sets of order k.

Suppose k is even. If the order of the residual set is odd, then since we started with $(2n-3)k$ integers, the number of 0-singletons must also be odd. We combine the residual set with one 0-singleton, and if necessary, an appropriate number of 0-doubletons to form a 0-set of order k. The remaining doubletons and singletons can then be combined into 0-sets, each of which is of order k. The same can be accomplished if order of the residual set is even.

Suppose k is odd. In this case, since $4 \le k \le n+2$, we have $k \ge 5$, $2k-b \ge 7$ and $n \ge 5$. We need to have $(2n-3)$ odd order 0-sets with order not exceeding k, and then add on doubletons to obtained the required 0-sets of order k. Take 6 complete passes of $[\boldsymbol{n}]$, and tabulate them as follows:

1	2	$\cdots$	$\frac{n-3}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+3}{2}$	$\cdots$	$n-2$	$n-1$	n
$n-1$	$n-2$	$\cdots$	$\frac{n+3}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n-3}{2}$	$\cdots$	2	1	n
1	2	$\cdots$	$\frac{n-3}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+3}{2}$	$\cdots$	$n-2$	$n-1$	n
$n-1$	$n-2$	$\cdots$	$\frac{n+3}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n-3}{2}$	$\cdots$	2	1	n
2	4	$\cdots$	$n-3$	$n-1$	1	3	$\cdots$	$n-4$	$n-2$	n
$n-2$	$n-4$	$\cdots$	3	1	$n-1$	$n-3$	$\cdots$	4	2	n

Each column consists of three 0-doubletons. From a column, we may take the elements in the first, third and sixth row to form one set; and elements in the second, fourth and fifth to form another set. Both of these sets are 0-tripletons. Suppose we have to perform the reduction process s times, $0 \le s \le 4$. That is, we have s 0-sets of order k and also may have a 0-set N of order less than k. Since up to s of the columns may have contributed a 0-doubleton during the reduction process, the number of 0-sets of order k or 0-tripletons that may be produced from these 6 passes is at least $2(n-1-s)+s = 2n-2-s$. The set N may be of odd order. So the number of odd order 0-sets is between $2n-6$ and $2n-1$. If this number is greater than $2n-3$, then $s = 1, 2$. We can use a column, which is not the last column, to form three 0-doubletons to instead of two 0-tripletons. So that we can control the number of odd order 0-sets lying between $2n-6$ and $2n-4$. Moreover, there is one 0-singleton from each complete pass $[\boldsymbol{n}]$. We can use these (at least $2k-b-s \ge 3$) 0-singletons to construct $2n-3$ odd order 0-sets. $\square$

Let us use the following example to demonstrate the last case of the proof above.

Example 5.1 Let $n = 7$ and $k = 5$. It is easy to see that the conditions of Lemma 5.4. $[\mathbf{55}] \equiv [\mathbf{7}] \times 7 \cup [\mathbf{6}]$. We need eleven 0-sets of odd order first. Taking 6 complete passes of $[\mathbf{7}]$ we have

1	2	3	4	5	6	7
6	5	4	3	2	1	7
1	2	3	4	5	6	7
6	5	4	3	2	1	7
2	4	6	1	*3*	5	7
5	3	1	6	*4*	2	7

Let $S = \{3, 4, 5, 6\}$ which is a 4-set. So we take the 3 (indicated by italic) from the fifth row and the fifth column of the above table. The set $S \cup \{3\}$ is a 0-set of order 5. Then $N = \{1, 2, 4\}$ which is still a 0-set. If all other 5 columns are used to form 0-tripletons, then we have twelve 0-sets of odd order which is excess 11. Therefore, we only use the first 4 columns to form eight 0-tripletons. Up to now we have ten 0-sets of odd order, five 0-doubletons, six 0-singletons, and a complete pass $[\mathbf{7}]$. Grouping the complete pass into three 0-doubletons and one 0-singleton. So we obtain ten 0-sets of odd order, eight 0-doubletons, and seven 0-singletons. It suffices to construct eleven 0-sets of order 5. Namely,
$\{3,4,5,6\} \cup \{3\}$, $\{1,2,4\} \cup \{5,2\}$, $\{1,1,5\} \cup \{5,2\}$, $\{6,6,2\} \cup \{6,1\}$,
$\{2,2,3\} \cup \{6,1\}$, $\{5,5,4\} \cup \{5,2\}$, $\{3,3,1\} \cup \{1,6\}$, $\{4,4,6\} \cup \{2,5\}$,

$\{4,4,6\} \cup \{3,4\}$, $\{3,3,1\} \cup \{7,7\}$, $\{7,7,7,7,7\}$.

