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Fractional differential equations for modelling financial processes with jumps

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Fractional Differential Equations for Modelling Financial Processes with Jumps

GUO Xu

A thesis submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

Principal Supervisor: Dr. LING Leevan

Hong Kong Baptist University

August 2015
DECLARATION

I hereby declare that this thesis represents my own work which has been done after registration for the degree of PhD at Hong Kong Baptist University, and has not been previously included in a thesis or dissertation submitted to this or any other institution for a degree, diploma or other qualifications.

Signature: _______________
Date: August 2015
ABSTRACT

The standard Black-Scholes model is under the assumption of geometric Brownian motion, and the log-returns for Black-Scholes model are independent and Gaussian. However, most of the recent literature on the statistical properties of the log-returns makes this hypothesis not always consistent. One of the ongoing research topics is to find a better financial pricing model instead of the Black-Scholes model.

In the present work, we concentrate on two typical 1-D option pricing models under the general exponential Lévy processes, namely the finite moment log-stable (FMLS) model and the the Carr-Geman-Madan-Yor-eta (CGMYe) model, and we also propose a multivariate CGMYe model. Both the frameworks, and the numerical estimations and simulations are studied in this thesis.

In the future work, we shall continue to study the fractional partial differential equations (FPDEs) of the financial models, and seek for the efficient numerical algorithms of the American pricing problems.

Keywords: fractional partial differential equation; option pricing models; exponential Lévy process; approximate solution.
ACKNOWLEDGEMENTS

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Chapter 1

Introduction

1.1 General introduction for option pricing problems

In finance, an option is a contract which offers the owner the right, but not the obligation, to buy (for call option) or sell (for put option) a security or other financial asset at certain period of time or on a specific date (exercise date). Options can protect or enhance an investor’s portfolio in rising, falling and neutral markets. Depending on the strategy, option trading can provide a variety of benefits that including the security of limited risk and the advantage of leverage.

Options can be classified in a few ways. According to the option rights, we have call options and put options: an option that conveys to the owner the right to buy something at a specific price is referred to as a call option; while an option that conveys the right of the owner to sell something at a specific price is referred to as a put option. Then according to the underlying assets, options can be classified as equity options, bond options, index options, future options, currency options and commodity options. We can also classify options due to their option styles. European option has the most basic option style, and it can be only exercised on expiration; American option can be exercised at any trading day on or before the expiry; Bermudan option can be exercised at some specified dates on or before the expiration; and Asian option is an option whose payoff is determined by the average underlying price over some preset time period. Except these four kinds of options, we also have some exotic
options, such as barrier options, basket option, etc. Barrier options are such kind of the options whose payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price. A barrier option can be a knock-out, meaning the option is worthless if the underlying asset price exceeds a fixed level. It can also be a knock-in, meaning it has no value until the underlying asset price reaches a fixed level. We will evaluate the prices of the European knock-out options in the numerical examples of Section 2.4.2. Besides, we will also price the European basket put option values in Section 4.4.2. Unlike the ordinary options, a basket option’s payoff is linked to a portfolio or “basket” of underlier values. The basket can be any weighted sum of underlier values so long as the weights are all positive.

Options valuation is a topic of ongoing research in both analytical and practical finance. Although option contracts have been known for centuries and the valuation of options has been studied since the nineteenth century, both research interest and the trading activity for options increased since 1973. Today many options are created in a standardized form and traded through clearing houses on regulated options exchanges.

1.2 Traditional option pricing models

The value of an option can be estimated by using a variety of quantitative techniques based on using stochastic calculus and the concept of risk neutral pricing. Option traders utilize various option pricing models to set a current theoretical value. Models often use certain fixed variables – factors, such as the underlying price, the strike price, and the time to maturity – along with forecasts for factors like implied volatility, to compute the theoretical value for a specific option at some time point. Variables will fluctuate over the whole life of the option, and the theoretical value of the option position will thus adapt to reflect these changes.
1.2.1 Black-Scholes model for European options

The most basic and traditional option pricing model is the Black-Scholes model. Following the early research work by L. Bachelier, and later research work by R. C. Merton, F. Black and M. Scholes \cite{14} made a major breakthrough by deriving a differential equation for the European-styled option values that depends on a non-dividend-paying stock.

The Black-Scholes model assumes that the market consists of at least one risky asset and one risk-free asset; there is no arbitrage opportunity (i.e. there is no way to make a risk-free profit); it is possible to borrow and lend any amount, even fractional, of cash at the risk-free rate; it is possible to buy and sell any amount, even fractional, of the stock (this includes short selling); and all the above transactions do not incur any fees or costs (i.e. frictionless market).

By employing the idea of constructing a risk neutral portfolio that replicates the returns of holding an option, Black and Scholes gave a closed-form solution for the theoretical price of European put and call options. This well-known pricing model was derived under the assumption that the stock price $S_t$ at time $t$, follows a geometric Brownian motion, that is, the natural logarithm of the stock price $\ln S_t$ follows a Brownian motion with a deterministic drift $\mu$:

$$d(\ln S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t.$$  \hspace{1cm} (1.1)

Here $\sigma > 0$ is the volatility of the stock returns, $\mu > 0$ is the average compounded growth of the stock price $S_t$, and $dB_t$ is the increment of the Brownian motion.

On the basis of the traditional Black-Scholes model, the price of a European-styled option, $V(S,t)$, which is written on the traded asset $S_t$, may also be demonstrated as an advection-diffusion type equation by making the change of the variable $\ln S_t = x_t$, together with the supposition that the accession of the underlying Brownian motion should under the equivalent martingale measure, or equally, the risk-neutral
probability measure:

\[
\frac{\partial V(x, t)}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V(x, t)}{\partial x^2} + \left( r - q - \frac{1}{2} \sigma^2 \right) \frac{\partial V(x, t)}{\partial x} = r V(x, t), \tag{1.2}
\]

where \( r \) is the risk-free interest rate, \( q \) is the continuous dividends yield, and the log-returns should be independent and Gaussian.

“The Greeks” measure the sensitivity of the value of a derivative or a portfolio to changes in parameter values while holding the other parameters fixed. The Greeks are important not only in the mathematical theory of finance, but also for those actively trading. For Black-Scholes model, the Greeks are given in closed form below. They can be obtained by differentiation of the Black-Scholes formula.

\[
\begin{align*}
\text{delta} & : \frac{\partial V}{\partial S}, \\
\text{gamma} & : \frac{\partial^2 V}{\partial S^2}, \\
\text{vega} & : \frac{\partial V}{\partial \sigma}, \\
\text{theta} & : \frac{\partial V}{\partial t}, \\
\text{rho} & : \frac{\partial V}{\partial r}.
\end{align*}
\]

The Black-Scholes model generates hedging strategy for effective risk management of option holders. Although the techniques behind the Black-Scholes model were ground-breaking, the application of this model is clumsy in actual options trading, because of the too idealistic assumptions of the constant volatility, efficient markets, continuous trading, and the returns on the underlying are normally distributed.

1.2.2 Binomial option pricing model for American options

The Cox-Rubenstein (or Cox-Ross-Rubenstein) binomial option pricing model is a variation of the original Black-Scholes model. It was first proposed in 1979 by John Carrington Cox, Stephen Ross and Mark Edward Rubenstein \[28\]. The model is popular because it considers the underlying instrument over a period of time, instead
of just at one point in time, by using a discrete-time (or lattice based) model of the varying price of the underlying financial instrument. As a consequence, this method is used to value American options that are exercisable at any time in a given interval as well as Bermudan options that are exercisable at specific instances of time. Moreover, since this method is able to provide a mathematical valuation of the option at each specified time, it leads a more accurate estimate of option prices than created by models that consider only one point in time.

The Cox-Ross-Rubenstein model is also used in a risk neutral environment. Valuation is performed iteratively, starting at each of the final nodes (those that may be reached at the time of expiry date), and then working backwards through the tree towards the first node (the valuation date). The value computed at each stage is the value of the option at that point in time. Option valuation using binomial method is a three-step process:

- Step 1: Create the binomial price tree, and price tree generation,
- Step 2: Find the option values at each final node,
- Step 3: Sequential calculation of the option value at each preceding node.

1.3 Jump process models

There are still many models are developed for option pricing. In the following section, we will first have an overview on the jump process option pricing models, then review the concept of Lévy processes, and introduce several pricing models according to its different distribution processes.

1.3.1 Overview of jump process models

Recent studies suggested that mathematical models based upon Gaussian processes or distributions with finite variance do not seem to describe the prices of financial
assets properly [31], because Gaussian distributions, to some extent, underestimate the probability of the appearance of jumps or large movements in stock prices over small time steps [76].

Generally, the data of option price follows a Markovian process, and from the spatial point of view, it satisfies a classical Fickian diffusion process. Loosely speaking, the Markov process can be thought of as 'memoryless', that is to say, the future probabilities of option data are determined by its most recent values. Moreover, Fick’s second law predicts how diffusion causes the concentration to change with time. It is a partial differential equation which in one dimension reads:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2},$$

where $D$ is the diffusion coefficient, and in two or more dimensions, we have to use the Laplacian $\Delta = \nabla^2$, which generalises the second derivative, obtaining the equation

$$\frac{\partial u}{\partial t} = D \Delta u.$$

The discrepancies by the assumption of the Gaussian distribution can occur both in the time relaxation that can deviate from the classical exponential pattern or Markovian processes, and in the spatial coordinates that can deviate the Fick’s second law. In other words, the resulting diffusion processes can no longer be represented by the canonical Black-Scholes equation. The financial crisis from 2007 to 2010 also shows that the prices of financial assets cannot be described properly by the models that are currently in use [92].

So far, a lot of efforts have been made to develop mathematical models that can better describe the prices of financial assets. Roughly speaking, there are two kinds of models were developed, namely, stochastic volatility models and jump process models. The stochastic volatility models turned out to be successful in many circumstances (see Heston [45], Hull and White [46], and Stein and Stein [90]), however, the appearances of jumps in markets can only be explained as a huge and non-realistic volatility.
On the other hand, jump process models allows us to model large price changes due to sudden exogenous events on information, and can explain some systematic empirical biases [1]. The main idea of jump process models is to adopt a Lévy process instead of the standard Brownian motion to model asset prices.

According to the definition, a jump process is a type of stochastic process, which has discrete movements, i.e. the jumps, rather than small continuous movements. Merton [69] constructed a hybrid model known as jump diffusion model to the asset prices, this model states that the prices have large jumps followed by small continuous movements. With the similar idea, Kou described a double exponential jump-diffusion model in [50], and Bates [10] discussed a model with compound Poisson jumps and stochastic volatility. All of these three models treat the jumps as rare events. Another kinds of jump process models has infinite number of jumps in every random interval. Example of such models are the variance gamma models [60, 62], the Normal inverse Gaussian process models [6, 7, 79], generalized hyperbolic distribution models [5, 44, 77] and tempered stable process models [49, 15, 18]. The Carr-Geman-Madan-Yor (CGMY) process model, which was first proposed by Carr et. at. [18], is a well-known tempered stable process model and has been widely applied. Comparing with the other jump models, CGMY process model has a greater modelling freedom and flexibility, and outperforms in preventing smirk flattening across the maturity dimension. Based on CGMY processes, Carr and Wu develop the finite moment log-stable (FMLS) model in [19], this model can keep the same tail behavior due to its self-similarity.

1.3.2 Lévy processes

We introduce the content of Lévy processes in this subsection. Since a Lévy process can be regarded as a combination of a Brownian motion, a compound Poisson process, and a square integrable pure jump martingale, we introduce the following three notations, so that we can better describe a Lévy process. They are
• Brownian motion,

• Poisson process,

• Compound Poisson process.

The probability term “almost surely” will be used to define the above processes. Hence for the better comprehension of the definitions of the processes, we introduce the concept of this term below.

**Definition 1.1 (Almost sure).** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In probability theory, an event $E \in \mathcal{F}$ happens almost surely if it happens with probability 1, i.e. $\mathbb{P}(E) = 1$.

If an event will always happen, and no outcome not in this event can possibly occur, then the event is called “sure”. Note that the term “sure” is not exactly equal to the term “almost sure”. Because $\mathbb{P}(E) = 1$ only implies that the probability of the outcome in $E$ occurring is larger than any fixed positive probability, but we cannot say that the outcomes not in $E$ will never occur.

The definitions of the characteristic functions for Lévy processes are also given in this part. Based on the characteristic functions, we introduce some properties of Lévy processes, i.e. the Lévy-Khintchine representation and the Lévy-Itô decomposition at the end of this subsection.

### 1.3.2.1 Brownian motion

Brownian motion process, also known as the Wiener process in finance, is of central importance in probability, and is one of the most interesting stochastic processes. It was named after a famous physical phenomena that first observed by the botanist R. Brown in 1827, and was first constructed as a mathematical random process in rigorous way by N. Wiener in a series of papers starting from 1918. Currently, Brownian motion and its related processes are used in a wide applications ranging from physics to statistics and economics.
**Definition 1.2** (Brownian motion). A stochastic process \( B_t \) is called a Brownian motion if it satisfies the following properties,

(i) \( B_0 = 0 \) almost surely,

(ii) \( t \mapsto B_t \) is continuous almost surely,

(iii) for \( 0 \leq t_1 \leq \cdots \leq t_n \), \( \{B_{t_k} - B_{t_{k-1}}, \ldots, B_{t_2} - B_{t_1}\} \) are independent,

(iv) for \( s < t \), \( B_t - B_s \) follows the normal distribution with \( \mu = 0 \) and \( \sigma = t - s \), i.e. \( N(0, t - s) \).

**1.3.2.2 Poisson process**

The Poisson process is one of the most important stochastic processes, and has a lot of applications in finance. The idea behind the Poisson process is to model the arrivals with a Poisson random variable at some rate \( \lambda \). In order to define the Poisson process, we shall first introduce the concept of Poisson distribution, and the definition of the Poisson process can be hence obtained.

**Definition 1.3** (Poisson distribution). A discrete random variable \( X \) is said to follow a Poisson distribution with the parameter \( \lambda > 0 \), i.e. \( X \sim \text{Pois}(\lambda) \), if the probability mass function of \( X \) has the form

\[
f(k; \lambda) = \mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!},
\]

where \( k \in \{0, 1, 2, \ldots\} \).

**Definition 1.4** (Poisson process). A Poisson process \( N_t \) is a stochastic process that satisfies the following points,

(i) \( N_0 = 0 \) almost surely, i.e. \( \mathbb{P}(N_0 = 0) = 1 \)

(ii) \( t \mapsto N_t \) is almost surely right continuous with left limits,

(iii) for \( 0 \leq t_1 \leq \cdots \leq t_n \), \( \{N_{t_k} - N_{t_{k-1}}, \ldots, N_{t_2} - N_{t_1}\} \) are independent,

(iv) for \( s < t \), \( N_t - N_s \) follows the distribution \( \text{Pois}((t - s)\lambda) \).
1.3.2.3 Compound Poisson process

Based on the above definition, we may construct a Poisson process in the following way.

Let \( \{X_n\}_{n=0}^{\infty} \) be a random walk with \( X_1 \) distributed as an exponential of rate \( \lambda \). Now the Poisson process \( N_t \) satisfies,

\[
N_t(\omega) = n \quad \text{if and only if} \quad X_n(\omega) \leq t < X_{n+1}(\omega).
\]

Follows this idea, we will now introduce the compound Poisson process.

**Definition 1.5** (Compound Poisson process). Let \( N_t \) be a Poisson process and \( \{\xi_i\}_{i=1}^{\infty} \) be a set of independent identically distributed (i.i.d.) random variables independent of \( N_t \). Then a compound Poisson process \( Y_t \) is defined as

\[
Y_t = \sum_{i=1}^{N_t} (\xi_i).
\]

1.3.2.4 Lévy process

Both Brownian motions and Poisson processes share some common ground. They all have stationary and independent increments. Notice that even though Brownian motions are continuous and Poisson processes are not continuous, they are all right continuous with left limits. This gives rise to a very general class of processes, i.e. Lévy processes.

**Definition 1.6** (Lévy process). A stochastic process \( L_t \) is said to be a Lévy process if it satisfies the following properties

(i) \( L_0 = 0 \) almost surely,

(ii) \( t \mapsto L_t \) is almost surely right continuous with left limits,

(iii) for \( 0 \leq t_1 \leq \cdots \leq t_n \), \( \{L_{t_k} - L_{t_{k-1}}, \ldots, L_{t_2} - L_{t_1}\} \) are independent,

(iv) for \( s < t \), \( L_t - L_s \) is equal in distribution to \( L_{t-s} \).
Any process that satisfying the properties (i), (iii) and (iv) is called a Lévy process in law.

1.3.2.5 Basic properties of characteristic functions

In this section we shall assume that $X$ is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and give a basic introduction to characteristic functions. In mathematics, we can use Fourier transforms on a distribution function to simplify the expression. For some random variables the distribution cannot be explicitly known whereas we can often know the characteristic function.

**Definition 1.7 (Characteristic function).** A characteristic function $\Phi$ of $X$ is defined as

$$\Phi(\xi) = \int_{\mathbb{R}} e^{ix\xi} \mathbb{P}(X \in dx),$$

and the function $\Psi = \ln \Phi$ is referred to as the characteristic exponent of $X$.

The next theorem will play an important role in the present thesis. It describes the basic properties of sequence of characteristic functions.

**Theorem 1.1 (Lévy continuity theorem).** Let $\{X_n : n = 1, 2, \ldots\}$ be a sequence of random variables (not necessarily on the same probability space) with characteristic functions $\Phi_n$. If $\Phi_n \to \Phi$ is point-wise, then the following statements are equivalent,

(i) $\Phi$ is the characteristic function of some random variable $X$,

(ii) $\Phi$ is the characteristic function of $X$ where $X_n \to X$,

(iii) $\Phi$ is continuous,

(iv) $\Phi$ is continuous in some neighbourhood of 0.

For the proof of the above theorem, we refer the readers to Fristedt and Gray [39]. Moreover, in order to see which functions are characteristic functions, we provide the following theorem from analysis.
Theorem 1.2 (Bochner). A function $\Phi$ is a characteristic function if and only if the following points hold,

(i) $\Phi(0) = 1,$

(ii) $\Phi(\xi)$ is continuous,

(iii) for any $\{u_i\}_{i=1}^n \subset \mathbb{R}$ and $\{v_i\}_{i=1}^n \subset \mathbb{C}, \sum_{i=1}^n \sum_{j=1}^n \Phi(u_i - u_j)v_i v_j \geq 0.$

The proof of Bochner’s theorem can be found in the book of Reed and Simon [78]. In the next few chapters of this thesis, we will use the notation of characteristic exponent $\Psi$, which is the exponent of the characteristic function $\Phi$, to work on the framework of the pricing models.

1.3.2.6 Lévy-Khintchine representation

Roughly speaking, a Lévy process can be viewed as a superposition of a deterministic drift component and two independent random processes: a Brownian motion process and a jump process, which are completely determined by the Lévy-Khintchine triplet $(\mu, \sigma^2, W)$. Here the drift $\mu$ and the volatility $\sigma$ are the same as those in the Brownian motion (1.1). The new term $W$ is the Lévy measure, which accounts for the jump behavior of the Lévy process $X_t$ and determines the frequency and magnitude of jumps.

Definition 1.8 (Lévy-Khintchine representation). A stochastic process $(X_t)_{t \geq 0}$ is a Lévy process if it has independent and stationary increments with a real-value characteristic exponent $\Psi$ given by the Lévy-Khintchine representation [37, 26]:

$$\ln \mathbb{E}[e^{i\xi X_t}] = t\Psi(\xi) \equiv imt\xi - \frac{1}{2}\sigma^2 t\xi^2 + t \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\xi x} - 1 - i\xi x I_{|x|<1}\right) W(dx),$$

where $m \in \mathbb{R}, i = \sqrt{-1}, I$ is the indicator function, and the Lévy measure $W$ satisfies

$$\int_{\mathbb{R}\setminus\{0\}} \min\{1, x^2\} W(dx) < \infty.$$
We shall assume the Lévy measure $W(dx)$ is absolutely continuous, so that it can be expressed as

$$W(dx) = \omega(x)dx,$$  \hspace{1cm} (1.5)

where $\omega(x)$ is known as the Lévy density.

The Lévy-Khintchine representation gives a simple and elegant way of working with Lévy processes, it allows us to have a general form of a characteristic exponent $\Psi$ (or characteristic function $\Phi$), which is as good as having the law of the process.

1.3.2.7 Lévy-Itô decomposition

The Lévy-Khintchine representation gives us the characteristic exponent of for a given Lévy process. Now we are considering to obtain the general form of the Lévy process with such a characteristic exponent.

Our next theorem gives deep insight into the workings of a Lévy process. This was first developed by Itô in 1942. It states that a Lévy process can be thought as a combination of a Brownian motion, a compound Poisson process and a pure jump process.

**Theorem 1.3** (Lévy-Itô decomposition). Let $(a, \sigma^2, \Pi)$ be a Lévy triplet, then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which three processes $L^{(1)}$, $L^{(2)}$ and $L^{(3)}$ exist, where $L^{(1)}$ is a Brownian motion with drift, $L^{(2)}$ is a compound Poisson process, and $L^{(3)}$ is a square integrable pure jump martingale that almost surely has a countable number of jumps on each finite interval. $L$ defined by $L = L^{(1)} + L^{(2)} + L^{(3)}$ is a Lévy process.

1.3.3 Some typical pricing models on Lévy process

1.3.3.1 Stable processes

Now, let us consider the stable processes, also called the Lévy-$\alpha$-stable processes. Real valued Lévy-$\alpha$-stable processes $S(\sigma, \alpha, \beta)$ with parameters $(\sigma, \alpha, \beta)$ are Lévy
processes with no continuous martingale part (for details, see DuMouchel, Samorodnitsky and Taqqu, Bertoin, Nolan, Itô), and the Lévy measure $W(dx) = \omega(x)dx$ has the form
\[ \omega(x) = \frac{c_p}{x^{1+\alpha}}I_{x>0} + \frac{c_n}{|x|^{1+\alpha}}I_{x<0}, \]  
(1.6)

where $\alpha \in (0, 2]$ is the fractional order, $\beta = \frac{c_p - c_n}{c_p + c_n} \in [-1, 1]$ is the skew index, and we denote
\[ \sigma = \left[ \frac{c_p + c_n}{2} \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1 - \frac{\alpha}{2}\right)}{\Gamma\left(1 + \alpha\right)} \right]^{\frac{1}{\alpha}} > 0. \]  
(1.7)

The characteristic exponent $\Psi(\xi)$ of $S(\sigma, \alpha, \beta)$ at $t = 1$ given by
\[ \Psi(\xi) = \begin{cases} -\sigma^\alpha|\xi|^\alpha \left(1 - i\beta \text{sgn}(\xi) \tan\left(\frac{\pi\alpha}{2}\right)\right) + i\mu \xi, & \alpha \neq 1, \\ -\sigma|\xi| \left(1 + \frac{2}{\pi} i\beta \text{sgn}(\xi) \log(|\xi|)\right) + i\mu \xi, & \alpha = 1, \end{cases} \]  
(1.8)

with the parameter $\mu \in \mathbb{R}$ standing for the drift; see [63].

The finite moment log-stable (FMLS) process model, which was proposed by Carr and Wu in [19], is a special case of the the Lévy-$\alpha$-stable process. In the FMLS model, the skew index $\beta$ is set to be $-1$, which implies that jumps only occur downwards and all the upward paths are continuous. For such a case, the characteristic exponent of the process simplifies to be
\[ \Psi_{\text{FMLS}}(\xi) = -\frac{1}{2} \sigma^\alpha \sec\left(\alpha \pi/2\right)(i\xi)^\alpha, \]  
(1.9)

where $1 < \alpha < 2$, $\sigma$ is the volatility, and the convexity adjustment $\nu$ has the form
\[ \nu = -\frac{1}{2} \sigma^\alpha \sec\left(\alpha \pi/2\right). \]  
(1.10)

Note that the “convexity adjustment” we used here refers to the difference between the forward interest rate and the future interest rate; this difference has to be added to the former to arrive at the latter. The need for this adjustment arises because of the non-linear relationship between our asset prices and the dividend yields.
1.3.3.2 Tempered stable processes

Unlike the standard stable processes, which can only be defined for the case $\alpha > 0$, the tempered stable process has no natural lower bound on $\alpha$, and is obtained by taking a 1-D stable process, and then multiplying the Lévy measure with a decreasing exponential factor on each half of the real axis. A one-dimensional generalized tempered stable process is a Lévy process on $\mathbb{R}$ with no Gaussian component, has the Lévy density as

$$\omega(x) = \frac{c_+ e^{-\lambda_+ x} I_{x > 0}}{x^{1+\alpha_+}} + \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+\alpha_-}} I_{x < 0},$$

where the parameters $c_-, c_+, \lambda_-, \lambda_+ > 0$, and $0 < \alpha_-, \alpha_+ < 2$. Note that the Lévy density of a tempered stable process (1.11) is obtained by multiplying an exponentially decaying factor to the stable one (1.6). After the exponential softening, the small jumps keep their stable-like behavior whereas the large jumps become much less violent (see the comment by Cont and Tankov [20, S4.5]).

The CGMY process is a special case of the tempered stable process with the specific choice of parameters $c_+ = c_-$ and $\alpha_+ = \alpha_-$. This process was introduced by Carr et. at. [18] to model the stock prices and capture the heavy-tailed behavior. The Lévy density of the CGMY process is thus given by

$$\omega_{CGMY}(x) = C \left[ \frac{\exp(-G|x|)}{|x|^{1+Y}} I_{x < 0} + \frac{\exp(-Mx)}{x^{1+Y}} I_{x > 0} \right],$$

with $C > 0$, $G \geq 0$, $M \geq 0$, and $Y < 2$.

From the Lévy-Khintchine formula

$$\ln \mathbb{E}[e^{i\xi X_t}] = i \text{mt} \xi + t \int_{\mathbb{R} \setminus \{0\}} (e^{i\xi x} - 1) W(dx),$$

the characteristic exponent of the CGMY process at $t = 1$ is given by

$$\Psi_{CGMY}(\xi) = CT(-Y) [(G + i\xi)^Y - G^Y + (M - i\xi)^Y - M^Y],$$

for $Y \neq 0, 1$, where $\Gamma(\cdot)$ is the gamma function, see [18]. Note that $\Gamma(-Y)$ has poles at $Y = 0, 1$, in which case the characteristic exponent of the CGMY process can be
obtained by setting $Y$ to 0 or 1, which are given by:

$$
\Psi_{CGMY}(\xi) = \begin{cases} 
C \left[ (G + i\xi) \ln(G + i\xi) - G \ln G 
+ (M - i\xi) \ln(M - i\xi) - M \ln M \right], & Y = 1, \\
- C \left[ \ln(G + i\xi) - \ln G + \ln(M - i\xi) - \ln M \right], & Y = 0.
\end{cases}
$$

The variance gamma (VG) model, proposed by Madan and Seneta in [61], adds more density to the tails of the normal distribution by replacing the deterministic time in the Black-Scholes model. It can be regarded as a special case of the CGMY model with $Y = 0$. The characteristic exponent of the VG model is given by

$$
\Psi_{VG}(\xi) = -\frac{1}{\kappa} \ln \left( 1 + \frac{\sigma^2 \kappa^2 \xi^2}{2} - i \theta \kappa \xi \right), \quad (1.15)
$$

where $\sigma$ and $\theta$ are the volatility and drift of a standard Brownian motion, $\kappa$ is the variance of the Gamma process with mean rate utility. For the relations between $\sigma$, $\theta$, $\kappa$ and $C$, $G$, $M$ see Carr et. at. [18], Eq. (8)-(10).

### 1.3.3.3 The Carr-Geman-Madan-Yor-eta (CGMYe) model

Motivated by the Lévy-Itô decomposition (see [26]), the log-return, $\ln S_t$, can be regarded as a sum of a Brownian motion and a CGMY process, we shall refer it as a CGMYe process following [18], and denote it by

$$
\ln S_t \sim \tilde{X}_t = X_{t,CGMYe}(C, G, M, Y, \eta) = X_{t,CGMY}(C, G, M, Y) + \eta W_t,
$$

where $W_t$ stands for the standard Brownian motion independent of the process $X_{CGMY}$. The characteristic exponent of the log-return relative in the CGMYe process is hence given by

$$
\Psi_{\tilde{X}_t}(\xi) = i(r - q - v)\xi - \frac{\eta^2}{2} \xi^2 + \Psi_{CGMY}(\xi), \quad (1.16)
$$

where $v$ is the convexity adjustment,

$$
v = \Psi_{CGMY}(-i) + \frac{\eta^2}{2} = CT(-Y) \left[ (G + 1)^Y - G^Y + (M - 1)^Y - M^Y \right] + \frac{\eta^2}{2}. \quad (1.17)
$$
Note that, by adopting the “risk neutral” measure $Q$, an European option has the value $V(x, t) = e^{-r(T-t)}E^Q[\Pi(X_T, T)]$, with $\Pi(X_T, T)$ being the final payoff, whose expectation can be evaluated through the complex Fourier transform introduced by Lewis [53], see also Lee [52], Chiarella and Ziogas [25].

1.4 Fourier transform method for pricing problems

In this section we introduce the Fourier transform option pricing approach given by Carr and Madan [17]. To illustrate the idea clearly and simply, we present the Black-Scholes model with Fourier transform pricing methodology. Based on this, we summarize a general work flow for the frameworks of options pricing problems, which is suitable for the analytical studies in Chapters 2 [3] and [4].

1.4.1 Motivation

Let $\{f_t : 0 \leq t \leq T\}$ be an information flow of the asset price (i.e. filtration). In an arbitrage-free market, prices of any assets can be calculated as expected terminal payoffs under $Q$ discounted by a risk-free interest rater $r$:

$$S_t = e^{-r(T-t)}E^Q[s_t | f_t],$$

which is the martingale condition.

Let $K$ be the strike price, $T$ be the expiration of a contingent claim, and $Q(S_T | f_t)$ be the probability density function of $S_T$ under $Q$ conditional on $f_t$. Then vanilla call and put option prices are computed as discounted risk-neutral conditional expectations of the terminal payoffs $(S_T - K)^+ = \max(S_T - K, 0)$, and $(K - S_T)^+ =$
max(K − ST, 0) as

\[
C(t, S_t) = e^{r(T-t)} \int_{K}^{\infty} (S_T - K)Q(S_T | f_t) dS_T,
\]

\[
P(t, S_t) = e^{r(T-t)} \int_{0}^{K} (K - S_T)Q(S_T | f_t)dS_T.
\]

(1.18)

(1.19)

Black-Scholes model assumes that a terminal stock price \( S_T \) conditional on \( f_t \) is a log-normal random variable with its density given by:

\[
Q(S_T | f_t) = \frac{1}{S_T \sqrt{2\pi \sigma^2(T-t)}} \exp \left[ -\frac{\left( \ln S_T - \left( \ln S_T + \left( r - \frac{1}{2} \sigma^2 \right) (T-t) \right) \right)^2}{2\sigma^2(T-t)} \right].
\]

Therefore, Black-Scholes pricing formula comes down to single integration problem with respect to \( S_T \) since all parameters and variables are known. This implies that as far as a conditional risk-neutral density of the terminal stock price \( Q(S_T | f_t) \) is given, vanilla option pricing reduces to single integration problem by Eq. (1.18).

However, \( Q(S_T | f_t) \) cannot be expressed using some special functions of mathematics for general exponential Lévy models, or is not known. Therefore, we cannot price vanilla options by simple using Eq. (1.18). Then how to price options under the general exponential Lévy processes? The answer is to use a very interesting fact that characteristic functions of general exponential Lévy processes are always known in closed-forms, and can be expressed in terms of special functions of mathematics. Note there exist one-to-one relationship between a probability density and a characteristic function through Fourier transform, and both of probability density and characteristic function uniquely determine a probability distribution.

### 1.4.2 Derivation of a call option price for Black-Scholes model with Fourier transform method

For simplicity, assume \( t = 0 \) without loss of generality. By using a change of variable technique from \( S_T \) to \( \ln S_T \), Eq. (1.18) can be rewrite as

\[
C(T, \ln S_T) = e^{-rT} \int_{\ln K}^{\infty} \left( e^{\ln S_T} - e^{\ln K} \right)Q(\ln S_T | f_0) d\ln S_T,
\]

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where \( Q(\ln S_T) = Q(\ln S_T \mid f_0) \) is a risk-neutral density of the log-return conditional on filtration \( f_0 \). Note that, a characteristic function of \( \ln S_T \) is a Fourier transform of its density function \( Q(\ln S_T) \)

\[
\Phi_T(\omega) \equiv \mathcal{F}[Q(\ln S_T)](\omega) = \int_{-\infty}^{\infty} e^{i\omega \ln S_T} Q(\ln S_T) d\ln S_T, \tag{1.20}
\]

and a characteristic function of \( \ln S_t \) for Black-Scholes model can be hence obtained as

\[
\Phi_{BS}(\omega) = \exp \left[ i \left( \ln S_0 + (r - q - \frac{1}{2} \sigma^2)T \right) \omega - \frac{(\sigma^2 T)\omega^2}{2} \right], \tag{1.21}
\]

and the convexity adjustment for Black-Scholes model is \( \upsilon = \frac{1}{2} \sigma^2 \).

Note that a call option price is not integrable, and cannot be Fourier transformed. So to solve this problem, Carr and Madan defines a modified call option price as

\[
\hat{C}(T, \ln K) \equiv e^{\alpha \ln K} C(T, \ln K), \tag{1.22}
\]

where \( \hat{C}(T, k) \) is expected to satisfy the following integrability condition by carefully choosing \( \alpha > 0 \)

\[
\int_{-\infty}^{\infty} |\hat{C}(T, \ln K)| d\ln K < \infty.
\]

Consider a Fourier transform of a modified call price

\[
\hat{\mathcal{F}}_T(\omega) = \int_{-\infty}^{\infty} e^{i\omega \ln K} \hat{C}(T, \ln K) d\ln K. \tag{1.23}
\]

Substitute Eq. (1.22) into Eq. (1.23), and interchange integrals, we have

\[
\hat{\mathcal{F}}_T(\omega) = \frac{e^{-rT} \Phi_T(\omega - (\alpha + 1)i)}{\alpha^2 + \alpha - \omega^2 + i(2\alpha + 1)\omega}. \tag{1.24}
\]

Finally, by substituting the characteristic function (1.21) into the general Fourier transform formula, we obtain the following pricing formula for vanilla call options

\[
C_{BS}(T, \ln K) = \frac{e^{-\alpha \ln K}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega \ln K} \frac{e^{-rT} \Phi_{BS}(\omega - (\alpha + 1)i)}{\alpha^2 + \alpha - \omega^2 + i(2\alpha + 1)\omega} d\omega. \tag{1.25}
\]
1.4.3 The fundamental solution of the Black-Scholes

The value of a European put option that satisfies Black-Scholes model can be evaluated as

\[ V_E(x, t) = \int_{-\infty}^{\infty} G(x, t; u, T)\Pi(u, T)du, \]  

where \( \Pi(u, T) \) is the final payoff of the option, and the function,

\[ G(x, t; u, T) = e^{-r(T-t)} \int_{-\infty}^{\infty} Q(\ln S_T | \mathcal{F}_t) d\ln S_T \]

\[ = e^{-r(T-t)} \int_{-\infty}^{\infty} \mathcal{F}^{-1}(\Phi_{BS}(\omega)) d\ln S_T, \]

is called the fundamental solution, or the heat kernel, or the transition density, with \( \mathcal{F}^{-1} \) being the inverse Fourier transform.

1.4.4 The general work flow of option pricing framework

We will summarize a general work flow for the frameworks of options pricing problems in this subsection. This work flow is suitable for the analytical studies in the whole thesis.

The general work flow has the form as:

- **Step 1:** Find the characteristic function \( \Phi \) (or the characteristic exponent \( \Psi \)) and the convexity adjustment \( \nu \) for the working model,

- **Step 2:** Derive the FPDE by using the Fourier (and inverse Fourier) transform techniques on \( \Phi \),

- **Step 3:** Derive the fundamental solution from the derivation of Step 2,

- **Step 4:** Derive the partial inequality and the Doob-Merer decomposition formula for American option values,

- **Step 5:** Derive the integral equation for the optimal-exercise boundary of American option values.
Note that, the first 3 steps are the framework for the European option pricing problems, and the last two steps are for the corresponding American option pricing problems.

1.5 Other important notations in the thesis

1.5.1 Definitions of fractional derivatives

There are several different ways to define the fractional derivatives, and a fractional
derivative is defined by a fractional integral. The most commonly used fractional
derivatives are the Riemann-Liouville derivative, the Grünwald-Letnikov derivative,
and the Caputo derivative, see Samko et al. [82] and Podlubny [74].

Definition 1.9 (Riemann-Liouville fractional integral). The fractional integral (or
the Riemann-Liouville integral) with order $\alpha > 0$ of the given function $f(x)$ is defined
as
\[
a D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \xi)^{\alpha-1} f(\xi) d\xi,
\]
where $a$ is the lower boundary of $x$, and $\Gamma(\cdot)$ is the Gamma function.

The corresponding Riemann-Liouville fractional derivative is based on Cauchy’s
formula for calculating the iterated integrals, and the fractional derivative of order $\alpha$
can be obtained by computing the $n$-th order derivative over the integral of order $(n -\alpha)$.

Definition 1.10 (Riemann-Liouville fractional derivatives). The left and right Riemann-
Liouville derivatives with order $\alpha > 0$ of the given function $f(x)$ are defined as
\[
a D_x^\alpha f(x) := RL_{a,x}^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_a^x (x - \xi)^{n-\alpha-1} f(\xi) d\xi
\]
and
\[
x D_x^\alpha f(x) := RL_{x,b}^\alpha f(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_x^b (\xi - x)^{n-\alpha-1} f(\xi) d\xi
\]
respectively, where $n$ is a non-negative integer and $n - 1 \leq \alpha < n$.  

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Another choice for the computation of fractional derivatives is the Grünwald-Letnikov fractional derivatives, which was introduced by A. K. Grünwald in 1867, and by A. V. Letnikov in 1868. The initial definition of Grünwald-Letnikov derivatives is given by a limit, and it is often used for a numerical approximation.

**Definition 1.11** (Grünwald-Letnikov fractional derivatives). The left and right Grünwald-Letnikov derivatives with order $\alpha > 0$ of the given function $f(x)$ are defined as

\[
aD^\alpha_x f(x) := GLD^\alpha_{a,x} f(x) = \lim_{h \to 0} h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x - kh), \quad (1.30)\]

and

\[
xD^\alpha_b f(x) := GLD^\alpha_{x,b} f(x) = \lim_{h \to 0} h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + kh) \quad (1.31)\]

respectively.

From Definitions 1.10 and 1.11, we can verify the following equations by integration by parts,

\[
\begin{align*}
RLD^\alpha_{a,x} f(x) &= GLD^\alpha_{a,x} f(x), \quad \text{if } f(x) \in C^n[a,x], \\
RLD^\alpha_{x,b} f(x) &= GLD^\alpha_{x,b} f(x), \quad \text{if } f(x) \in C^n[x,b].
\end{align*}
\]

This fact provide a numerical method for the fractional differential equations with Riemann-Liouville derivatives. Therefore, the Riemann-Liouville definition is suitable for the problem formulation, while the Grünwald-Letnikov definition is utilized to obtain the numerical solution, see Podlubny [74].

Caputo fractional derivative was defined by M. Caputo in 1967. It seems that the Caputo derivative is more welcome than the other two fractional derivatives, since the initial value of fractional differential equation with Caputo derivative is the same as that of integer differential equation.

**Definition 1.12** (Caputo fractional derivatives). The left and right Caputo derivatives with order $\alpha > 0$ of the given function $f(x)$ are defined as

\[
aD^\alpha_x f(x) := CD^\alpha_{a,x} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x - \xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi \quad (1.32)\]
respectively, where \( n \) is a non-negative integer and \( n - 1 \leq \alpha < n \).

Note that the Riemann-Liouville derivatives and Caputo derivatives can be mutually transformed by using the following formulas:

\[
\text{RL}D^\alpha_{a,x} f(x) = CD^\alpha_{a,x} f(x) + \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a),
\]

\[
\text{RL}D^\alpha_{x,b} f(x) = CD^\alpha_{x,b} f(x) + \sum_{k=0}^{n-1} (-1)^k \frac{(b-x)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(b),
\]

and these two derivatives will be the same if \( f^{(k)}(a) = f^{(k)}(b) = 0 \). Actually, all of these three fractional derivatives are equivalent under some conditions.

Moreover, a key issue in the solution of FDEs is the discretization of the integral operators involved. By the definition of the Grünwald-Letnikov derivative, it is natural to use Eqs. (1.30) and (1.31) to approximate the left and right Riemann-Liouville derivatives as

\[
\text{RL}D^\alpha_{a,x} f(x) = \lim_{h \to 0^+} h^{-\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(x - kh),
\]

\[
\text{RL}D^\alpha_{x,b} f(x) = \lim_{h \to 0^+} h^{-\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(x + kh),
\]

where \( h \) is the space step and \( g_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} \) with \( \binom{\alpha}{k} \) being the fractional binomial coefficients.

The coefficients \( g_k^{(\alpha)} \) can be evaluated recursively and satisfy the following properties [74, 66, 67]:

\[ g_0^{(\alpha)} = 1, \quad \text{and} \quad g_k^{(\alpha)} = \left(1 - \frac{\alpha + 1}{k}\right) g_{k-1}^{(\alpha)} \text{ for } k \geq 1. \]
1.5.2 Doob-Meyer decomposition

The idea of the Doob-Meyer decomposition will be used in the next three chapters. Here we will briefly introduce this theorem.

The Doob-Meyer decomposition is a very important result in the development of stochastic calculus. This theorem states that every sub-martingale, which is right continuous with left limit, can be uniquely decomposed as the sum of a local martingale and an increasing predictable process.