By a similar proof of Lemma 5.4, we have the following lemma.

Lemma 5.5 *Let n be an even integer and k an integer, where $4 \le k \le n-1$. If $3k(3k-1) \equiv 0 \pmod{2n}$, then $[\boldsymbol{(2n-3)k}]$ has a $(2n-3,k)$-partition all of which are consisting only of 0-sets.*

Note that, $k = n$ is not a case of Lemma 5.5. We consider this case as follows:

Lemma 5.6 *Let n be an even integer and k an integer, where $4 \le k \le n-1$. If $3k(3k-1) \equiv n \pmod{2n}$, then $[\boldsymbol{(2n-3)k}]$ may be partitioned into $(2n-3)$ sets of order k. $(\frac{3n}{2}-3)$ of them are 0-sets and $\frac{n}{2}$ of them are 1-sets.*

Proof: The starting point of the proof is the same as that of Lemma 5.1. We use a complete pass $[\boldsymbol{n}]$ to obtain $\frac{n}{2}$ one-sets. Each of the other complete passes may be grouped into $\frac{n-1}{2}$ 0-doubletons, one 0-singleton and one $\frac{n}{2}$-singleton.

The incomplete pass $[\boldsymbol{r}]$ may be dealt with in similar manner as in Lemma 5.4 to obtain s zero-sets of order k and is reduced to a residual set R of order at most k by using s complete passes, where $0 \le s \le 4$.

Additional reduction: Since $n|3k(3k-1)$, R is either a 0-set or an $\frac{n}{2}$-set. If R is a 0-set, no additional reduction is needed. Suppose R is an $\frac{n}{2}$-set. If $|R| < k$, then take any $\frac{n}{2}$-singleton from a complete pass $[\boldsymbol{n}]$ to form a 0-set of order at most k. Suppose $|R| = k$. Since $[\boldsymbol{r}]$ starts with at most one $\frac{n}{2}$, and the element $\frac{n}{2}$ is never returned to the residual set during the reduction process, so there exists x_1 $(\neq \frac{n}{2}) \in R$. Take the sets $\{\frac{n}{2}+x_1, \frac{n}{2}-x_1\}$, $\{\frac{n}{2}\}$ and $\{n\}$ from a complete pass of $[\boldsymbol{n}]$ to form two 0-sets $[R \setminus \{x_1\}] \cup \{\frac{n}{2}+x_1\}$ and $\{x_1, \frac{n}{2}-x_1, \frac{n}{2}, n\}$.

This process might affect up to $s+t$ complete passes of $[\boldsymbol{n}]$, where $0 \le s \le 4$ and $t = 0,1$ (for the additional reduction). Note that $0 \le s+t \le 5$. So far we have s zero-sets of order k, at most two zero-sets of order not exceeding k, $\frac{n}{2}$ one-doubletons, a number of 0-doubletons, a number of 0-singletons and a number of $\frac{n}{2}$-singletons. Since $[\boldsymbol{(2n-3)k}]$ is an $\frac{n}{2}$-set, the number of $\frac{n}{2}$-singletons is even. Thus we can pair up the $\frac{n}{2}$-singletons to produce 0-doubletons.

Now suppose k is even. Then $k \ge 4$ and the number of complete passes is $2k-b \ge 8-b$ for $1 \le b \le 3$. If $1 \le b \le 2$, then there are 6 or more passes of $[\boldsymbol{n}]$, and the number of complete passes affected by the reduction process, additional reduction and the rectifying process is at most 6. If $b = 3$, then there are 5 or more passes of $[\boldsymbol{n}]$. But it follows from $r = 3n - 3k$, that $k > 2n/3$, and therefore the order of the residual set will be strictly less than k after 1 reduction. Consequently, at most 3 complete passes of $[\boldsymbol{n}]$ are affected by the reduction process, additional reduction and the rectifying process. In all case, once that is done, we can form $\frac{n}{2}$ one-sets and $\frac{3n}{2} - 3$ zero-sets, all of which are of order k.