**Lemma 1.1.** Let $X$ be a right continuous martingale with left limit. Then, for each $t \geq 0$, the set

$$\{X_\tau : \tau \leq t \text{ is a stopping time}\}$$

is uniformly integrable.

This suggests the following generalized concepts of uniform integrability for stochastic processes.

**Definition 1.13.** Let $X$ be a jointly measurable stochastic process. Then, it is

- of class (D) if $\{X_\tau : \tau < \infty \text{ is a stopping time}\}$ is uniformly integrable.
- of class (DL) if, for each $t \geq 0$, $\{X_\tau : \tau \leq t \text{ is a stopping time}\}$ is uniformly integrable.

The term ‘class (D)’ sub-martingales are commonly used for the Doob-Meyer decomposition.

**Theorem 1.4** (Doob-Meyer decomposition). Any local sub-martingale $X$ has a unique decomposition

$$X = M + A,$$

where $M$ is a local martingale, $A$ is a predictable increasing process starting from zero.
Furthermore, we also have the following properties for the Doob-Meyer decomposition theorem:

1. if $X$ is a proper sub-martingale, then $A$ is integrable and satisfies

$$
E[A_\tau] \leq E[X_\tau - X_0],
$$
for all uniformly bounded stopping times $\tau$.

2. $X$ is of class (DL) if and only if $M$ is a proper martingale, and $A$ is integrable, in which case

$$
E[A_\tau] = E[X_\tau - X_0],
$$
for all uniformly bounded stopping times $\tau$.

3. $X$ is of class (D) if and only if $M$ is a uniformly integrable martingale and $A_\infty$ is integrable. Then, $X_\infty = \lim_{t \to \infty} X_t$ and $M_\infty = \lim_{t \to \infty} M_t$ exist almost surely.

1.5.3 Quadratic approximation method for American option problems

We introduce the quadratic approximation method in this part, since we will use this method for finding the analytical formula of American option prices for both short and long maturities.

The quadratic approximation method was first developed by MacMillan [59] for the pricing problem of non-dividend paying enquiry options and later extended to commodity options by Barone-Adesi and Whaley [8]. This method has been proved to be quite efficient with good accuracy, especially for shorter-lived option pricing problems.

For illustration, we take the Black-Scholes equation as an example. The governing equation can be given as

$$
\frac{\partial V}{\partial \tau} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \left( r - q - \frac{1}{2} \sigma^2 \right) \frac{\partial V}{\partial S} - rV,
$$

(1.34)
where $S$ is the stock price, $\tau = T - t$ is the time to maturity, $r$ is the risk-free interest rate, $q$ is the continuous dividends yield, and $\sigma$ is the volatility of $S$. Now let $P(S, \tau)$ be the American put option value, and $p(S, \tau)$ be the corresponding European put option value, then the early exercise premium can be written as

$$e(S, \tau) = P(S, \tau) - p(S, \tau).$$

Inside the continuation region, Eq. (1.34) holds for both $P(S, \tau)$ and $p(S, \tau)$. Since the differential equation is linear, the same equation holds for $e(S, \tau)$. Letting the parameters $k_1 = 2r/\sigma^2$, $k_2 = 2\left(r - q - \frac{1}{2}\sigma^2\right)/\sigma^2$, and defining

$$e(S, \tau) = A(\tau)f(S, A(\tau)),$$

with $A(\tau)$ to be determined. Then Eq. (1.34) can be transformed to be

$$S^2 \frac{\partial^2 f}{\partial S^2} + k_2 S \frac{\partial f}{\partial S} - k_1 f \left[1 + \frac{\partial A}{\partial \tau} \left(1 + \frac{A \partial f}{f \partial A}\right)\right] = 0.$$  \hspace{1cm} (1.35)

A judicious choice for $A(\tau)$ is $A(\tau) = 1 - e^{-r\tau}$. Thus Eq. (1.35) becomes

$$S^2 \frac{\partial^2 f}{\partial S^2} + k_2 S \frac{\partial f}{\partial S} - \frac{k_1}{A} \left[f + (1 - A)A \frac{\partial f}{\partial A}\right] = 0.$$  \hspace{1cm} (1.36)

Note that the last term in the last equation include the factor $(1 - A)A$, which will turns to be 0 when $\tau = 0$ and $\tau \to \infty$. Suppose we drop the the last term of Eq. (1.36), it hence can be reduced to an ordinary differential equation, and the error is controlled by the magnitude of the quadratic term $(1 - A)A$. This is how the name of this approximation method is derived. The approximate equation for $f$ now becomes

$$S^2 \frac{\partial^2 f}{\partial S^2} + k_2 S \frac{\partial f}{\partial S} - \frac{k_1}{A} f = 0,$$  \hspace{1cm} (1.37)

where $A$ is assumed to be nonzero. The general solution of Eq. (1.37) is

$$f(S) = c_1 S^{b_1} + c_2 S^{b_2},$$  \hspace{1cm} (1.38)
where $c_1, c_2$ are arbitrary constants, $b_1 < 0, b_2 > 0$ are the roots of the following equation

$$b^2 + (k_2 - 1)b - \frac{k_1}{A} = 0.$$ 

The term $c_1 S^{b_1}$ in Eq. (1.38) can be discarded since $f(S)$ tends to zero when $S$ approaches 0. Additionally, the arbitrary constant $c_2$ can be obtained by applying the value matching condition at the critical asset value $S^*$, and considering the high-contact condition $\frac{\partial P}{\partial S}(S^*, \tau) = -1$ along the optimal exercise boundary.

So finally, the approximate value of the American put option has the form as

$$P(S, \tau) = p(S, \tau) + \frac{S^*}{b_2} \left[1 - e^{-(q + \frac{1}{2} \sigma^2)\tau} N(d_1(S^*))\right] \left(\frac{S}{S^*}\right)^{b_2}, \quad S < S^*,$$

with

$$K - S^* = p(S^*, \tau) + \left[1 - e^{-(q + \frac{1}{2} \sigma^2)\tau} N(d_1(S^*))\right] \frac{S^*}{b_2},$$

and

$$d_1(S^*) = \frac{\ln(S^*/K) + (r - q)\tau}{\sigma \sqrt{\tau}}.$$ 

The content of the present thesis is arranged as follows: In Chapter 2 and 3, we construct the fractional partial differential equations (FPDEs) for European put option values of both FMLS model and CGMYe model, and derive the fundamental solution of the FPDEs. We also concentrate on the analytical study of American options under the two models. The decomposition formulas of the American option prices and the integral equations for the optimal-exercise boundaries are established. Moreover, the analytical approximation formulas are obtained for the option values, which are valid for both short and long maturities.

Besides the framework, we also give the numerical simulations after each model. We use Gauss-Jacobi spectral method to evaluate the option prices under the FMLS model, and compare the simulation results the finite difference method. Then we apply the maximum likelihood estimation (MLE) as the parameter estimator, and
use Gauss quadrature to simulate the optimal-exercise boundary of American options of the CGMYe model, and compares the effect with the standard Black-Scholes model.

In Chapter 4, we extend the 1-D CGMYe model to a multi-asset case. Similar framework, such as the derivation of the fundamental solution and the FPDE for European put options, are provided for the multi-asset CGMYe model. The general case of American options are also mentioned in this part. We also estimate parameters of 2-D CGMYe model in this section, and simulate a European basket put option values by using the estimated parameters. Finally, concluding comments and future studies are given in Chapter 5.
Chapter 2

Finite moment log-stable (FMLS) process model

In this chapter, we first overview the framework of the European option pricing problems given by Carr and Wu [19], Cartea and del Castillo-Negrete [20] for a typical Lévy-$\alpha$-stable process model, namely the finite moment log-stable (FMLS) model. In this part, we review the fractional partial differential equation (FPDE) of the European put option values and derive the fundamental solution of the FPDE.

Moreover, we concentrate on the analytical study of American options. We give the decomposition formula of the American option value as a sum of the corresponding European option value and an early exercise premium. Based on this, an integral equation is derived for the optimal-exercise boundary. Besides, an analytical approximation for the American put option value is also established, which is valid for both long and short maturities.

Additionally, we use Gauss-Jacobi spectral method for numerical simulations, and illustrate the flexibility and accuracy of the method by several numerical examples.

The presentation of this section will be organized in the following three subsections

(1) Construction of the fractional partial differential equation (FPDE) for European put option values of the FMLS model, and also provide the derivation of the fundamental solution of the FPDE;

(2) The focus in which subsection is the decomposition formula of the American option price for FMLS model, and the integral equation for the optimal-exercise
boundary in the explicit form. The near-expiry behavior of the optimal-exercise boundary and the analytical approximation formula for both short and long maturities are also derived.

(3) Describe the numerical algorithm of the efficient spectral method, and give four examples for the illustration of the flexibility and accuracy of the numerical method, as well as the properties of the FMLS model.

2.1 Review of the European option model

2.1.1 The characteristic function and the FPDE

In this section, we shall briefly review the FMLS model that proposed by Carr and Wu [19].

A time-dependent stochastic process $X_t$ is a Lévy process if it has independent and stationary increments with log-characteristic function given by the Lévy-Khintchine representation:

$$\ln E[e^{i\xi X_t}] \equiv t\Psi(\xi) = mit\xi - \frac{1}{2}\sigma^2t\xi^2 + \int_{\mathbb{R}\setminus \{0\}} (e^{i\xi x} - 1 - i\xi x I_{|x|<1}) W(dx),$$  \hspace{1cm} (2.1)

where $m \in \mathbb{R}$, $i = \sqrt{-1}$, $I$ is the indicator function, $\Psi(\xi)$ is the characteristic exponent of the Lévy process, and the Lévy measure $W$ satisfies:

$$\int_{\mathbb{R}\setminus \{0\}} \min\{1, x^2\} W(dx) < \infty.$$  \hspace{1cm} (2.2)

We shall assume the Lévy measure $W(dx)$ is absolutely continuous, so that it can be expressed as

$$W(dx) = \omega(x)dx,$$  \hspace{1cm} (2.3)

with $\omega(x)$ being the Lévy density.

As an extension to the Black-Scholes model, assuming the stock price $S_t$ follows an exponential Lévy process:

$$d(\ln S_t) = (r - q - \nu)dt + dL_t^Q,$$  \hspace{1cm} (2.4)
where the parameter $r$ is the risk-free interest rate, $q$ is the continuous dividends yield, $v$ denotes a convexity adjustment such that $\mathbb{E}^Q[S_T] = e^{(r-q)(T-t)}S_t$, and $dL_t^Q$ is the increment of a Lévy process under the equivalent martingale measure $Q$. Note that Eq. (2.4) has the solution given by
\[
S_T = S_t \exp \left( (r - q - v)(T - t) + \int_t^T dL_u^Q \right). \tag{2.5}
\]

Clearly, Eq. (2.4) allows the appearance of jumps, and naturally reduce to the traditional Black-Scholes model when no jumps are present: set $m = 0$, $W = 0$, and $\nu = \sigma^2/2$, Eq. (2.4) then simplifies to the following equation
\[
d(\ln S_t) = \left( r - q - \frac{1}{2} \sigma^2 \right) dt + \sigma dB_t^Q,
\]
which is exactly the Black-Scholes model.

Now, let us consider the Lévy-$\alpha$-stable process with density given by
\[
\omega_{LS}(x) = \begin{cases} 
\frac{1}{2} (1 - \beta) D |x|^{-1-\alpha}, & \text{for } x < 0, \\
\frac{1}{2} (1 + \beta) D x^{-1-\alpha}, & \text{for } x > 0,
\end{cases}
\]
where $D > 0$, $\beta \in [-1, 1]$ is the skew index, $1 < \alpha < 2$ is the stability index. By the above definitions in (2.1)-(2.3), the density in (2.6) yields a characteristic exponent of the Lévy-$\alpha$-stable process in terms of the parameters $\sigma$, $\alpha$, $\beta$ and $m$ as
\[
\Psi_{LS}(\xi) = -\frac{\sigma^\alpha}{4 \cos(\alpha \pi/2)} ((1 - \beta)(i\xi)^\alpha + (1 + \beta)(-i\xi)^\alpha) + im\xi. \tag{2.6}
\]

In the FMLS model, the skew index $\beta$ is set to be $-1$ (see [19]), which implies that the jumps in this model only occur downwards, and all the upwards paths are continuous. For such a case, the characteristic exponent of the process simplifies to be as
\[
\Psi_{FMLS}(\xi) = -\frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2)(i\xi)^\alpha, \tag{2.7}
\]
and the convexity adjustment $v$ in Eq. (2.4) reduces to $v = -\frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2)$. Next, by applying the Fourier transform, Cartea and del Castillo-Negrete [20] obtain the
corresponding FPDE of FMLS model

\[
\partial_t V(x,t) = \left( \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) aD_x^\alpha V(x,t) - \left( r - q + \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) \partial_x V(x,t) + r V(x,t),
\]

(2.8)

where \(-\infty \leq a \leq y = \ln S_t < \infty, 1 < \alpha < 2,\) and the left Riemann-Liouville (RL) fractional derivative is defined as

\[
aD_x^\alpha V(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_a^x (x-\xi)^{1-\alpha} V(\xi,t) d\xi.
\]

(2.9)

Here, \(a\) is the lower bound of \(\ln S_t\), which in principle is equal to \(-\infty\), but in practice, \(a\) is usually set to be finite number. Note that when \(\alpha\) approaches 2, Eq. (2.8) reduces to the traditional Black-Scholes model (see Eq. (1.2)), and \(\alpha = 1\) yields an over-specified wave equation that has no solution in general.

### 2.1.2 Derivation of the fundamental solution

The FPDE of the FMLS model for a European put option with its terminal condition can be written as

\[
\partial_t V(x,t) = \left( \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) aD_x^\alpha V(x,t) - \left( r - q + \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) \partial_x V(x,t) + r V(x,t),
\]

(2.10)

\[
V(x,T) = \max(K - e^x, 0) = f(x).
\]

(2.11)

Applying the Fourier transform on both sides of Eq. (2.10) and Eq. (2.11) gives

\[
\partial_t \hat{V}(\xi,t) = \left( \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) (i\xi)^\alpha \hat{V}(\xi,t) - \left( r - q + \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) (i\xi) \hat{V}(\xi,t) + r \hat{V}(\xi,t),
\]

(2.12)

\[
\hat{V}(\xi,T) = \hat{f}(\xi).
\]

(2.13)

Let \(\tau = T - t\), then upon solving Eq. (2.12), we can further write \(\hat{V}(\xi,t)\) as

\[
\hat{V}(\xi,t) = \hat{f}(\xi)e^{-\left[\frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2)(i\xi)^\alpha - (r - q + \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2))(i\xi) + r\right] \tau}.
\]

(2.14)
Taking the inverse Fourier transform to Eq. (2.14), we obtain the fundamental solution of the FPDE, $G(x, t; u, T)$, as below:

\[
V(x, t) = \frac{1}{2\pi} \int_{-\infty+i\xi}^{\infty+i\xi} \mathcal{V}(\xi, t) e^{i\xi x} d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\infty+i\xi}^{\infty+i\xi} \mathcal{V}(\xi, t) e^{i\xi x} \left[ \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) (i\xi)^\alpha - (r-q+\frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2))(i\xi)^\alpha + r \right] d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\infty+i\xi}^{\infty+i\xi} e^{i\xi x} \left[ \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2)(i\xi)^\alpha - (r-q+\frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2))(i\xi)^\alpha + r \right] \mathcal{V}(\xi, t) e^{i\xi x} d\xi
\]

\[
= \int_{-\infty+iu}^{\infty+iu} G(x, t; u, T) f(u) du.
\] (2.15)

**Lemma 2.1.** The value of a European put option that satisfies FMLS model can be evaluated as

\[
V_E(x, t) = \int_{-\infty+i\xi}^{\infty+i\xi} G(x, t; u, T) \max(K - e^u, 0) du,
\] (2.16)

where $u^i \equiv \text{Im} u$, $G(x, t; u, T)$ is the fundamental solution,

\[
G(x, t; u, T) = \frac{1}{2\pi} e^{-rT} \int_{-\infty+i\xi}^{\infty+i\xi} e^{i\xi x - u + (r-q+\frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2))(i\xi)^\alpha - \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2)(i\xi)^\alpha} d\xi.
\] (2.17)

## 2.2 Analytic approximation for American options

In this part, we shall describe the pricing problem of the American put options under FMLS model.

### 2.2.1 Characterization of American option

For American option holders, there is an optimal-exercise policy that maximize the option value. Mathematically, it can be reduced to a free-boundary problem, that is,
to fix the free-boundary $B(t)$ (also known as optimal exercise boundary in finance), which divides the region $\{0 \leq S < \infty, 0 \leq t \leq T\}$ into two parts, one is the continuation region $\Sigma_1 = \{B_t \leq S < \infty\}$, and the other is the stopping region $\Sigma_2 = \{0 \leq S < B_t\}$.

Suppose the option value $V(S,t)$ and delta $\partial V/\partial S$ are continuous for $S \geq 0$, i.e. the high-contract condition. The log-return of the stock follows a FMLS process, and hence the American option value satisfies Eq. (2.8) in the continuation region $\Sigma_1$. In what follows, we shall use the log-return $x = \ln S$, and set $b(t) = \ln B(t)$ for convenience. The option value is still denoted by $V(x,t)$. Denote the operator $\mathcal{L}$ by

$$\mathcal{L}V(x,t) = \partial_t V(x,t) - \left(\frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2)\right)_a D_x^\alpha V(x,t)$$

$$+ \left(\frac{r - q + \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2)}{\partial_x V(x,t)}\right) - rV(x,t),$$

where $a$ is the lower bound of the log-return, which in principle is equal to $-\infty$, but in practice, $a$ is usually set to be a finite number.

Then in the continuation region $\Sigma_1$, we obtain

$$V(x,t) > \max(K - e^x, 0), \quad \text{and} \quad \mathcal{L}V(x,t) = 0. \quad (2.19)$$

Moreover, in the stopping region $\Sigma_2$, since the the optimal-exercise point $B_t < K$, we have

$$V(x,t) = \max(K - e^x, 0) = K - e^x, \quad (2.20)$$

$$\mathcal{L}V(x,t) = -\frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) a D_x^\alpha V(x,t) - \left(\frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) - q\right) e^x - rK$$

$$= \frac{\sigma^\alpha \sec(\alpha \pi/2)}{\Gamma(2 - \alpha)} \left\{ \int_a^x (x - \xi)^{1-\alpha} e^\xi d\xi + (x - a)^{-\alpha} [e^a(x - a + 1 - \alpha)$$

$$- K(1 - \alpha)]\right\} - \left(\frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) - q\right) e^x - rK$$

$$:= -H(x) < 0. \quad (2.21)$$

Takeing both (2.19) and (2.20) into account, we obtain the following inequality for
American options:

\[ \partial_t V(x,t) \leq \left( \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) aD_\alpha^a V(x,t) \]

\[ - \left( r - q + \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) \partial_x V(x,t) + r V(x,t). \]

Then at time \( t \), the value of an American option under FMLS model satisfies the following problem

\[
\begin{align*}
\mathcal{L} V(x, t) &= 0, \quad b(t) < x < \infty, \quad 0 \leq t < T, \\
V(b(t), t) &= K - e^{b(t)}, \quad 0 \leq t \leq T \\
\frac{\partial}{\partial x} V(b(t), t) &= -e^{b(t)}, \quad 0 \leq t \leq T \\
V(x, T) &= \max(K - e^x, 0), \quad -\infty < x < \infty.
\end{align*}
\] (2.22)

### 2.2.2 Derivation of the decomposition formula and the integral equation of the optimal-exercise boundary

In this part, we establish the formula for the decomposition of an American put option under FPDE framework, which is similar to the classical Black-Scholes model.

In the domain \( \Sigma = \Sigma_1 \cup \Sigma_2 = \{-\infty < x < \infty, 0 \leq t \leq T\} \), the price \( V(x, t) \) of an American put option is continuous and has continuous delta (the high-contact condition). Moreover, in the regions \( \Sigma_1 \) and \( \Sigma_2 \), \( V(x, t) \) satisfies Eqs. (2.19) and (2.20) respectively. That is

\[-\mathcal{L} V(x, t) = \begin{cases} 
0, & (x, t) \in \Sigma_1, \\
H(x), & (x, t) \in \Sigma_2.
\end{cases}\]

Here for the FMLS model,

\[
H(x) = -\frac{\sigma^\alpha \sec(\alpha \pi/2)}{\Gamma(2 - \alpha)} \left[ \int^x_a (x - \xi)^{1-\alpha} e^\xi d\xi ight. \\
+ (x - a)^{-\alpha} e^a(x - a + 1 - \alpha) - K(1 - \alpha) \\
+ rK - \left( q - \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) e^x. \] (2.23)

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Below we shall apply the method used by Jiang [48] to derive the optimal-exercise boundary for the American put options. Note that the adjoint operator $L^*$ of $L$ as

\[ L^* \nu(u, \gamma) = \frac{\partial}{\partial \gamma} \nu - \frac{1}{2} \sigma \sec(\alpha \pi / 2) \partial^\alpha \nu \] (2.24)

\[ - (r - q + \frac{1}{2} \sigma \sec(\alpha \pi / 2) \partial^\alpha \nu - r \nu \] (2.25)

\[ = 0, \quad \text{if } \gamma > t, \]

\[ \nu(u, \gamma) = \delta(u - x), \quad \text{if } \gamma = t, \] (2.26)

where $\nu(u, \gamma) = G(x, t; u, \gamma)$.

**Proposition 2.1.** Let $G^*(u, \gamma; x, t)$ denote the fundamental solution of $L^*$, then

\[ G(x, t; u, \gamma) = G^*(u, \gamma; x, t). \] (2.27)

**Proof.** By considering the property of the adjoint operator,

\[ 0 = \int_{-\infty}^{\infty} dy \int_{t+\epsilon}^{\gamma-\epsilon} \left[ G^*(y, z; x, t) L G(y, z; u, \gamma) \right. \]

\[ - G(y, z; u, \gamma) L^* G^*(y, z; x, t) \] \[ \left. dz \right] \]

\[ = \int_{-\infty}^{\infty} dy \int_{t+\epsilon}^{\gamma-\epsilon} \left[ \partial_z (G^* G) - \frac{1}{2} \sigma \sec(\alpha \pi / 2) G^* \frac{\partial^\alpha G}{\partial y^\alpha} G \right. \]

\[ + (-1)^\alpha \frac{1}{2} \sigma \sec(\alpha \pi / 2) G \frac{\partial^\alpha G*}{\partial y^\alpha} G^* \]

\[ + (r - q + \frac{1}{2} \sigma \sec(\alpha \pi / 2) \partial y (G^* G)) \partial y (G^* G) \right] dz. \] (2.28)

Note that $G^* \frac{\partial^\alpha}{\partial y^\alpha} G \to 0$, $G \frac{\partial^\alpha}{\partial y^\alpha} G^* \to 0$ and $\partial y (G^* G) \to 0$, when $y \to \pm \infty$. We have

\[ \int_{-\infty}^{\infty} G^*(y, \gamma - \epsilon; x, t) G(y, \gamma - \epsilon; u, \gamma) dy \]

\[ = \int_{-\infty}^{\infty} G^*(y, t + \epsilon; x, t) G(y, t + \epsilon; u, \gamma) dy. \] (2.29)

Letting $\epsilon \to 0$ in the above equation leads to

\[ \int_{-\infty}^{\infty} G^*(y, \gamma; x, t) \delta(y - u) dy = \int_{-\infty}^{\infty} G(y, t; u, \gamma) \delta(y - x) dy, \] (2.30)

which is equivalent to $G^*(u, \gamma; x, t) = G(x, t; u, \gamma)$.
Now we multiply both sides of \((2.23)\) by \(G^*(u, \gamma; x, t)\), and integrating over the domain \(\{ -\infty \leq a \leq u < \infty, t + \epsilon \leq \gamma \leq T \}\),

\[
\int_{t+\epsilon}^{T} d\gamma \int_{-\infty}^{b(\gamma)} H(u)G^*(u, \gamma; x, t)du = -\int_{t+\epsilon}^{T} d\gamma \int_{-\infty}^{\infty} G^*(u, \gamma; x, t)\mathcal{L}V du
\]

\[
= -\int_{t+\epsilon}^{T} d\gamma \int_{-\infty}^{\infty} \left[ G^*(u, \gamma; x, t)\mathcal{L}V(u, \gamma) - V(u, \gamma)\mathcal{L}G^*(u, \gamma; x, t) \right] du
\]

\[
= -\int_{t+\epsilon}^{T} d\gamma \int_{-\infty}^{\infty} \left[ \partial_{\gamma}(G^* V) - \frac{1}{2}\sigma^2 \sec(\alpha\pi/2)G^* \frac{\partial^\alpha}{\partial u^\alpha} V \\
+(-1)^{\alpha/2}\sigma^2 \sec(\alpha\pi/2)V \frac{\partial^\alpha}{\partial u^\alpha} G^* \\
+(r - q + \frac{1}{2}\sigma^2 \sec(\alpha\pi/2)) \partial_u (G^* V) \right] du.
\]  

(2.32)

Here, we have made use of the fact that \(b(\gamma) = \ln B(\gamma)\) is monotonic. Note that \(G^* \frac{\partial^\alpha}{\partial u^\alpha} V \to 0, V \frac{\partial^\alpha}{\partial u^\alpha} G^* \to 0\) and \(\partial_u (G^* V) \to 0\), as \(u \to \pm \infty\). We have

\[
\int_{-\infty}^{\infty} G^*(u, t + \epsilon; x, t) V(u, t + \epsilon) du = \int_{-\infty}^{\infty} G^*(u, T; x, t) V(u, T) du \\
+ \int_{t+\epsilon}^{T} d\gamma \int_{-\infty}^{b(\gamma)} H(u)G^*(u, \gamma; x, t)du.
\]  

(2.33)

Now, letting \(\epsilon \to 0\) in Eq. \((2.33)\), and recalling Proposition \(2.1\) we get

\[
V(x, t) = \int_{-\infty}^{\infty} G(x, t; u, T) \max(K - e^u, 0) du \\
+ \int_{t}^{T} d\gamma \int_{-\infty}^{b(\gamma)} H(u)G(x, t; u, \gamma)du \\
= V_E(x, t) + e(x, t),
\]  

(2.34)

where \(V_E(x, t)\) is price of the corresponding European put option, and \(e(x, t)\) is the early exercise premium. The above decomposition formula shows that, given the optimal-exercise boundary \(x = b(t)\), the American option price can be determined by Eq. \((2.34)\).

We can also derive an integral equation for the optimal-exercise boundary \(b(t) = \)
\[ V(b(t), t) = K - e^{b(t)}, \] 

(2.35)

we have a Volterra integral equation of the second kind,

\[
\begin{align*}
    b(t) &= \ln \left( K - V_E(b(t), t) - e(b(t), t) \right) \\
    &= \ln \left[ K - \int_{-\infty}^{\infty} G(b(t), t; u, T) \max(K - e^u, 0) du \right. \\
    &\quad - \int_{t}^{T} d\gamma \int_{-\infty}^{b(\gamma)} H(u) G(b(t), t; u, \gamma) du \bigg].
\end{align*}
\] 

(2.36)

We shall apply the collocation method with Newton’s iteration (or secant method) to solve Eq. (2.36) numerically.

### 2.2.3 Asymptotic behavior close to expiry

Despite of the complexity of the integral equation (2.36), we may derive the asymptotic behavior of \( b(t) \) as \( t \to T \).

At the expiry, the option value is given by the payoff, that is \( V(x, T) = K - e^x \).

If the American put option is alive, then its value satisfies Eq. (2.8). By substituting the above put value into Eq. (2.8), given that \((x, t)\) lie in the continuation region, we have

\[ \partial_t V(x, t) \big|_{t=T} = H(x), \] 

(2.37)

where \( H(x) \) is given in (2.23). Moreover, the left-hand side \( \partial_t V(x, T) \) must be non-negative in order that the American put option is kept alive until the time close to expiry. The value of \( x \) at which \( \partial_t V(x, T) \) changes sign can be approximated by numerical methods.

If we rewrite the integral term in the fractional derivative \(-\infty D^\alpha_x e^x\) into an incomplete Gamma function by the change of variables \( z = x - \xi \),

\[
\begin{align*}
    \int_{a}^{x} (x - \xi)^{1-\alpha} e^{\xi} d\xi &= e^x \int_{0}^{x-a} z^{2-\alpha-1} e^{-z} dz \\
    &= e^x \gamma(2 - \alpha, x - a),
\end{align*}
\] 

(2.38)
where the incomplete Gamma function \(\gamma(a, z)\) is defined as \(\gamma(a, z) := \int_0^z t^{a-1}e^{-t}dt\).

It is readily seen that \(\lim_{z \to \infty} \gamma(a, z) = \Gamma(a)\). Indeed, \(\Gamma(a) - \gamma(a, z) = O(e^{-z}z^{a-1})\), when \(z\) goes to \(\infty\).

Thus we can rewrite \(H(x)\) as

\[
H(x) = rK - (q - c)e^x - \frac{\sigma^a \sec(\alpha \pi/2) \Gamma(2 - \alpha)}{\Gamma(2 - \alpha)} [e^x \gamma(2 - \alpha, x - a) \\
+ (x - a)^{-\alpha} [e^a(x-a + 1-\alpha) - K(1 - \alpha)] \\
- \frac{\sigma^a \sec(\alpha \pi/2) \Gamma(2 - \alpha)}{\Gamma(2 - \alpha)} (x-a)^{-\alpha} [K(\alpha - 1) + e^a(x-a + 1-\alpha)] \\
+ \sigma^a \sec(\alpha \pi/2) [1 - \frac{\gamma(2 - \alpha, x - a)}{\Gamma(2 - \alpha)}] e^x + rK - qe^x]
\]

\[= H_1(x) + H_2(x) + H_3(x), \]

where

\[
H_1(x) = -\frac{\sigma^a \sec(\alpha \pi/2)}{\Gamma(2 - \alpha)} (x-a)^{-\alpha} [K(\alpha - 1) + e^a(x-a + 1-\alpha)], \\
H_2(x) = \sigma^a \sec(\alpha \pi/2) [1 - \frac{\gamma(2 - \alpha, x - a)}{\Gamma(2 - \alpha)}] e^x, \\
H_3(x) = rK - qe^x.
\]

Note that \(H_3(x)\) is just the term from the standard Black-Scholes model. Moreover,

\[
H_1(x) \sim -\frac{\sigma^a \sec(\alpha \pi/2)e^a}{\Gamma(2 - \alpha)} (x-a)^{1-\alpha} \to 0, \quad x-a \to \infty, \quad (2.39) \\
H_2(x) \sim O(e^a(x-a)^{1-\alpha}) \to 0, \quad x-a \to \infty. \quad (2.40)
\]

We below provide some typical pricing examples. For the numerical simulation, we shall choose \(a = -100\). The numerical result for \(b(T)\) agrees with that of Black-Scholes model up to 3-digits for \(K \geq 1\). Consider such an American put option evaluation with \(\sigma = 0.25\), \(T = 1\) and the fractional parameter \(\alpha = 1.6\). We shall let \(H(x) = 0\), and use Newton’s iteration method to approximate the value of \(x\) and \(x^*(T)\). The Table 2.1 shows that, no matter \(r \geq q\) or \(r < q\), the value \(x^*(T)\) is very close to \(b(T)\) of Black-Scholes model.

As \(a \to -\infty\), the \(H_2(x)\) is exponentially small, and the \(H_1(x)\) is of order \((x-a)^{-\alpha}\) only. Therefore, the close-expiry behavior of the optimal-exercise boundary is similar.
Table 2.1: Asymptotic behavior of $b(t)$ close to expiry

<table>
<thead>
<tr>
<th>$r \geq q$</th>
<th>$K = 1$</th>
<th>$x = 0.0002$</th>
<th>$x^*(T) = 0$</th>
<th>$b(T) = \ln K = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(r = q = 0.05)$</td>
<td>$K = 50$</td>
<td>$x = 3.912$</td>
<td>$x^*(T) = 3.912$</td>
<td>$b(T) = \ln K = 3.912$</td>
</tr>
<tr>
<td>$r &lt; q$</td>
<td>$K = 1$</td>
<td>$x = -1.608$</td>
<td>$x^*(T) = -1.608$</td>
<td>$b(T) = \ln \frac{rK}{q} = -1.609$</td>
</tr>
<tr>
<td>$(r = 0.01, q = 0.05)$</td>
<td>$K = 50$</td>
<td>$x = 2.304$</td>
<td>$x^*(T) = 2.304$</td>
<td>$b(T) = \ln \frac{rK}{q} = 2.304$</td>
</tr>
</tbody>
</table>

To that of Black-Scholes model. In particular, if $a = -\infty$, $b(T)$ is exactly the same as the Black-Scholes model. We know the optimal-exercise boundary for Black-Scholes model is

$$b(t) \sim \begin{cases} 
\ln K, & r \geq q, \\
\ln K + \ln \frac{r}{q}, & r < q,
\end{cases}$$

as $t \to T$ (see [51]).

### 2.2.4 Finding the American option values using quadratic approximation method

As a matter of fact, analytic price formulas do not exist for most American options, except for a few special cases, such as the American call on an asset with no dividend or discrete dividends and the perpetual American options. In this section, we present a well-known effective analytic approximation method, called quadratic approximation method, for finding the American put option values.

The quadratic approximation method for the Black-Scholes model was first proposed by MacMillan [59] for non-dividend paying asset option and later extended to commodity options by Barone-Adesi and Whaley [8]. Their approximation formula turns out to be quite accurate for both short- and long-maturity pricing problems.

Recall the governing FPDE for the American option in the continuation region
Σ₁ is the same as Eq. (2.8), which has the form as

\[
\partial_t V(x, t) = \left( \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) a D_2^\alpha V(x, t) \\
- \left( r - q + \frac{1}{2} \sigma^\alpha \sec(\alpha \pi/2) \right) \partial_x V(x, t) + rV(x, t),
\]

By the decomposition formula for American put options, we define the early exercise premium as

\[
e(x, t) = V(x, t) - V_E(x, t).
\]

Note that the Eq. (2.41) holds for both \( V(x, t) \) and \( V_E(b(t), t) \) in the continuation region Σ₁. Since the FPDE is linear, the same equation holds for the early exercise premium \( e(x, t) \) as well. We now let \( k_1 = 2r/\sigma^\alpha \sec(\alpha \pi/2) \) and \( k_2 = 2(r - q)/\sigma^\alpha \sec(\alpha \pi/2) \), and assume \( e(x, t) \) takes the form

\[
e(x, t) = X(t)f(x, X(t)),
\]

with \( X(t) \) to be determined.

Now Eq. (2.41) can be transformed into

\[
a D_x^\alpha f - (k_2 + 1) \partial_x f + k_1 f \left[ 1 - \frac{\partial_t X}{rX} \left( 1 + \frac{X}{f} \frac{\partial f}{\partial X} \right) \right] = 0.
\]

A proper choice for \( X = X(t) \) is \( X(t) = 1 - e^{-r(T-t)} \), so that (2.44) becomes

\[
a D_x^\alpha f - (k_2 + 1) \partial_x f + \frac{k_1}{X} \left[ f - (1 - X)X \frac{\partial f}{\partial X} \right] = 0.
\]

Note that the factor \((1 - X)X\) tends to zero as \( \tau = T - t \) approaches zero or infinity. Hence, for short or long maturity, we may drop the last term in Eq. (2.45) so that it reduces to an ordinary differential equation (ODE), which is

\[
a D_x^\alpha f - (k_2 + 1) \partial_x f + \frac{k_1}{X} f = 0.
\]

Here, \( X = X(t) \) is assumed to be nonzero. Now we set \( y = x - a \), so that \( a D_x^\alpha f(x) = a D_y^\alpha f(y + a) \), and denote \( f(x) = f(y + a) =: g(y) \). Then Eq. (2.46) can be rewritten as

\[
a D_y^\alpha g(y) - (k_2 + 1) \partial_y g(y) + \frac{k_1}{X} g(y) = 0.
\]
The last equation can be solved by Laplace transform. According to Podlubny \[74\], the Laplace transform of the Riemann-Liouville fractional derivative of order \(\alpha\) is

\[
L\{0D_y^\alpha g(y); s\} = s^\alpha G(s) - [0D_y^{\alpha-1} g(y)]_{y=0} - s [0D_y^{\alpha-2} g(y)]_{y=0} \tag{2.48}
\]

\[
= s^\alpha G(s) - C,
\]

where

\[
C = [0D_y^{\alpha-1} g(y)]_{y=0} + s [0D_y^{\alpha-2} g(y)]_{y=0} \tag{2.49}
\]

\[
= \left[ \frac{1}{\Gamma(3-\alpha)} \frac{\partial^2}{\partial y^2} \int_0^y (y-\xi)^{2-\alpha} g(\xi) d\xi \right]_{y=0}
+ \left[ \frac{s}{\Gamma(4-\alpha)} \frac{\partial^2}{\partial y^2} \int_0^y (y-\xi)^{3-\alpha} g(\xi) d\xi \right]_{y=0}.
\]

Thus the Laplace transform of Eq. (2.47) can be expressed as

\[
s^\alpha G(s) - C - (k_2 + 1) [sG(s) - g(0)] + \frac{k_1}{X} G(s) = 0, \tag{2.50}
\]

where

\[
g(0) = f(a) = \frac{e(a, t)}{X} = \frac{K - e^a - V_E(a, t)}{X}. \tag{2.51}
\]

It is then straightforward to obtain

\[
G(s) = \frac{CX - (k_2 + 1)(K - e^a - V_E(a, t))}{X s^\alpha - (k_2 + 1)X s + k_1}. \tag{2.52}
\]

Taking the inverse Laplace transform of \(G(s)\) leads to

\[
f(x) = g(y) = L^{-1}\{G(s); y\}
= \int_{c-i\infty}^{c+i\infty} e^{sy} G(s) ds
= \int_{c-i\infty}^{c+i\infty} e^{sy} \left[ \frac{CX - (k_2 + 1)(K - e^a - V_E(a, t))}{X s^\alpha - (k_2 + 1)X s + k_1} \right] ds
= \int_{c-i\infty}^{c+i\infty} e^{sx-a} \left[ \frac{CX - (k_2 + 1)(K - e^a - V_E(a, t))}{X s^\alpha - (k_2 + 1)X s + k_1} \right] ds, \tag{2.53}
\]

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where $c = Re(s) > c_0$, and $c_0$ is an arbitrary point that lies in the right half plane of the absolute convergence of $G(s)$.

So we finally have

$$V(x, t) = V_E(x, t) + e(x, t)$$

$$= V_E(x, t) + \int_{c-i\infty}^{c+i\infty} e^{s(x-a)} \left[ \frac{CX - (k_2 + 1)(K - e^a - V_E(a, t))}{Xs^a - (k_2 + 1)Xs + k_1} \right] ds,$$

where $V_E$ is the corresponding European option value given in Eq. 2.16, $k_1 = 2r/\sigma^a \sec(\alpha \pi/2)$, $k_2 = 2(r - q)/\sigma^a \sec(\alpha \pi/2)$, and $X = 1 - e^{-r(T-t)}$.

### 2.3 Numerical simulations by approximating the fundamental solution

Barrier options are such kind of the options whose payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price. In this section, we will show some numerical results of both European call and put barrier options that follow the finite moment log-stable processes by simulating the integral and the fundamental solution in Eq. (2.16) directly.

Depending on the boundary conditions, a total of three different cases exist: up-and-out options, down-and-out options, and double-knock-out options. The values of a European up-and-out call options with barrier located at $S_{min}$ and $S_{max}$ should be:

$$V(x, t) = \begin{cases} 
0, & \text{for } e^x \geq S_{max}, 0 \leq t < T, \\
\max(e^x - K, 0), & \text{for } 0 \leq e^x < S_{max}, t = T.
\end{cases}$$

Similarly, we can give the values of European down-and-out call options:

$$V(x, t) = \begin{cases} 
0, & \text{for } e^x \leq S_{min}, 0 \leq t < T, \\
\max(e^x - K, 0), & \text{for } S_{min} < e^x, t = T.
\end{cases}$$
Then, by considering the two barriers’ conditions together, we obtain the values of European double-and-out call options below:

\[ V(x, t) = \begin{cases} 
0, & \text{for } e^x \leq S_{\text{min}} \text{ or } e^x \geq S_{\text{max}}, \, 0 \leq t < T, \\
\max(e^x - K, 0), & \text{for } S_{\text{min}} < e^x < S_{\text{max}}, \, t = T.
\end{cases} \]

The values of the corresponding European barrier put options are defined very similar to the call options. We can just use \( \max(K - e^x, 0) \) instead of \( \max(e^x - K, 0) \) in Eqs. (2.54)–(2.54). For example, the value of a European down-and-out put option with barrier located at \( S_{\text{min}} \) is defined as

\[ V(x, t) = \begin{cases} 
0, & \text{for } e^x \leq S_{\text{min}}, \, 0 \leq t < T, \\
\max(K - e^x, 0), & \text{for } S_{\text{min}} < e^x, \, t = T.
\end{cases} \]

For the illustration, we set the strike price \( K = 50 \), the risk-free interest rate \( r = 0.05 \), the continuous dividend yield \( q = 0 \), the volatility \( \sigma = 0.25 \), and fractional order \( \alpha = 1.5 \), and for different maturity time \( T = \{3/12, 2/12, 1/12, 2/52, 1/52, 0\} \). The results for the simulation of the European down-and-out call and put option values are plotted in Figs 2.1 and 2.2 respectively.

Unlike to evaluate the option prices by approximating the fundamental solution and the integrals, below we will introduce an alternative numerical method, namely
Figure 2.2: European down-and-out put option evaluation with lower barrier at $
S_{\text{min}} = 30$

the Gauss-Jacobi spectral method, to solve the FPDE (2.8) to obtain the option values. This method is more efficient than to approximate the fundamental solution especially in lower-dimensional problems.

2.4 Alternative numerical methods and simulation examples

In this section, we use Gauss-Jacobi spectral method to approach the space-fractional derivative in the FPDE (2.8). We shall first introduce the numerical algorithm of the efficient spectral method, then by simulating the pricing of European and American options, we illustrate the flexibility and accuracy of the method by comparing to some previously proposed finite difference schemes.