Now suppose k is odd. Similar to Lemma 5.4 we need $(\frac{3n}{2} - 3)$ odd order 0-sets with order not exceeding k and $\frac{n}{2}$ odd order 0-sets with order not exceeding $k - 2$. Take 6 complete passes of $[\boldsymbol{n}]$, and tabulate them as follows:

1	2	$\cdots$	$\frac{n-2}{2}$	$\frac{n+2}{2}$	$\frac{n+4}{2}$	$\cdots$	$n-2$	$n-1$	n	$\frac{n}{2}$
$n-1$	$n-2$	$\cdots$	$\frac{n+2}{2}$	$\frac{n-2}{2}$	$\frac{n-4}{2}$	$\cdots$	2	1	n	$\frac{n}{2}$
1	2	$\cdots$	$\frac{n-2}{2}$	$\frac{n}{2}$	$\frac{n+2}{2}$	$\cdots$	$n-3$	$n-2$	$n-1$	n
$n-1$	$n-2$	$\cdots$	$\frac{n+2}{2}$	$\frac{n}{2}$	$\frac{n-2}{2}$	$\cdots$	3	2	1	n
2	4	$\cdots$	$n-2$	1	3	$\cdots$	$n-5$	$n-3$	$n-1$	n
$n-2$	$n-4$	$\cdots$	2	$n-1$	$n-3$	$\cdots$	5	3	1	n

With the exception of the rightmost one, each column consists of three 0-doubletons which can be regrouped into two 0-tripletons. This can be done by taking elements in the first, third and sixth row to form one 0-tripleton; and elements in the second, the fourth and the fifth row to form another. If the number of 0-doubletons used for reduction purposes is $s+t$, then at least $2(n-1-s-t)$ zero-tripletons and three 0-singletons $\{n\}$ may be formed from these six passes of $[\boldsymbol{n}]$. We only include three 0-singletons $\{n\}$ because one might have been used in the additional reduction above. However, in the reduction process, at least s zero-sets of order k are produced, therefore the possible total number of odd order 0-sets is at least $2n-s-2t+1 \geq 2n-5$. Note that, we have $\frac{n}{2}$ odd order 0-sets with order not exceeding $k - 2$.

Since k is odd, $k \geq 5$ and the number of complete passes is $2k-b \geq 10-b$, where $1 \leq b \leq 3$. If $b = 1$, then there are 9 or more passes of $[\boldsymbol{n}]$, after providing 6 passes for producing the odd order sets above and 1 pass for rectifying process, there are at least two unaffected complete passes of $[\boldsymbol{n}]$. Each of these two passes contains at least one odd order 0-set, namely $\{n\}$. So we can construct up to $2n - s - 2t + 1 + 2 \geq 2n - 3$ odd order 0-sets (including at least five 0-singletons). If $b = 2$, then the number of complete passes is not less than 8. However, in this case $3k = 2n-r \geq n+1 \geq r+2$, and we have $s \leq 3$. We have at least one undisturbed complete pass, which will contribute at least one 0-singleton. So the possible total number of odd order 0-sets that can be formed is at least $2n - s - 2t + 1 + 1 \geq 2n - 3$ (including at least four 0-singletons). If $b = 3$, then the number of complete passes is not less than 7. there are enough complete passes to product the odd order sets and to perform the rectifying process. However, in this case $3k = 3n - r \geq 2n$, we have $s \leq 2$. So the possible total number of odd order 0-sets that can be formed is at least $2n - s - 2t + 1 \geq 2n - 3$ (including at least three 0-singletons). □

Theorem 5.7 *If $n|3k(3k-1)$ and $k \geq 4$, then $P_n^2[k]$ is edge-magic.*

Proof: Combining Remark 1.1, Lemmas 5.5 and 5.6, Corollaries 3.3, 3.4 and 5.2, we have the theorem. □

Combining the above theorem with the results in Section 4, we have

Theorem 5.8 *For $n \geq 3$ and $n \neq 6$, $I_m(P_n^2) = \{k \geq 2 \mid 3k(3k-1) \equiv 0 \pmod{n}\}$ and $I_m(P_6^2) = \{k \geq 1 \mid 3k(3k-1) \equiv 0 \pmod{6}\}$.*

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