Our results of the numerical examples indicate that the global character of the proposed method is well-suited to fractional partial integral-differential equations and can naturally engage the global behavior of the solution. When moving from integer-order to fractional-order pricing models, the proposed method is an attractive alternative without extra computational cost.
2.4.1 Overview of Gauss-Jacobi spectral method

It has been noted for decades that, in research, the partial integral-differential equations are introduced to capture the non-locality, which is induced by the jumps in the Lévy process \[87\] \[27\]. The key issue we are going to discuss in this section is to discretize the integral operators involved in the solution of FPDEs.

In the case of approximating the fractional operators, Lynch et al. \[58\] used numerical approximations of the fractional integral appearing in the definition of the Riemann-Liouville derivative; Cartea et al. \[20\] was on the basis of the Grünwald-Letnikov definition of fractional operators; whereas both of them implemented the finite difference numerical methods to approach the space-fractional derivatives, which are mainly based on the local representations of functions by low order polynomials for the most part.

Compared with the finite difference methods, alternative, spectral methods can greatly reduce the spatial dispersion errors, and generally provide a better numerical accuracy. Additionally, the same model can sometimes be simulated with a reduced number of grid points, which further reduces the memory requirement and computational complexity of the application. Consequently, we choose to apply the Gauss-Jacobi spectral method in this subsection to derive the numerical solutions of FPDE (2.8).

Gauss-Jacobi quadrature is a well-known method for numerical quadrature problems. Generally, the \( n \)-point Gauss-Jacobi quadrature rule has the form as

\[
\int_{-1}^{1} G(x)(1-x)^p(1+x)^q dx \approx \sum_{i=1}^{n} \omega_i G(x_i), \tag{2.54}
\]

where \( G \) is a smooth function in \([-1, 1]\), \( \{x_i\}_{i=1}^{n} \) are the roots of Jacobi polynomials of degree \( n \) called Jacobi points, and the weights \( \omega_i \) are giving by

\[
\omega_i = -\frac{2n + p + q + 2}{n + p + q + 1} \frac{\Gamma(n + p + 1)\Gamma(n + q + 1)}{\Gamma(n + p + q + 1)(n + 1)!} \frac{2^{p+q}}{P_{n}^{(p,q)}(x_i)P_{n+1}^{(p,q)}(x_i)}, \tag{2.55}
\]
where $\Gamma(\cdot)$ denotes the Gamma function. This method is a special case of Gaussian quadrature over the interval $[-1, 1]$ and has the weight function $\omega(x) = (1-x)^p(1+x)^q$, where $p, q > -1$. The orthogonal polynomials associated to the weight function $\omega(x)$ consist of Jacobi polynomials. Moreover, the Jacobi polynomials $P_n^{(p,q)}$ are solutions of the equation

$$(1 - x^2)y'' + (q - p - (p + q + 2)x)y' + n(n + p + q + 1)y = 0, \quad (2.56)$$

which has explicit expressions and orthogonals.

Note that the interval $[-1, 1]$ can be replaced by any other interval by a linear transformation. Thus, Gauss-Jacobi quadrature can be used to approximate integrals with singularities at the end points. Chebyshev-Gauss quadrature is a special case of Gauss-Jacobi quadrature with the parameter $p = q = \pm \frac{1}{2}$. Similarly, Gauss-Legendre quadrature arises when $p = q = 0$. More generally, the special case of $p = q$ turns Jacobi polynomials into Gegenbauer polynomials, and the technique is thus called the Gauss-Gegenbauer quadrature.

We assume the underlying function $V(x, t)$ is sufficiently smooth so that it allows us to reformulate the fractional problem by using the Grünwald-Letnikov definitions. Furthermore, we regard the function $V(x, t)$ as two-times continuously differentiable, in the interval $[-1, 1]$, see Mainardi et al. [65]. Thus, the Riemann-Liouville fractional derivative coincides with the Grünwald-Letnikov derivative

$$aD_x^\alpha V(x, t) = \frac{V(a, t)(x-a)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{V'(a, t)(x-a)^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$+ \frac{1}{\Gamma(2-\alpha)} \int_a^x (x - \xi)^{1-\alpha} V''(\xi, t) d\xi. \quad (2.57)$$

where $-\infty < x = \ln S_t < \infty$, $1 < \alpha \leq 2$, $a$ is the lower bound of $x$, which in principle is equal to $-\infty$, but in practice, $a$ is usually set to be a finite number.

According to the work of Ling and Yamamoto [54], we implement a simple transformation which maps the integration limits to the interval $[-1, 1]$

$$\xi(\eta) = \frac{x - a}{2} - \frac{1 + \eta}{2} x + \frac{1 - \eta}{2} a, \quad \eta \in [-1, 1], \quad (2.58)$$
in which the variable \( x \) can be considered as fixed.

Hence the integral in Eq. \( (2.57) \) becomes

\[
\frac{1}{\Gamma(2-\alpha)} \int_a^x (x-\xi)^{1-\alpha} \frac{\partial^2 V(\xi,t)}{\partial \xi^2} d\xi
\]

\[
= \frac{1}{\Gamma(2-\alpha)} \left( \frac{x-a}{2} \right)^{2-\alpha} \int_{-1}^1 (1-\tau)^{1-\alpha} \frac{\partial^2 V(\xi(\eta),t)}{\partial \eta^2} d\eta.
\]

Applying the Gauss-Jacobi quadrature formula on the \( n \) samples of \( \frac{\partial^2 V(\xi(\eta),t)}{\partial \eta^2} \) at the Jacobi points \( \{\eta_k\}_{k=1}^n \), the integral can be approximated by

\[
\int_{-1}^1 (1-\tau)^{1-\alpha} \frac{\partial^2 V(\xi(\eta),t)}{\partial \xi^2} d\eta \approx \sum_{k=1}^n \left( \frac{\partial^2 V(\xi(\eta_k),t)}{\partial \eta^2} \omega_k \right).
\]

To carry forward, we eliminate the derivative terms by approximating \( V \) by

\[
V(x,t) = \sum_{i=1}^n V(x_i,t)L_i(x),
\]

with \( L_i(x) \) being a cardinal basis function associated with the same Jacobi points \( \{\eta_i\}_{i=1}^n \) as above, and

\[
L_i(x) = \prod_{j=1,j\neq i}^n \left( \frac{x-\xi(\eta_j)}{x_i-\xi(\eta_j)} \right).
\]

Therefore, we obtain a semi-discretized form of the Riemann-Liouville derivative

\[
a D_2^\alpha V(x,t) = \frac{V(a,t)(x-a)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^n V(x_i,t) \left( \frac{\partial L_i(\xi(-1))}{\partial \eta} \right)
\]

\[
+ \frac{1}{\Gamma(2-\alpha)} \left( \frac{x-a}{2} \right)^{2-\alpha} \sum_{k=1}^n \left( \sum_{i=1}^n V(x_i,t) \left( \frac{\partial^2 L_i(\xi(\eta_k))}{\partial \eta^2} \right) \omega_k \right).
\]

Lastly, we discrete the other two terms in the Eq. \( (2.8) \), as in the standard Gauss-Jacobi spectral method for integer-order PDE, to obtain the final algebraic system. For the first order derivative in the space, \( \partial_x V(x,t) \), it was shown in [43] that the generalized Jacobi polynomials satisfy the derivative recurrence relation

\[
\partial_x J_n^{k,l}(x) = C_n^{k,l} J_{n-1}^{k,l}(x),
\]

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with $k, l \in \mathbb{Z}$, and we use the classical 4th order Runge-Kutta (RK4) method [24] for time discretization to ensure good accuracy in time.

$$C_{n}^{k,l} = \begin{cases} 
-2(n+k+l+1), & \text{if } k,l \leq -1, \\
-n, & \text{if } k \leq -1, \ l > -1, \\
-n, & \text{if } l \leq -1, \ k > -1, \\
1/2(n+k+l+1), & \text{if } k,l > -1.
\end{cases}$$

Hence, in this way, we can write the first order derivative term as $\partial_{x}V(x,t) = \hat{C}V(x,t)$ with the coefficients matrix $\hat{C}$, and the Eq. (2.8) can be modified as the following form into the bargain:

$$\frac{\partial V(x,t)}{\partial x} = \hat{C}V(x,t), \quad (2.64)$$

It is worthy to mention that, although this iterative method reaches fourth-order accuracy, it may require only three levels of storage in total.

Generally, spectral methods will indeed improve the accuracy towards the low-order finite difference method, while it also has a strict requirement of smoothness for the target functions at each collocation point. Readers might have already noticed that the initial conditions of our target options were not smooth. Therefore, we hereby intend to adopt some simple approximations, in order to polish the initial values of the pricing problems.

For further parameter analysis, let us take a typical barrier option, the up-and-out European call option, for example. The initial values for up-and-out call options at the expiry date should be $\max(e^x - K, 0)$, which equals

$$\frac{1}{2}(e^x - K) + \frac{1}{2}|e^x - K|. \quad (2.65)$$

We consider applying such a transformation to approach the absolute value function $|e^x - K|$ as

$$|e^x - K| = (e^x - K) \times \text{sgn}(e^x - K) \approx (e^x - K) \times \tanh \frac{e^x - K}{\varepsilon}, \quad (2.66)$$
Table 2.2: Difference between the values of European options with smooth or non-smooth initial conditions

<table>
<thead>
<tr>
<th>Stock Price at $t = 0$</th>
<th>Spectral Method with non-smooth initial data</th>
<th>N=14</th>
<th>N=28</th>
<th>N=42</th>
<th>N=56</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>M=40</td>
<td>M=160</td>
<td>M=360</td>
<td>M=640</td>
</tr>
<tr>
<td>ε = $\frac{1}{64}$</td>
<td>0.0082879</td>
<td>0.0223676</td>
<td>0.0067097</td>
<td>0.0011704</td>
<td>0.0009325</td>
</tr>
<tr>
<td>ε = $\frac{1}{168}$</td>
<td>0.0093077</td>
<td>0.0282339</td>
<td>0.0079551</td>
<td>0.0027655</td>
<td>0.0012946</td>
</tr>
<tr>
<td>ε = $\frac{1}{232}$</td>
<td>0.0125763</td>
<td>0.0375082</td>
<td>0.0099851</td>
<td>0.0054281</td>
<td>0.0023173</td>
</tr>
<tr>
<td>ε = $\frac{1}{336}$</td>
<td>0.0165686</td>
<td>0.0520743</td>
<td>0.0129736</td>
<td>0.0066698</td>
<td>0.0029492</td>
</tr>
<tr>
<td>ε = $\frac{1}{560}$</td>
<td>0.0219713</td>
<td>0.0697516</td>
<td>0.0169171</td>
<td>0.0056600</td>
<td>0.0031434</td>
</tr>
<tr>
<td>ε = $\frac{1}{1080}$</td>
<td>0.0282532</td>
<td>0.0899806</td>
<td>0.0213010</td>
<td>0.0081604</td>
<td>0.0048370</td>
</tr>
</tbody>
</table>

where the parameter $\varepsilon$ should be positive and related to the parameter $N$ [24]. Note that, in order to ensure the accuracy, the number of time-steps, $M$, should also depend on the size of $N$. Thus, is can be observed that the approximation error

$$\left| e^x - K \right| - (e^x - K) \tanh \frac{e^x - K}{\varepsilon}$$

(2.67)

tends downward along with the increasing of $N$, whereas the error would be different slightly for distinct values of $\varepsilon$.

The smooth of singularity points could improve the accuracy of the method to some extent. In a rather specific expression, it greatly reduces the memory requirements, using less than 100 units against more than 1000 units, and together with a higher accuracy from $O(N^{-1})$ to $O(e^{-cN})$ for some $c > 0$ [24]. It is worth to mention that, we only consider to improve the initial values with jumps, so that significant results could be obtained. Below we set $K = 50$, $\alpha = 1.5$, $r = 0.05$, $q = 0$, $\sigma = 0.25$, $T = \frac{1}{12}$, and $S < 50$. Let $\varepsilon = \frac{1}{6N}$ and $M \approx T \times N^2$, and test the errors of $\left| \left| e^x - K \right| - (e^x - K) \tanh \frac{e^x - K}{\varepsilon} \right|$, under different initial conditions.

Table 2.2 reflects the changing of difference between the values of European up-and-out call options with smooth or non-smooth initial conditions, corresponding to
the parameter $N$ equals 14, 28, 42, 56, and $M$ equals 40, 160, 360, 640 respectively. Clearly, if we compare the numbers shown in the last 4 columns of the above Table 2.2, the effect of decreasing of the difference can be observed.

### 2.4.2 Numerical examples

In this section, we will show some numerical results of both European and American options that follow the finite moment log-stable processes by using the Gauss-Jacobi spectral method. When smoothing (2.66) is applied, we use fixed parameter $\varepsilon = \frac{1}{6N}$ and let $M \sim N^2T$ for all simulations throughout the section.

To be specific, we use the notation $V_{model\ method}^{model\ method}$ to denote the numerical approximation of solving model, either the traditional Black-Scholes model $BS$ as in Eq. (1.2) or the FMLS model of the fractional order $\alpha$ as in Eq. (2.8), by method, either finite difference method $FD(N)$ or Gauss-Jacobi $GJ(N)$ method with $N$ points.

**Example 2.4.2.1: Comparing results with FD scheme**

Cartea and del Castillo-Negrete [20] applied the first-order finite difference scheme to solve the FPDE (2.8). The results they obtained were positive, but the truncation errors seemed to be relatively significant. Compared with $FD$ method, a main advantage of spectral method is that, the latter requires much fewer unknowns, which will significantly save the storage and CPU time. For example, a rule of thumb, which was introduced by Gottlieb and Orszag in 1981 [42], is that to achieve an engineering precision of 1%, spectral method only needs $\pi$ points per wave-length, as opposed to roughly ten points per wave-length required by a low-order method. Another important feature of spectral methods is that the derivatives of discrete functions are usually computed exactly [3]. Therefore, spectral methods are usually free of phase errors, which can be very problematic for long-time integrations of partial differential equations [24].
Figure 2.3: Infinity norms between European up-and-out call options values obtained by $FD(N = 1500)$ and $GJ(N)$ against various $N$.

In the following example, we shall verify European barrier call options with parameters used in [20]. The starting price of the stock at time $t = 0$, $S_0 = 50$, and the other parameters hold the same as in Section 2.3 that is: let $K = 50$, $\alpha = 1.5$, $r = 0.05$, $q = 0$, and $\sigma = 0.25$.

To compare simulated results of $FD$ and $GJ$, we use relatively large $N = 1500$ in $FD$ to ensure small errors, and the option values obtained by $FD(N = 750)$ and $FD(N = 1500)$ are less than $10^{-4}$. Fig. 2.3 demonstrates the infinite-norm differences between $t \in [0, 1/52]$ of European up-and-out call option values

$$\| V_{FD}^{FMLS(\alpha=1.5)} - V_{GJ(N)}^{FMLS(\alpha=1.5)} \|_{\infty}$$

for various $N$ used in $GJ$.

It is observed that $V_{GJ(N)}^{FMLS(\alpha=1.5)}$ rapidly converges but stays away from the $FD$ solution by a distance of $10^{-2}$. This suggests that the $GJ$ solutions converge but not towards the $FD$ solution.

To better compare the two methods, we turn our analysis to a theoretical fact: $FMLS(\alpha) \to BS$ as $\alpha \nearrow 2$. Although the numerical procedure for solving $BS$ and
Figure 2.4: Difference of European up-and-out call option values by $BS$ and $FMLS(\alpha = 2)$, verse various stock prices.

Figure 2.5: Various European barrier call option evaluations with lower barrier at $S_{\text{min}} = 30$ and upper barrier at $S_{\text{max}} = 83$ by $GJ$. 

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FMLS are different, one should expect

\[ V_{BS}^{method} \approx V_{method}^{FMLS(\alpha=2)} \]

if the employed numerical method is consistent and accurate. From Fig. 2.4, we can see that GJ outperforms FD in such a sense, although both methods do not yield perfect agreement on results of two models. Here all of the figures develop the values for different maturity time \( T \), \( T = \{3/12, 2/12, 1/12, 2/52, 1/52, 0\} \), which should be corresponding with 3 months, 2 months, 1 month, 2 weeks, 1 week and at the expiry date. The GJ is much more higher consistent in approximated option values around the strike price \( K = 50 \), whereas the FD yields significant differences for a wide range of stock prices. As we generally agree that FD would solve BS correctly, this leaves us to doubt the accuracy of FD when solving FMLS.

In the end of this example, we will price all three-typed European barrier call options by GJ with \( N = 30 \), \( M = 360 \) respectively. These figures match nicely with the ones reported in [20] due to the relatively small difference relative to option price. But as shown above, they are not identical to each other and such difference in option prices could have significant implications in real applications. Moreover, the down-and-out call option values are very close to what we obtained by the method of approximating the fundamental solution the the integral. The 2-D figure for the European down-and-out call option values in Fig. 2.5 is almost the same as the Fig. 2.4 in Section 2.3.

Example 2.4.2.2: Effects of \( \alpha \) in FPDE (2.8)

In this example, we attempt to study the behaviour of \( V^{FMLS(\alpha)} \) with different \( 1 < \alpha < 2 \) by studying the pricing problem of the European up-and-out call options. The maturity date \( T \) changes from \( \frac{1}{48} \) to \( \frac{3}{4} \) respectively. Figure 2.6 provides a color scale of

\[ \| V_{GJ}^{FMLS(\alpha)} - V_{GJ}^{BS} \|_{\infty} \]
Figure 2.6: The behaviour of $V^{FMLS}(\alpha)$ with varying $\alpha$ and expiry dates $T$. (a) the infinity norms of $V^{FMLS}(\alpha) - V^{BS}$, (b) a more detailed perspective for $T$ between 0 and 0.25, (from top to bottom: $T = 3/12, 2/12, 1/12, 2/52, 1/52$, and 0 respectively.)

to give a general impression of the effects of parameters $\alpha$ and $T$.

The top part of Fig. 2.6(a) coherent with our observation in Fig. 2.4(a) that the $GJ$ method yields is consistent with results for $BS$ and $FMLS(\alpha \nearrow 2)$. Moreover, the effect of $\alpha$ on option pricing are more significant when $\alpha$ is away from 2 and for longer expiry date $T$. It is shown that the maximum difference for $V^{FMLS}(\alpha=1.5)$ and $V^{FMLS}(\alpha=2)$ fluctuates between $-0.6$ and $0.4$ in general. For a better visuality of the difference between the two models, we provide a detailed version of the option values for $S$ between 30 and 83 and $T$ from 0 to 0.25 in Fig. 2.6(b).
Example 2.4.2.3: Time-varying interest rate $r$

Generally speaking, the GJ approach is more flexible for pricing options. In this example, we concentrate on the pricing problem for a European double-and-out call option, and we will demonstrate some results by changing the constant interest rate $r$ in FPDE (2.8) to a random variable. We consider a 5-month European double-and-out call option with the fractional term $\alpha = 1.5$ and the other conditions remain unchanged as in Example 2.4.2.1. For the stochastic interest rate, we assume that:

$$
\begin{align*}
  r_1 &= f(r_f), \\
  r_n &= f(r_{n-1}), \quad \text{for } n = 2, 3, \ldots, M,
\end{align*}
$$

where $r_f$ is risk-free interest rate value. We choose $r_f = 0.05$, $f(r) = r(1 + \epsilon\%)$, and assume the random numbers $\epsilon$ follow the normal distribution with $\mu = 0$ and $\sigma = 1$.

Fig. 2.7(a) shows all ten random $r$, ranging from 0.01 to 0.09, used in this simulation. The resulting option values are shown in Fig. 2.7(b). Despite the stochastic nature of $r$, the option values remain smooth over time. To better visualize the effect of these random $r$, Fig. 2.7(c) shows their differences to that of the constant risk-free interest rate. Note that there are three specific stock prices that are independent of $r$, namely 38, 66, and 83. Our numerical evidence by modifying the mean and standard deviation of the interest rate suggests that there will always exist three intersections in Fig. 2.7(c) at different stock prices. This interesting fact tells us that such intersections, especially the middle one, might be independent of interest rate movement, but affected by some other factors.

Example 2.4.2.4: American put options evaluation

In the following example, we consider such an American put option evaluation with strike price $K = 100$, risk-free interest rate $r = 0.15$, $q = 0.05$, $\sigma = 0.30$, and $T = 1$ as given by Wu and Kwok [99]. Let the stock prices belong to the interval $[1, e^6]$. In the following computations, we try to compare the results obtained by the binomial
(a) Different $r$ values

(b) European double-and-out call option evaluation

(c) Difference in option values for random and constant $r$

Figure 2.7: Differences of European double-and-out call options values with random $r$. (a) stochastic interest rates used, (b) plots of resulting option values, (c) difference between option prices of random and constant $r$. 
Table 2.3: American put options evaluation for FMLS model

<table>
<thead>
<tr>
<th>Stock Price S</th>
<th>80</th>
<th>85</th>
<th>90</th>
<th>95</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{GJ(\alpha=2)}$</td>
<td>20.1642</td>
<td>16.2664</td>
<td>12.8386</td>
<td>10.4143</td>
<td>8.4106</td>
</tr>
<tr>
<td>$</td>
<td></td>
<td>V_{GJ(\alpha=2)} - V_{BS}</td>
<td></td>
<td>_\infty$</td>
<td>0.1047</td>
</tr>
</tbody>
</table>

It can be observed clearly from the above table that the difference between the GJ method and the classical binomial option price method are controlled within an allowed limit: about less than 1% in general, and with a minimum difference of about $10^{-2}$ level at $S$ equals 95. Additionally, although Fig 2.8 demonstrates a significant tendency for the differences between option values with different $\alpha_s$, the largest difference should be less than 3%. That basically illustrated that the GJ method can be concluded as a feasible approach to evaluate the prices of American put options for similar issues, which are complicated for the FD method to handle; and the $\alpha$ value plays a negligible role in American put option evaluations.

However, we also have to point out that currently a wide range of American options are still difficult to price, subject to the decision to choose the relevant boundary and initial conditions. In some cases, the smooth pasting condition fails, and the unknown optimal exercise boundary is not trivial as well.
Figure 2.8: Infinity norms for $V_{GJ}$ between $\alpha = 2$ and $\alpha = 1.5$, 1.7, and 1.9 for American put options
Chapter 3

Carr-Geman-Madan-Yor-eta (CGMYe) model

The presentation of this section will be organized in the following three subsections

(1) Construction of the FPDE for European put option values for CGMYe models. In addition, the fundamental solution and some relative properties for this FPDE will be discussed;

(2) We concentrate on the pricing problem of the American put options in the framework of CGMYe model in this subsection. We show that the American option price has an integral representation, and hence the optimal-exercise boundary satisfies an nonlinear integral equation. Then, the near-expiry behavior of the optimal-exercise boundary is derived. An analytical approximation formula is also obtained for the option value, which is valid for both short and long maturities.

(3) Use 1-D maximum likelihood estimation (MLE) method to estimate the parameters in CGMYe model, and evaluate the European option values and the optimal exercise boundary of the American options based on the estimated parameters.
3.1 Framework of European option pricing problems

3.1.1 The characteristic function and the FPDE

3.1.1.1 CGMY process

The CGMYe process can be regarded as a combination of a pure CGMY process and a Brownian motion.

As one part of the CGMYe process, The CGMY process, which was introduced by Carr et. al. [18] to model the stock prices and to capture the heavy-tailed behavior, is a Lévy process on \( \mathbb{R} \) with no Gaussian component. The Lévy density of the CGMY process is thus given by

\[
\omega_{\text{CGMY}}(x) = C \left[ \exp(-|x|) \mathbf{I}_{x<0} + \frac{\exp(-Mx)}{|x|^{1+Y}} \mathbf{I}_{x>0} \right],
\]

with \( C > 0, \ G \geq 0, \ M \geq 0, \) and \( Y < 2. \) From the Lévy-Khintchine formula (2.1), the characteristic exponent of the CGMY process at \( t = 1 \) is given by

\[
\Psi_{\text{CGMY}}(\xi) = C \Gamma(-Y) \left[ (G + i\xi)^Y - G^Y + (M - i\xi)^Y - M^Y \right]
\]

for \( Y \neq 0, 1, \) where \( \Gamma(\cdot) \) is the gamma function, see Carr et. al. [18]. Note that \( \Gamma(-Y) \) has poles at \( Y = 0, 1, \) in which case the characteristic exponent of the CGMY process can be obtained by sending \( Y \) to 0 or 1, which are given by:

\[
\Psi_{\text{CGMY}}(\xi) = \begin{cases} 
C \left[ (G + i\xi) \ln(G + i\xi) - G \ln G \\
+ (M - i\xi) \ln(M - i\xi) - M \ln M \right], & Y = 1, \quad (3.3) \\
- C \left[ \ln(G + i\xi) - \ln G + \ln(M - i\xi) - \ln M \right], & Y = 0.
\end{cases}
\]

By adopting the “risk neutral” measure \( Q, \) an European option with the final payoff \( \Pi(X_T, T) \) has the value

\[
V(x, t) = e^{-r(T-t)} \mathbb{E}^Q[\Pi(X_T, T)].
\]

(3.4)
This expectation can be evaluated through the complex Fourier transform (see Lewis [53]). Using this technique, Cartea and del Castillo-Negrete [20] obtain the following fractional partial differential equation (FPDE) for the CGMY processes

\[
\frac{\partial}{\partial t}V(x,t) = -\frac{\eta^2}{2} \frac{\partial^2 V(x,t)}{\partial x^2} - C\Gamma(-Y) \left[ e^{-Gx} \int_{-\infty}^{x} (e^{Gx} e^{Gx} V(x,t)) + e^{Mx} \int_{x}^{\infty} (e^{-Mx} V(x,t)) \right] - (r - q - \upsilon) \frac{\partial V(x,t)}{\partial x} + \left[ r + C\Gamma(-Y)(M^Y + G^Y) \right] V(x,t),
\]

where \(-\infty < x = \ln S_t < \infty, C > 0, G \geq 0, M \geq 0, Y < 2, and the convexity adjustment

\[
u_{CGMY} = C\Gamma(-Y)[(G + 1)^Y - G^Y + (M - 1)^Y - M^Y]. \tag{3.6}\]

The operators \(-\infty D_x^Y\) and \(\int D_x^Y\) are the left and right Riemann-Liouville (RL) fractional derivatives defined as

\[
-\infty D_x^Y f(x) = \frac{1}{\Gamma(n-Y)} \int_{-\infty}^{x} (x-y)^{n-Y-1} f(y) dy \approx \frac{1}{\Gamma(n-Y)} \int_{-\infty}^{x} (x-y)^{n-Y-1} f^{(n)}(y) dy,
\]

\[
\int D_x^Y f(x) = \frac{1}{\Gamma(n-Y)} \int_{x}^{\infty} (y-x)^{n-Y-1} f(y) dy \approx \frac{1}{\Gamma(n-Y)} \int_{x}^{\infty} (y-x)^{n-Y-1} f^{(n)}(y) dy. \tag{3.7}\]

where \(n = \lfloor Y \rfloor + 1\) is the nearest integer larger than \(Y\).

### 3.1.1.2 Extend to CGMYe process model

As an extension of a CGMY model, the log-return of a CGMYe model, \(\ln S_t\), can be regarded as a sum of a Brownian motion and a CGMY process. The characteristic exponent of the log-return relative in the CGMYe process is

\[
\Psi_{\tilde{X}_t}(\xi) = i(r - q - \upsilon)\xi - \frac{\eta^2}{2} \xi^2 + \Psi_{CGMY}(\xi), \tag{3.8}\]

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where \( r - q \) is the mean rate of return on the stock, and the convexity adjustment \( \nu \) has the form

\[
\nu = \Psi_{CGMY}(-i) + \frac{\eta^2}{2} = \nu_{CGMY} + \frac{\eta^2}{2}
\]

\[
= C \Gamma(-Y) [(G + 1)^Y - G^Y + (M - 1)^Y - M^Y] + \frac{\eta^2}{2}.
\]  

(3.9)

Next, applying the same technique used by Cartea and del Castillo-Negrete in [20], we obtain the following FPDE for the 1-D CGMYe model as

\[
\partial_t V(x, t) = -\frac{\eta^2}{2} \frac{\partial^2 V(x, t)}{\partial x^2} - C \Gamma(-Y) \left[ e^{-Gx} \int_{-\infty}^{\infty} D_y^Y \left( e^{Gx} V(x, t) \right) dy + e^{Mx} \int_{-\infty}^{\infty} D_y^Y \left( e^{-Mx} V(x, t) \right) dy \right] - (r - q - \nu) \frac{\partial V(x, t)}{\partial x} + \left[ r + C \Gamma(-Y) (M^Y + G^Y) \right] V(x, t),
\]

with \( \eta \geq 0 \), and the other parameters, i.e. \( x, C, G, M, Y \), are the same as the CGMY model.

Notice that FPDE (3.10) has a slight difference with FPDE (3.5), since we added a Brownian motion part into the model. In the real cases, the Brownian motion part is usually nonzero, and hence the CGMYe model and its governing equation (3.10) obtained above are more appropriate than the only CGMY model that considered by Cartea and del Castillo-Negrete in [20]. And when \( \eta = 0 \), Eq. (3.10) just simplifies to the Eq. (25) in [20]. We should also mention that Cont and Tankov [26] derived the partial integro-differential equation (PIDE) for the option value, and the FPDE (3.10) is essentially equivalent to that PIDE. The FPDE (3.10) is more convenient for the quantitative and qualitative study of the models.

### 3.1.2 Derivation of the fundamental solution

Below we shall give the fundamental solution of FPDE (3.10). The complex Fourier transform of the final payoff \( \Pi(X_T, T) \) is

\[
\hat{\Pi}(\xi, T) = \int_{-\infty - i\xi^1}^{\infty + i\xi^1} e^{-i\xi u} \Pi(u, T) du,
\]

(3.11)

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with $\xi^i \equiv \text{Im} \xi$, and the inverse Fourier transform gives

$$V(x, t) = \frac{e^{-r(T-t)}}{2\pi} \text{E}^Q \left[ \int_{-\infty+iu^i}^{\infty+iu^i} e^{i\xi x_T} \hat{\Pi}(\xi, T) d\xi \right]$$

$$= \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty+iu^i}^{\infty+iu^i} \text{E}^Q \left[ e^{i\xi x_T} \right] \hat{\Pi}(\xi, T) d\xi$$

$$= \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty+iu^i}^{\infty+iu^i} e^{i\xi x + (T-t)\Psi_{\tilde{X}_t}(\xi)} \hat{\Pi}(\xi, T) d\xi, \quad (3.12)$$

where $\Psi_{\tilde{X}_t}(\xi)$ is the characteristic exponent of the CGMYe process $\tilde{X}_t$, which is given in Eq. (3.8), see Lewis [53]. It follows from Eq. (3.12) that

$$\frac{\partial \hat{V}(\xi, t)}{\partial t} = \left( r - \Psi_{\tilde{X}_t}(\xi) \right) \hat{V}(\xi, t), \quad (3.13)$$

with the solution given by

$$\hat{V}(\xi, t) = e^{\left[-r+\Psi_{\tilde{X}_t}(\xi)\right](T-t)} \hat{\Pi}(\xi, T). \quad (3.14)$$

Taking the inverse Fourier transform to Eq. (3.14), we obtain the fundamental solution $F(x, t; u, T)$ of the FPDE as follows:

$$V(x, t) = \frac{1}{2\pi} \int_{-\infty+iu^i}^{\infty+iu^i} \hat{V}(\xi, t)e^{i\xi x} d\xi$$

$$= \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty+iu^i}^{\infty+iu^i} e^{i\xi x + (T-t)\Psi_{\tilde{X}_t}(\xi)} \left[ \int_{-\infty+iu^i}^{\infty+iu^i} \frac{1}{2\pi} e^{i\xi (x-u)} \Pi(u, T) du \right] d\xi$$

$$= \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty+iu^i}^{\infty+iu^i} \left[ \int_{-\infty+iu^i}^{\infty+iu^i} \frac{1}{2\pi} e^{i\xi (x-u)} \Pi(u, T) du \right] d\xi$$

$$:= \int_{-\infty+iu^i}^{\infty+iu^i} F(x, t; u, T) \Pi(u, T) du. \quad (3.15)$$

**Lemma 3.1.** The value of a European put option that satisfies the FPDE (3.10) of CGMYe model is given by

$$V_E(x, t) = \int_{-\infty+iu^i}^{\infty+iu^i} F(x, t; u, T) \max(K - e^u, 0) du, \quad (3.16)$$

where $u^i \equiv \text{Im} u$, $F(x, t; u, T)$ is the fundamental solution,

$$F(x, t; u, T) = \frac{1}{2\pi} e^{-r(T-t)} \int_{-\infty+iu^i}^{\infty+iu^i} e^{i\xi (x-u) + (T-t)\Psi_{\tilde{X}_t}(\xi)} d\xi. \quad (3.17)$$
3.2 Analytic approximation for American options

In this part, we shall describe the pricing problem of the American put options in the framework of the CGMYe model.

3.2.1 Decomposition formula and the integral equation for the optimal-exercise boundary

3.2.1.1 Fractional inequality of American option values

As we mentioned in the first chapter, the American option pricing problem can be regarded as a free boundary problem, that is, to fix the free-boundary $B(t)$ (also known as the optimal-exercise boundary), which divides the region $\{0 \leq S < \infty, 0 \leq t \leq T\}$ into two parts, one is the continuation region $\Sigma_1 = \{B(t) < S < \infty\}$, and the other is the stopping region $\Sigma_2 = \{0 \leq S \leq B(t)\}$.

Suppose the option value $V(S,t)$ and delta $\partial V/\partial S$ are continuous for $S \geq 0$, i.e. the high-contract condition, the log-return of the stock price follows a CGMYe process, and hence follows the FPDE (3.10) in the continuation region $\Sigma_1$. In what follows, we shall use the log-return $x = \ln S$, and set $b(t) = \ln B(t)$ for convenience. The option value is still denoted by $V(x,t)$. Denote the operator $\mathcal{L}$ by

$$
\mathcal{L}V(x,t) = \partial_t V(x,t) + \frac{\eta^2}{2} \frac{\partial^2 V(x,t)}{\partial x^2} + CT(-Y) \left[ e^{-Gx} \left( e^{Gx} V(x,t) \right) + e^{Mx} D^{-Y} Y \left( e^{Mx} V(x,t) \right) \right] \\
+ (r - q - \nu) \frac{\partial V(x,t)}{\partial x} - \left[ r + CT(-Y) (G^Y + M^Y) \right] V(x,t).
$$

In $\Sigma_1$, the option value is greater than the expiration revenue,

$$
V(x,t) > \max(K - e^x, 0), \quad \text{and} \quad \mathcal{L}V(x,t) = 0.
$$
Moreover, in $\Sigma_2$, since the the optimal-exercise point $B(t) < K$, we have

\[ V(x, t) = \max(K - e^x, 0) = K - e^x, \]

\[ \mathcal{L}V(x, t) = CT(-Y)\left[e^{-Gx}\int_{-\infty}^{x} D_x^Y(e^{Gx}V(x, t)) + e^{Mx}\int_{x}^{\infty} D_x^Y(e^{-Mx}V(x, t))\right] \]

\[ + CT(-Y)((G + 1)^Y + (M - 1)^Y) - \eta^2 + q \right] e^x \]

\[ - \left[ r + CT(-Y)(G^Y + M^Y)\right] K \]

\[ = e^x q - Kr := -H(x) < 0, \]

Here, we have made use of the fact that

\[ -\infty D_x^Y e^{\lambda x} = \lambda^Y e^{\lambda x}, \quad \text{and} \quad x D_\infty^Y e^{\lambda x} = (-\lambda)^Y e^{\lambda x}. \quad (3.18) \]

Combining the cases for both continuation region and stopping region, we obtain the following inequality for American options:

\[ \left[ r + CT(-Y)(G^Y + M^Y)\right] V(x, t) \]

\[ \geq \partial_t V(x, t) + \eta^2 \frac{\partial^2 V(x, t)}{\partial x^2} + (r - q - v) \frac{\partial V(x, t)}{\partial x} \]

\[ + CT(-Y)\left[e^{-Gx}\int_{-\infty}^{x} D_x^Y(e^{Gx}V(x, t)) + e^{Mx}\int_{x}^{\infty} D_x^Y(e^{-Mx}V(x, t))\right]. \]

Notice that Cartea and del Castillo-Negrete [20] give a similar result for the geometric KoBoL process model, and the above inequality is a generalization of Eq. (42) in [20] to the CGMYe model.

To summarise, given time $t$, the value of an American option under CGMYe model satisfies the following problem

\[
\begin{cases}
\mathcal{L}V(x, t) = 0, & b(t) < x < \infty, \quad 0 \leq t < T, \\
V(b(t), t) = K - e^{b(t)}, & 0 \leq t \leq T \\
\frac{\partial}{\partial x} V(b(t), t) = -e^{b(t)}, & 0 \leq t \leq T \\
V(x, T) = \max(K - e^x, 0), & -\infty < x < \infty.
\end{cases}
\]
3.2.1.2 The integral equation for the optimal-exercise boundary

In this part, we establish the formula for the decomposition of an American put option under the FPDE framework, which is similar to the classical Black-Scholes model.

In the domain \( \Sigma = \Sigma_1 \cup \Sigma_2 = \{ -\infty < x < \infty, 0 \leq t \leq T \} \), the price \( V(x,t) \) of an American put option is continuous and has continuous delta (the high-contact condition). Moreover, considering the values of \( V(x,t) \) in both regions \( \Sigma_1 \) and \( \Sigma_2 \), we have

\[
-\mathcal{L}V(x,t) = g(x) := \begin{cases} 
0, & (x,t) \in \Sigma_1, \\
H(x), & (x,t) \in \Sigma_2.
\end{cases}
\]

The pricing problem \((3.19)\) hence can be reduced to

\[
\begin{cases}
-\mathcal{L}V(x,t) = g(x), & -\infty < x < \infty, \ 0 \leq t < T, \\
V(x,T) = \max(K - e^x, 0), & -\infty < x < \infty,
\end{cases}
\]

which is a Cauchy problem for a inhomogeneous parabolic equation. To solve this problem, one can apply Duhamel’s principle to obtain

\[
V(x,t) = \int_{-\infty}^{\infty} F(x,t;u,T) \max(K - e^u, 0) du \\
+ \int_{t}^{T} d\gamma \int_{-\infty}^{b(\gamma)} H(u) F(x,t;u,\gamma) du \\
:= V_E(x,t) + e(x,t), \tag{3.20}
\]

where \( V_E(x,t) \) is price of the corresponding European put option given in \((3.16)\), and \( e(x,t) \) is the early exercise premium. This is also known as the Doob-Meyer decomposition for American options. The decomposition formula shows that, given the optimal-exercise boundary \( x = b(t) \), the American option price can be determined by Eq. \((3.20)\).

We can also derive an integral equation for the optimal-exercise boundary \( b(t) = \ln B(t) \) from Eq. \((3.20)\). Recall the condition at this boundary

\[
V(b(t),t) = K - e^{b(t)}, \tag{3.21}
\]
then combine the Eq. (3.19), we have

\[
e^{b(t)} = K - V_E(b(t), t) - e(b(t), t) = K - \int_{-\infty}^{\infty} F(b(t), t; u, T) \max(K - e^u, 0) du - \int_{t}^{T} d\gamma \int_{-\infty}^{b(\gamma)} H(u) F(b(t), t; u, \gamma) du.
\]

(3.22)

This is a Volterra integral equation of the second kind, which is highly nonlinear. For the implementation, we shall apply the collocation method with Newton’s iteration (or secant method) to solve Eq. (3.22) numerically.

### 3.2.2 Near expiry behavior

Despite of the complexity of the integral equation (3.22), we may derive the asymptotic behavior of \( b(t) \) as \( t \) approaches the expiry in this section.

The value of an American put option satisfies Eq. (3.10), and at the expiry, the option value is given by the payoff, that is \( V(x, T) = K - e^x \). If the American put option is alive, then its value satisfies Eq. (3.10). Given that \( (x, t) \) is in the continuation region, we next substitute the above put option value into Eq. (3.10), and have

\[
\partial_t V(x, t) \big|_{t=T} = H(x),
\]

(3.23)

where \( H(x) \) is given in (3.18). On the other hand, the left-hand side \( \partial_t V(x, T) \) must be non-negative, so that the American put option keeps alive until the time close to expiry. The value of \( x \) at which \( \partial_t V(x, T) \) changes sign can be approximated by numerical methods.

Note that \( H(x) \) is the same as in the Black-Scholes model, and so the leading-term asymptotic behavior of the optimal-exercise boundary \( b(t) \) coincides with the Black-Scholes model as \( t \to T \),

\[
b(t) \sim \begin{cases} 
\ln K, & r \geq q, \\
\ln K + \ln \frac{r}{q}, & r < q,
\end{cases}
\]

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The next term in the asymptotic approximation is different from the Black-Scholes model. For a refined asymptotic result, we can use the methods in [36] to obtain the next term in the asymptotic expansion.

### 3.2.3 Finding the American option values by quadratic approximation method

Closed-form price formulas do not exist for most American options, except for a few special cases, such as the American call on an asset with no dividend or discrete dividends and the perpetual American options. In this section, we present a well-known and efficient analytic approximation method, namely the quadratic approximation method, for finding the American put option values.

Below we shall use the quadratic approximation method to find the American put option values for both short- and long-maturity pricing problems.

Recall the governing FPDE for the American option in the continuation region $\Sigma_1$:

$$
\partial_t V(x, t) = - \frac{\eta^2}{2} \frac{\partial^2 V(x, t)}{\partial x^2} - CT(-Y) \left[ e^{-Gx} \int_{-\infty}^x D_x^Y (e^{Gx} V(x, t)) + e^{Mx} D_x^Y (e^{-Mx} V(x, t)) \right] \\
- (r - q - \nu) \frac{\partial V(x, t)}{\partial x} + \left[ r + CT(-Y)(G^Y + M^Y) \right] V(x, t),
$$

(3.24)

By the decomposition formula for American put options, we define the early exercise premium as $e(x, t) = V(x, t) - V_E(x, t)$.

Note that the Eq. (3.24) holds for both $V(x, t)$ and $V_E(x(t), t)$ in the continuation region $\Sigma_1$. Since the FPDE is linear, the same equation holds for the early exercise premium $e(x, t)$ as well. Set $k_1 = r + CT(-Y)(G^Y + M^Y)$, $k_2 = r - q - \nu$, and $k_3 = CT(-Y)$. Here we assume that $e(x, t)$ takes the form $e(x, t) = A(t)f(x, A)$, with $A(t)$ to be determined later.
Now Eq. (3.24) can be transformed into
\[
\frac{\eta^2}{2} \frac{\partial^2 f}{\partial x^2} + k_2 \frac{\partial f}{\partial x} + k_3 \left[ e^{-Gx} \int_{-\infty}^{0} D_x Y(e^{Gx} f) + e^{Mx} \int_{0}^{\infty} D_x Y(e^{-Mx} f) \right] \\
- k_1 f \left[ 1 - \frac{1}{k_1 A} \frac{dA}{dt} \left( 1 + \frac{A \partial f}{f \partial A} \right) \right] = 0. \tag{3.25}
\]

A proper choice for \(A = A(t)\) is \(A(t) = 1 - e^{-k_1(T-t)}\), so that Eq. (3.25) becomes
\[
\frac{\eta^2}{2} \frac{\partial^2 f}{\partial x^2} + k_2 \frac{\partial f}{\partial x} + k_3 \left[ e^{-Gx} \int_{-\infty}^{0} D_x Y(e^{Gx} f) + e^{Mx} \int_{0}^{\infty} D_x Y(e^{-Mx} f) \right] \\
- k_1 \left[ f - (1 - A)A \frac{\partial f}{\partial A} \right] = 0. \tag{3.26}
\]

Note that the factor \((1 - A)A\) tends to zero as \(\tau = T - t\) approaches zero or infinity. Hence, for short or long maturity, we may drop the last quadratic term in Eq. (3.26) so that it reduces to an ordinary differential equation (ODE), which is
\[
\frac{\eta^2}{2} \frac{\partial^2 f}{\partial x^2} + k_2 \frac{\partial f}{\partial x} + k_3 \left[ e^{-Gx} \int_{-\infty}^{0} D_x Y(e^{Gx} f) + e^{Mx} \int_{0}^{\infty} D_x Y(e^{-Mx} f) \right] - \frac{k_1}{A} f = 0. \tag{3.27}
\]

Here, \(A = A(t)\) is assumed to be nonzero. The function \(A\) can be regarded as a parameter in Eq. (3.27). The general solution of \(f(x)\) is given by
\[
f(x) = a e^{cx}, \tag{3.28}
\]
where \(a\) is an arbitrary constant, and \(c\) is the root of the following equation
\[
\frac{\eta^2}{2} c^2 + k_3 \left[ (G + c)^Y + (-1)^n (M - c)^Y \right] + k_2 c - \frac{k_1}{A} = 0, \tag{3.29}
\]
with \(n = [Y] + 1\) be the nearest integer larger than \(Y\).

Note that the arbitrary constant \(a\) can be determined by applying the value matching condition and the smooth pasting condition at the optimal-exercise boundary \(b(t)\),
\[
K - e^{b(t)} = V_E(b(t), t) + Aae^{cb(t)}, \tag{3.30}
\]
\[
-e^{b(t)} = e^{b(t)} \frac{\partial V_E(b(t), t)}{\partial S} + cAe^{cb(t)}. \tag{3.31}
\]

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By eliminating $a$ in Eq. (3.31), we obtain the following nonlinear algebraic equation for $b(t)$

\[
\left[ c - 1 - \frac{\partial V_E(b(t), t)}{\partial S} \right] e^{b(t)} = cK - cV_E(b(t), t). \tag{3.32}
\]

Here we can also derive the expression of $a$ from Eq. (3.31)

\[
a = -\frac{1}{cAe^{(c-1)b(t)}} \left( 1 + \frac{\partial V_E(b(t), t)}{\partial S} \right). \tag{3.33}
\]

By substituting Eq. (3.33) into Eq. (3.28), we finally obtain the following formula for American put options

\[
V(x, t) = V_E(x, t) - \frac{1}{c} e^{cx+(1-c)b(t)} \left[ 1 + e^{-x} \frac{\partial V_E(b(t), t)}{\partial x} \right], \tag{3.34}
\]

where $c$ is the root of Eq. (3.29) and can be obtained numerically.

### 3.3 Numerical simulations

In the following section, we shall show some numerical results for both European and American options.

#### 3.3.1 Evaluation of European option values by Gauss quadrature

The analytic formula of European put option price for the CGMYe model is given in (3.16). Based on these formulas, we can calculate the European options efficiently.

##### 3.3.1.1 Fast evaluation by Gauss quadrature

The first example consists of computing European put option prices with different values of the fractional order $Y$ and parameter $C$, because these two parameters always appear together in the CGMYe model. We use Gauss-Chebyshev quadrature (Shen et al. [88]) to evaluate the integrals in Eq. (3.16).
Table 3.1: Computation of European put option values with different $C$ and $Y$

<table>
<thead>
<tr>
<th>$C$</th>
<th>$Y$</th>
<th>option values</th>
<th>ref. values by FFT</th>
<th>relative error</th>
<th>time for G-C</th>
<th>time for FFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.06354046</td>
<td>0.06353405</td>
<td>1.01e-4</td>
<td>0.1331 s</td>
<td>0.2911 s</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.10296690</td>
<td>0.10296691</td>
<td>2.06e-8</td>
<td>0.1226 s</td>
<td>0.2888 s</td>
</tr>
<tr>
<td>1</td>
<td>0.8</td>
<td>0.14789423</td>
<td>0.14789424</td>
<td>7.32e-8</td>
<td>0.1240 s</td>
<td>0.2917 s</td>
</tr>
<tr>
<td>0.1</td>
<td>1.1</td>
<td>0.04273731</td>
<td>0.04273732</td>
<td>3.98e-7</td>
<td>0.1301 s</td>
<td>0.2934 s</td>
</tr>
<tr>
<td>0.1</td>
<td>1.5</td>
<td>0.10651585</td>
<td>0.10651585</td>
<td>6.87e-8</td>
<td>0.1226 s</td>
<td>0.2918 s</td>
</tr>
<tr>
<td>0.1</td>
<td>1.8</td>
<td>0.25612797</td>
<td>0.25613050</td>
<td>9.88e-6</td>
<td>0.1280 s</td>
<td>0.2980 s</td>
</tr>
</tbody>
</table>

For the illustration, we choose the strike price $K = 1$, $S = 1$, $T = 1$, $r = 0.1$, $q = 0.0$, $\eta = 0.0$, and $G = M = 5$. We take different values of $C$ and $Y$ for verification. From Table 3.1, we can see that the results by Gauss-Chebyshev quadrature approach are very close to the numerical values with the FFT method introduced by Carr and Madan [17]. Moreover, the computational time of Gauss-Chebyshev method is almost the half of the FFT method, which makes our computation more efficient.

The case for $Y < 0$ is not considered here, since the characteristic function decays very slow, and hence requires further numerical techniques.

**Algorithm 1: evaluation of the European option value**

- Step 1: Generate Gauss-Chebyshev points and weights for the integral in Eq. 3.16.

- Step 2: Evaluate the fundamental solution via Gauss-Chebyshev quadrature at the Chebyshev points given in step 1.

- Step 3: Evaluate the European option value by integrating Eq. 3.16 via quadrature.
3.3.1.2 Comparison between CGMYe model and the Black-Scholes model

In the following example, we first construct a time series of daily log-price relatives \( \ln \left( \frac{S_t}{S_0} \right) \), and estimate the parameters \( C, G, M, Y, \) and \( \eta \) for the underlying asset from the adjusted return data. The data is chosen from Hong Kong Stock Exchanges (HKEx), the NASDAQ Global Select (NasdaqGS) market, and the New York Stock Exchange (NYSE) from January 2012 to December 2014, and we set the time spacing as \( \Delta t := t_{m+1} - t_m = 1/245 \). Note that the transition density of \( \tilde{X}_t^{ij} \) is

\[
P(\tilde{X}_{t_{m+1}}^{ij} = u|\tilde{X}_{t_m}^{ij} = v) = \frac{1}{2\pi} \int_{-\infty+i\xi_j^i}^{\infty+i\xi_j^i} e^{i\xi_j^i(v-u)+\Delta t \Psi_j(\xi_j)} d\xi_j,
\]

which is an inverse Fourier transform of the characteristic function. Due to the lack of the closed-form expression of the transition density, we shall use the fast Fourier transform (FFT) to evaluate at some prescribed points, and then the transition density is approximated by linear interpolation on these obtained values.

By writing the transition density as a function of \( C_j, G_j, M_j, Y_j, \eta_j \), we next apply the maximum likelihood estimation (MLE) as the parameter estimator. The log-likelihood function has the form

\[
\ln L(C_j, G_j, M_j, Y_j, \eta_j) = \sum_{i=1}^{n} \ln P(\tilde{X}_{t_{m+1}}^{ij} = u|\tilde{X}_{t_m}^{ij} = x).
\]

The results of the estimates parameters for 8 stocks are presented in Table 3.2. We can see from the sixth column of the table that, the estimated values of \( \eta \) is very small, which implies that the influence of the Brownian motion of the CGMYe process is small. For the comparison purpose, we also estimated the parameters \( \mu \) and \( \sigma \) for the Black-Scholes model.

By using the parameters of CNPC, HSBC and SINOPEC listed in Table 3.2, we then compare the European put option values between the CGMYe model and the Black-Scholes model in Fig. 3.1. Obviously, the values under of the CGMYe model are larger than those of the Black-Scholes model when stock price is near the strike price, and the largest difference takes when \( S = K \).
Figure 3.1: Comparison between CGMYe model and BS model for European put
### Table 3.2: Results of MLE under the real data

<table>
<thead>
<tr>
<th>Stock</th>
<th>C</th>
<th>G</th>
<th>M</th>
<th>Y</th>
<th>η</th>
<th>μ</th>
<th>σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAPL</td>
<td>0.1464</td>
<td>11.6047</td>
<td>22.5197</td>
<td>1.3495</td>
<td>-0.0927</td>
<td>-9.10e-4</td>
<td>0.0169</td>
</tr>
<tr>
<td>CNPC</td>
<td>0.1924</td>
<td>70.8414</td>
<td>51.4211</td>
<td>1.4106</td>
<td>0.0000</td>
<td>5.87e-5</td>
<td>0.0146</td>
</tr>
<tr>
<td>HSBC</td>
<td>0.4577</td>
<td>68.6225</td>
<td>95.2302</td>
<td>1.1589</td>
<td>0.0000</td>
<td>-2.93e-4</td>
<td>0.0102</td>
</tr>
<tr>
<td>IBM</td>
<td>1.0773</td>
<td>28.1160</td>
<td>37.5846</td>
<td>0.6444</td>
<td>-0.1145</td>
<td>-1.19e-4</td>
<td>0.0110</td>
</tr>
<tr>
<td>INTC</td>
<td>0.3893</td>
<td>34.9065</td>
<td>45.2965</td>
<td>1.1927</td>
<td>0.0000</td>
<td>-6.58e-4</td>
<td>0.0135</td>
</tr>
<tr>
<td>MSFT</td>
<td>0.0776</td>
<td>25.7883</td>
<td>23.5054</td>
<td>1.4631</td>
<td>0.0000</td>
<td>-8.44e-4</td>
<td>0.0137</td>
</tr>
<tr>
<td>SINOPEC</td>
<td>23.9138</td>
<td>58.9514</td>
<td>52.7350</td>
<td>0.1925</td>
<td>0.1797</td>
<td>-1.21e-4</td>
<td>0.0161</td>
</tr>
<tr>
<td>YHOO</td>
<td>0.4166</td>
<td>21.7388</td>
<td>38.3684</td>
<td>1.1841</td>
<td>-0.1236</td>
<td>-1.50e-3</td>
<td>0.0171</td>
</tr>
</tbody>
</table>

### 3.3.2 Simulation of the optimal-exercise boundaries

The optimal-exercise boundary satisfies the Volterra integral equation \(3.22\), and the implementation of this equation is based on the collocation method and the secant method. In order to solve the integral equation \(3.22\) numerically, we first discretize the time interval by the graded mesh, that is \(\tau_n = T (n/N)^2\) with \(0 \leq n \leq N\) and \(N\) being the total number of nodes in space. We shall denote the value of the optimal-exercise boundary at these nodes by \(B_n = e^{b(\tau_n)}\). Note that \(B_0 = e^{b(0)} = K\) in the case when \(r > q\) (see Section \(3.2.2\)), and \(B_n\) satisfies a system of nonlinear algebraic equations

\[
F_j(B_1, \cdots, B_j) = 0, \quad 1 \leq j \leq N.
\]

The value of \(B_n\) is then obtained iteratively by solving the above nonlinear system via the secant method.

**Algorithm 2: evaluation of the optimal-exercise boundary**

- Step 1: Generate the nodes for the graded mesh, \(\tau_n = (n/N)^2 T\) for \(0 \leq n \leq N\),
- Step 2: Initialize the optimal-exercise boundary \(B_n = e^{b(\tau_n)}\) with \(B_0 = K\),
Table 3.3: The optimal-exercise price for some specific time

<table>
<thead>
<tr>
<th>time $t$</th>
<th>CNPC</th>
<th></th>
<th>HSBC</th>
<th></th>
<th>SINOPEC</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CGMYe</td>
<td>BS</td>
<td>CGMYe</td>
<td>BS</td>
<td>CGMYe</td>
<td>BS</td>
</tr>
<tr>
<td>0</td>
<td>0.9257</td>
<td>0.9787</td>
<td>0.9534</td>
<td>0.9840</td>
<td>0.9163</td>
<td>0.9760</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9322</td>
<td>0.9799</td>
<td>0.9587</td>
<td>0.9848</td>
<td>0.9238</td>
<td>0.9773</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9398</td>
<td>0.9814</td>
<td>0.9612</td>
<td>0.9860</td>
<td>0.9326</td>
<td>0.9790</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9494</td>
<td>0.9835</td>
<td>0.9668</td>
<td>0.9875</td>
<td>0.9435</td>
<td>0.9814</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9629</td>
<td>0.9868</td>
<td>0.9752</td>
<td>0.9899</td>
<td>0.9588</td>
<td>0.9850</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Step 3: Give two initial guesses for $B_1$, denoting by $B_1^0$ and $B_1^1$, and solve $F_1(B_1) = 0$ via the secant method. Update $B$ by setting $B_1$ equal to the value obtained,

- Step 4: Give two initial guesses for $B_2$, denoting by $B_2^0$ and $B_2^1$, and solve $F_2(B_1, B_2) = 0$ for $B_2$ via secant method, Update $B$ by setting $B_2$ equal to the value obtained,

- Step 5: Repeat the same procedure as in steps 3 and 4 for $B_n$ with $n = 3, 4, \cdots, N$, iteratively.

In the following example, we evaluate the optimal-exercise boundary of American put options for three different stocks under the CGMYe model. The parameters used here are the same as we set in section 3.3.1, here we choose the data of CNPC, HSBC, SINOPEC. Then we compare the optimal-exercise prices $S(t)$ between the CGMYe model and the Black-Scholes model for all the three stocks at some particular time levels in Table 3.3. From this table, we can see that the values of optimal-exercise boundary in the CGMYe model are smaller than the Black-Scholes model in general, and this discrepancy decreases when $t$ approaches the expiry date.
3.3.3 Analytic approximation for American put options

The American option value can be approximated by the formula in Eq. (3.34) when \( \tau = T - t \) tends to zero or infinity. The European option value \( V_E \) and the value of the optimal-exercise boundary \( b(t) \) can be evaluated by the methods described in Sections 3.3.1 and 3.3.2, respectively. The value of \( c = c(t) \) can be obtained by solving the nonlinear algebraic equation (3.29), and the delta value \( \Delta V_E \) can then be evaluated through equation (3.32).

Algorithm 3: approximation for the American option value

- Step 1: Given \( t \), compute \( c = c(t) \) by solving Eq. (3.29) via the secant method; the two initial guesses are chosen in the interval \((-G, M)\),
- Step 2: Compute the value of \( b(t) \) by Algorithm 2,
- Step 3: Compute the European option values \( V_E(b(t), t) \) and \( V_E(x, t) \) by Algorithm 1,
- Step 4: Compute the delta value \( \Delta V_E = \frac{\partial V_E}{\partial S} \) by solving equation (3.32),
- Step 5: The American option value is then obtained by equation (3.34).

To verify the validity of the approximate formula (3.34), we compare the results with the numerical solution obtained by the method of Almendral and Oosterlee [2], for different values of \( Y \). In the following example, the parameter are set to be as \( C = 1, G = M = 5, \eta = 0.0, r = 0.1, q = 0.0, K = 1, \tau = 0.0625, \) and \( Y \) takes 0.3 and 0.5. The comparison results are plotted in Fig. 3.2. It is clear that the analytic approximation in Section 3.2.3 is quite accurate when \( \tau = T - t \) is small.
Figure 3.2: Verification of the approximate formula (3.34).
Chapter 4

Extending CGMYe model to multi-asset

There is already a large, and still growing literature on option pricing with Lévy processes. However, most of the existing work deals with the problem in a univariate set-up. In order to price a multi-asset option, such as a rainbow option or a basket option, we may assume each asset follows a 1-D Lévy process. Moreover, we should also take consideration of the correlation between these assets. The asset prices can thus be regarded as a multi-dimensional Lévy process.

There are three ways to build a structure of multivariate Lévy processes, namely, multivariate subordination, linear combination and Lévy copulas (see Deelstra and Petkovic [29] (2010), Cont and Tankov [26] (2004)). Multivariate subordination has attractive analytical and statistical properties whereas robust and fast parameter estimations turn out to be difficult in higher dimensions [86]. Lévy copulas were originally introduced by Tankov in [93]. Compared with the standard copulas, Lévy copulas are defined on the Lévy measure $W(dx)$, so we can have a straightforward interpretation of the behavior of the multivariate process. However, the complexity of this method makes it only suitable for Monte Carlo simulations, and there are still a lot of untouched issues on Lévy copulas, for instance, there is no parameter estimation methods have been proposed yet [86].

In the present chapter, we concentrate on the analytical study of the framework of the multi-variate CGMYe process model. The decomposition formula of the fundamental solution and the fractional partial differential equation of the European option
value are established in the explicit form. Moreover, a parameter estimation is applied for the multi-dimensional problem. Numerical simulations are also provided for European basket put option values.

4.1 Framework of 2-D European option pricing problems

In this section, we use the idea of linear combination in the following work to formulate the multivariate model, see Sato [84] (1999, pp. 65-66), Deelstra and Petkovic [29] (2010).

4.1.1 The characteristic function and the FPDE

The characteristic exponent of the log-return relative in the 1-D CGMYe process is

\[ \Psi_{\tilde{X}_t}(\xi) = i(\mu + v)\xi - \frac{\eta^2}{2}\xi^2 + \Psi_{CGMY}(\xi), \]  

(4.1)

where \( \mu \) is the mean rate of return on the stock, and \( v \) is the convexity adjustment defined by

\[ v = \Psi_{CGMY}(-i) + \frac{\eta^2}{2}. \]  

(4.2)

Below we will give two theorems about the multi-dimensional Lévy process problems. Theorem 4.1 present the rule to construct a multi-dimensional Lévy process model by using linear combination method. The idea of this theorem was first introduced by Sato in [84]. Theorem 4.2 provides some statistical definitions of the \( n \)-dimensional CGMY process, including the mean vector, and the variance and covariance matrixes.

**Theorem 4.1** (Sato). Let \( \tilde{X}_t \) be an \( n \)-dimensional Lévy process with mutually independent components, \( A \in \mathbb{R}^{d\times n} \) a matrix. Then the characteristic exponent of
\( Z_t = A\tilde{X}_t \) is given by
\[
\Psi_{Z_t}(\xi) = t \sum_{s=1}^{n} \Psi_{\tilde{X}_s} \left( \sum_{j=1}^{d} \xi^j (A)_{js} \right),
\]
where \( \Psi_{\tilde{X}_s}(\xi) \) is the characteristic exponent of the marginal \( \tilde{X}_s \) given in Eq. (4.1), \((A)_{js}\) is the \((j,s)\)-th entry of the matrix \( A \).

By using the above theorem, we can derive the characteristic exponent of \( n \)-dimensional CGMYe process. We give the characteristic exponent of 2-D case right behind the Theorem 4.2 in this section, and propose the \( n \)-dimensional characteristic exponent, for \( n \geq 3 \), in Section 4.3.

**Theorem 4.2.** Let \( Z_t = A\tilde{X}_t \) and \( \tilde{X}_t \) be an \( n \)-dimensional CGMYe process with independent components. Then the mean vector and the covariance matrix at \( t = 1 \) are given by \( \mathbb{E}(Z) = A\mathbb{E}(\tilde{X}) \) and \( \text{Cov}(Z) = A'DA \) respectively, where
\[
\mathbb{E}(\tilde{X}^j) = C_j \Gamma(1 - Y_j)(M_{Y_j}^{Y_j-1} - G_{Y_j}^{Y_j-1}) + b_j \tag{4.4}
\]
with \( b_j = r - q_j - \upsilon_j \) and
\[
v = \Psi_{Z_t}(-i1_{n\times 1}), \tag{4.5}
\]
for \( 1_{n\times 1} = (1, \ldots, 1)' \), \( D = \text{diag}(\text{Var}(\tilde{X}^1), \ldots, \text{Var}(\tilde{X}^n)) \), and
\[
\text{Var}(\tilde{X}^j) = C_j \Gamma(2 - Y_j)(G_{Y_j}^{Y_j-2} + M_{Y_j}^{Y_j-2}) + \eta_j^2 \tag{4.6}
\]
for \( j = 1, 2, \ldots, n \).

Here, and in what follows, the prime on a vector stands for its transpose.

**Proof.** According to the definition (2.1) and (3.2), the expectation of a CGMY process is
\[
\mathbb{E}(X^j) = i^{-1} \varphi_{X^j}^{(1)}(0) = i^{-1} \frac{\partial}{\partial \xi} \left( \exp[t\Psi_{X^j}(\xi)] \right) \bigg|_{\xi=0}
= i^{-1} \left[ tC_j Y_j \Gamma(-Y_j)((G_j + i\xi)^{Y_j-1} - (M_j - i\xi)^{Y_j-1}) \right] \bigg|_{\xi=0}
= tC_j \Gamma(1 - Y_j)(M_{Y_j}^{Y_j-1} - G_{Y_j}^{Y_j-1}). \tag{4.7}
\]
Here we have also made use of the fact that $\Psi_{X_j}(0) = 0$. Note that

$$\tilde{X}_t = X_{t,CGMY}(C, G, M, Y) + \eta W_t,$$

so the expectation of a CGMYe process at $t = 1$ has the form

$$E(\tilde{X}^j) = C_j \Gamma(1 - Y_j)(M_j^{Y_j-1} - G_j^{Y_j-1}) + b_j$$

(4.8)

with $b_j = r - q_j - \nu_j$, $j = 1, 2, \ldots, n$.

We know that $Z_t = A' \tilde{X}_t$, then $E(Z) = E(A \tilde{X}) = A E(\tilde{X})$ at $t = 1$, and $E(\tilde{X}) = (E(\tilde{X}^1), \ldots, E(\tilde{X}^n))$.

By a similar manner as in (4.7), we can derive the variance of a CGMY process.

$$\text{Var}(X^j) = E[(X^j)^2] - [E(X^j)]^2$$

$$= \left[ [E(X^j)]^2 + t C_j Y_j (Y_j - 1) \Gamma(-Y_j) \right.$$  

$$\left. - [(G_j + i \xi)^{Y_j-2} + (M_j - i \xi)^{Y_j-2}] - [E(X^j)]^2 \right]_{\xi=0}$$

$$= t C_j \Gamma(2 - Y_j)(G_j^{Y_j-2} + M_j^{Y_j-2}),$$

for $j = 1, 2, \ldots, n$, and the variance of a CGMYe process can hence be given by

$$\text{Var}(\tilde{X}^j) = \text{Var}(X^j) + \eta_j^2.$$

Note that the covariance matrix of $Z$ include three parts: $\text{Var}(Z^1)$, $\text{Var}(Z^2)$, and $\text{Cov}(Z^1, Z^2)$. The expression of the three parts at $t = 1$ has the form as:

$$\text{Var}(Z^1) = \text{Var}((A)_{11} \tilde{X}^1 + (A)_{21} \tilde{X}^2) = (A)_{11}^2 \text{Var}(\tilde{X}^1) + (A)_{21}^2 \text{Var}(\tilde{X}^2),$$

$$\text{Var}(Z^2) = \text{Var}((A)_{12} \tilde{X}^1 + (A)_{22} \tilde{X}^2) = (A)_{12}^2 \text{Var}(\tilde{X}^1) + (A)_{22}^2 \text{Var}(\tilde{X}^2),$$

and

$$\text{Cov}(Z^1, Z^2) = E(Z^1 Z^2) - E(Z^1)E(Z^2)$$

$$= (A)_{11}(A)_{12} \text{Var}(\tilde{X}^1) + (A)_{21}(A)_{22} \text{Var}(\tilde{X}^2).$$

Thus we can simply write the covariance matrix of $Z$ as $\text{Cov}(Z) = A'DA$, with $D = \text{diag}(\text{Var}(\tilde{X}^1), \ldots, \text{Var}(\tilde{X}^n)).$
Let us now consider the two-dimensional case for CGMYe process through linear combination. By Theorem 4.1, the characteristic exponent of \(Z_t = A\tilde{X}_t\) at \(t = 1\) is

\[
\Psi(\xi) := \psi(\zeta) = \Psi_1(\xi_1(A)_{11} + \xi_2(A)_{21}) + \Psi_2(\xi_1(A)_{12} + \xi_2(A)_{22}) \tag{4.9}
\]

\[
= C_1 \Gamma(-Y_1) \left[ (M_1 - i\xi_1(A)_{11} - i\xi_2(A)_{21})^{Y_1} - M_1^{Y_1} \right] + \left[ (A)_{11} \eta_1 \xi_1^{22} + (A)_{21} \eta_2 \xi_2^{22} \right] + C_2 \Gamma(-Y_2) \left[ (M_2 - i\xi_1(A)_{12} - i\xi_2(A)_{22})^{Y_2} - M_2^{Y_2} \right] + \left[ (A)_{12} \eta_1 \xi_1^{22} + (A)_{22} \eta_2 \xi_2^{22} \right]
\]

where \(\xi = (\xi_1, \xi_2)'\) is the dual variable of \(x = (x_1, x_2)'\), \(\Psi_j\) is the characteristic exponent for \(\tilde{X}^j\) with \(j = 1, 2\), and \(\zeta = (\zeta_1, \zeta_2)'\) with \(\zeta_1 = \xi_1(A)_{11} + \xi_2(A)_{21}\) and \(\zeta_2 = \xi_1(A)_{12} + \xi_2(A)_{22}\)

The value of a European-style option is given by the ‘risk neutral’ expectation of the final payoff \(\Pi(Z_T, T)\),

\[
V(z, t) = e^{-r(T-t)}\mathbb{E}^Q[\Pi(Z_T, T)]. \tag{4.10}
\]

Assuming that the payoff \(\Pi(Z_T, T)\) has a complex Fourier transform

\[
\hat{\Pi}(\zeta, T) = \int_{-\infty+ic_1}^{\infty+ic_1} \int_{-\infty+ic_2}^{\infty+ic_2} e^{-i\zeta_1 u_1 - i\zeta_2 u_2} \Pi(u, T) du_1 du_2, \tag{4.11}
\]

with \(\zeta^i = \text{Im}\zeta\), and the inverse Fourier transform gives

\[
V(z, t) = \frac{e^{-r(T-t)}}{4\pi^2} \mathbb{E}^Q \left[ \int_{-\infty+ic_1}^{\infty+ic_1} \int_{-\infty+ic_2}^{\infty+ic_2} e^{i\zeta_1 z_1^1 + i\zeta_2 z_2^2} \hat{\Pi}(\zeta, T) d\zeta_1 d\zeta_2 \right] \tag{4.12}
\]

\[
= \frac{e^{-r(T-t)}}{4\pi^2} \int_{-\infty+ic_1}^{\infty+ic_1} \int_{-\infty+ic_2}^{\infty+ic_2} \mathbb{E}^Q \left[ e^{i\zeta_1 z_1^1 + i\zeta_2 z_2^2} \right] \hat{\Pi}(\zeta, T) d\zeta_1 d\zeta_2
\]

\[
= \frac{e^{-r(T-t)}}{4\pi^2} \int_{-\infty+ic_1}^{\infty+ic_1} \int_{-\infty+ic_2}^{\infty+ic_2} e^{i\zeta_1 z_1^1 + i\zeta_2 z_2^2 + (T-t)\psi(\zeta)} \hat{\Pi}(\zeta, T) d\zeta_1 d\zeta_2,
\]

where \(\psi(\zeta)\) is the characteristic exponent of the Lévy process \(Z_t\), which is given in Eq. (4.9). It follows from Eq. (4.12) that

\[
\frac{\partial \hat{V}(\zeta, t)}{\partial t} = \left( r - \psi(\zeta) \right) \hat{V}(\zeta, t), \tag{4.13}
\]

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with the solution given by
\[
\hat{V}(\zeta, t) = e^{[-r+\psi(\zeta)(T-t)]} \hat{\Pi}(\zeta, T).
\] (4.14)

Taking the inverse Fourier transform on Eq. (4.13), we finally obtain the following FPDE:
\[
\frac{\partial V(z, t)}{\partial t} + \frac{\eta_1^2}{2} \frac{\partial^2 V(z, t)}{\partial z_1^2} + \frac{\eta_2^2}{2} \frac{\partial^2 V(z, t)}{\partial z_2^2} + b_1 \frac{\partial V(z, t)}{\partial z_1} + b_2 \frac{\partial V(z, t)}{\partial z_2} + I + II
\]
\[
= \left[ r + C_1 \Gamma(-Y_1)(M_{11}^{Y_1} + G_{11}^{Y_1}) + C_2 \Gamma(-Y_2)(M_{21}^{Y_2} + G_{21}^{Y_2}) \right] V(z, t),
\] (4.15)
where \( z = Ax \), \( b_j = r - q_j - v_j \), \( v = \Psi(-i1_{2 \times 1}) \) for \( 1_{2 \times 1} = (1, 1)' \), \( j = 1, 2 \), and
\[
I = C_1 \Gamma(-Y_1) \left[ e^{-G_{z_1 \infty}^{Y_1}} D_{z_1}^{Y_1}(e^{G_{z_1 \infty}^{Y_1}} V(z, t)) + e^{M_{z_1 \infty}^{Y_1}} D_{z_1}^{Y_1}(e^{-M_{z_1 \infty}^{Y_1}} V(z, t)) \right],
\]
\[
II = C_2 \Gamma(-Y_2) \left[ e^{-G_{z_2 \infty}^{Y_2}} D_{z_2}^{Y_2}(e^{G_{z_2 \infty}^{Y_2}} V(z, t)) + e^{M_{z_2 \infty}^{Y_2}} D_{z_2}^{Y_2}(e^{-M_{z_2 \infty}^{Y_2}} V(z, t)) \right].
\]
Here \(-\infty D_{z_j}^{Y_j}(e^{G_j z_j} V(z, t)) \) and \( z_j D_{\infty}^{Y_j}(e^{-M_j z_j} V(z, t)) \), \( j = 1, 2 \), are the left and right Riemann- Liouville (RL) fractional derivatives defined as
\[
-\infty D_{z_j}^{Y_j} f(z) = \frac{1}{\Gamma(n - Y_j)} \frac{\partial^n}{\partial z_j^n} \int_{-\infty}^{z_j} (z_j - y_j)^{n - Y_j - 1} f(y) dy_j
\] (4.16)
and
\[
z_j D_{\infty}^{Y_j} f(z) = \frac{1}{\Gamma(n - Y_j)} \frac{\partial^n}{\partial z_j^n} \int_{z_j}^{\infty} (y_j - z_j)^{n - Y_j - 1} f(y) dy_j,
\] (4.17)
where \( n = [Y_j] + 1 \) is the nearest integer larger than \( Y_j \).

**Derivation of Eq. (4.15).** We start by taking the inverse Fourier transform of \( \psi(\zeta) \hat{V}(\zeta, t) \):
\[
\mathcal{F}^{-1}(\psi(\zeta) \hat{V}(\zeta, t)) = \frac{1}{(2\pi)^2} \int_{-\infty+i\zeta_1}^{\infty+i\zeta_1} \int_{-\infty+i\zeta_2}^{\infty+i\zeta_2} e^{i\zeta_1 \cdot 1 + i\zeta_2 \cdot 2} \psi(\zeta) \hat{V}(\zeta, t) d\zeta_1 d\zeta_2
\]
\[
:= \tilde{I} + \tilde{II}.
\] (4.18)

By changing variables, we let \( \zeta_1 = \xi_1(A)_{11} + \xi_2(A)_{21} \) and \( \zeta_2 = \xi_1(A)_{12} + \xi_2(A)_{22} \), and \( B = (A')^{-1} \), the inverse matrix of the transpose of \( A \). Then it is straightforward to write \( \xi \) in terms of \( \zeta \) as: \( \xi_1 = \zeta_1(B)_{11} + \zeta_2(B)_{12} \) and \( \xi_2 = \zeta_1(B)_{21} + \zeta_2(B)_{22} \), with \((B)_{ij}\) is the \((j, s)\)-th entry of the matrix \( B \). Set \( z = Ax \).
The change of variables leads to

\[ \tilde{I} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty+i\zeta_1} \int_{-\infty}^{\infty+i\zeta_2} \left[ C_1 \Gamma(-Y_1) e^{i\zeta_1 Y_1 + i\zeta_2 Y_2} \right. \left. \left( (M_1 - i\zeta_1) Y_1 - M_1 Y_1 + (G_1 + i\zeta_1) Y_1 - G_1 Y_1 \right) - \frac{\eta_1^2}{2} \zeta_1^2 + ib_1 \zeta_1 \right] \hat{V}(\zeta, t) d\zeta_1 d\zeta_2 \]

\[ = \frac{1}{2\pi} \int_{-\infty+i\zeta_1}^{\infty+i\zeta_1} \left[ C_1 \Gamma(-Y_1) e^{i\zeta_1 Y_1} \left( (M_1 - i\zeta_1) Y_1 - M_1 Y_1 + (G_1 + i\zeta_1) Y_1 - G_1 Y_1 \right) \right. \left. - \frac{\eta_1^2}{2} \zeta_1^2 + ib_1 \zeta_1 \right] \hat{V}(\zeta_1; z_2, t) d\zeta_1 \]

\[ = C_1 \Gamma(-Y_1) \left[ e^{M_1 z_1} D_{\infty} Y_1 (e^{-M_1 z_1} V(z, t)) + e^{-G_1 z_1} D_{\infty} Y_1 (e^{G_1 z_1} V(z, t)) \right] \]

\[ - (M_1^2 + G_1^2) V(z, t) \]

\[ + \eta_1^2 \frac{\partial^2}{\partial z_1^2} V(z, t) + b_1 \frac{\partial}{\partial z_1} V(z, t). \]

In a similar means, we also have

\[ \tilde{II} = C_2 \Gamma(-Y_2) \left[ e^{M_2 z_2} D_{\infty} Y_2 (e^{-M_2 z_2} V(z, t)) + e^{-G_2 z_2} D_{\infty} Y_2 (e^{G_2 z_2} V(z, t)) \right] \]

\[ - (M_2^2 + G_2^2) V(z, t) \]

\[ + \eta_2^2 \frac{\partial^2}{\partial z_2^2} V(z, t) + b_2 \frac{\partial}{\partial z_2} V(z, t). \]

Here \( b_j = r - q_j - v_j \) for \( j = 1, 2 \), and the convexity adjustment

\[ v = \Psi(-i1_{2x1}) \] (4.20)

where \( 1_{2x1} = (1, 1)' \), and \( \Psi \) is given in Eq. (4.9). Thus, the inverse Fourier transform of Eq. (4.13) can be expressed as

\[ \frac{\partial V(z, t)}{\partial t} + \tilde{I} + \tilde{II} = rV(z, t), \]

which is exactly the Eq. (4.15).
4.1.2 Derivation of the fundamental solution

Note that Eq. (4.12) can be also written as the form

\[ V(z,t) = e^{-r(T-t)} \left( \frac{1}{4\pi^2} \int \int e^{i\zeta_1 u_1 - i\zeta_2 u_2} \Pi(u,T) du_1 du_2 \right) d\zeta_1 d\zeta_2 \]

\[ = e^{-r(T-t)} \left( \frac{1}{4\pi^2} \int \int G(z,t;u,T) \Pi(u,T) du_1 du_2 \right) \]

where \( \zeta_j \equiv \text{Im}\zeta_j \) for \( j = 1, 2 \), and \( e^{-r(T-t)}G(z,t;u,T) \) is the fundamental solution of Eq. (4.15). Note that \( G(z,t_1;u,t_2) \) is exactly the same as the transition density for the associated process \( Z_t \), that is

\[ G(z,t_1;u,t_2) = \mathbb{P}(Z_{t_2} = u | Z_{t_1} = z), \]

and the parameter estimation of FPDE (4.15) is based on the computation of transition densities.

4.2 2-D American option pricing problems

In this section, we consider the problem of pricing American option under the multi-asset CGMY\( e \) process. Assuming the option is written on \( n \)-asset with the component denoted by \( S_j, 1 \leq j \leq n \). It is known that there is an optimal-exercise policy that maximizes the option value for American option holders. Mathematically, it is to fix the free-boundary \( B(t) = (B_1(t), \ldots, B_n(t))' \), which divides the region \( \{ 0 \leq S_j < \infty, 0 \leq t \leq T \} \) into the continuation region \( \Sigma_1 = \{ B_j(t) \leq S_j < \infty \} \), and the stopping region \( \Sigma_2 = \{ 0 \leq S_j < B_j(t) \} \).

Suppose the option value and the delta \( \frac{\partial V}{\partial S} \) are continuous for \( S \geq 0, \) \textit{i.e.} the high-contract condition. The log-return of the stocks follow a \( n \)-dimensional CGMY\( e \)
process, and hence the American option value satisfies Eq. (4.15) in the continuation
region $\Sigma_1$. In what follows, we shall use the log-return $z = \ln S$, and set $b(t) = \ln B(t)$
for convenience. Denote the operator $\mathcal{L}$ by

$$\mathcal{L}V(z,t) = \frac{\partial V(z,t)}{\partial t} + \frac{\eta_1^2}{2} \frac{\partial^2 V(z,t)}{\partial z_1^2} + \frac{\eta_2^2}{2} \frac{\partial^2 V(z,t)}{\partial z_2^2} + b_1 \frac{\partial V(z,t)}{\partial z_1} + b_2 \frac{\partial V(z,t)}{\partial z_2} + I + II$$

(4.23)

where $I$ and $II$ are given in Eq. (5.1).

In $\Sigma_1$, the option value is greater than the expiration revenue,

$$V(z,t) > \Pi(z,t), \quad \text{and} \quad \mathcal{L}V(z,t) = 0,$$

(4.24)

with $\Pi(z,t)$ be the payoff at time $t$. Moreover, in $\Sigma_2$, since the the optimal-exercise
point $B(t) < K$, we have

$$V(z,t) = \Pi(z,t), \quad \text{and} \quad \mathcal{L}V(z,t) < 0.$$

(4.25)

By combining the situations in both $\Sigma_1$ and $\Sigma_2$, we obtain the following inequality
for two-asset American options:

$$\frac{\partial V(z,t)}{\partial t} + \frac{\eta_1^2}{2} \frac{\partial^2 V(z,t)}{\partial z_1^2} + \frac{\eta_2^2}{2} \frac{\partial^2 V(z,t)}{\partial z_2^2} + b_1 \frac{\partial V(z,t)}{\partial z_1} + b_2 \frac{\partial V(z,t)}{\partial z_2} + I + II$$

$$\leq \left[ r + C_1 \Gamma(-Y_1)(M_1^Y_1 + G_1^Y_1) + C_2 \Gamma(-Y_2)(M_2^Y_2 + G_2^Y_2) \right] V(z,t).$$

Thus, given time $t$, the value of a multi-asset American put option under CGMYe
model satisfies the following problem

$$\begin{cases}
\mathcal{L}V(z,t) = 0, & b(t) < z < \infty, \quad 0 < t < T, \\
V(b(t),t) = \Pi(b(t),t), & 0 \leq t \leq T \\
\frac{\partial}{\partial z} V(b(t),t) = \frac{\partial}{\partial z} \Pi(b(t),t), & 0 \leq t \leq T \\
V(z,T) = \Pi(z,T), & -\infty < z < \infty.
\end{cases}$$

(4.26)
The multi-asset American option under the CGMYe model might also be treated analytically, for instance, the decomposition formulas and the integral equation for the optimal-exercise boundary, can be established as in the single-asset case, but these formulas are more complicated in the form, and hence we shall use numerical methods to solve the multi-asset American pricing problem directly in the future work.

### 4.3 Extend to high-dimensional problems

By applying the idea of affine transformation, we extend the 2-dimensional problem to high-dimension.

**Definition 4.1.** An n-dimensional CGMYe process \( Z_t \) is said to be multivariate CGMYe distributed if it has the following representation:

\[
Z_t = A\tilde{X}_t,
\]

where \( A \in \mathbb{R}^{n \times n} \), and the random vector \( \tilde{X}_t = (\tilde{X}_1^t, \ldots, \tilde{X}_n^t) \) consists of mutually independent random variables \( \tilde{X}_i^j \in \text{CGMYe}, j = 1, \ldots, n \).

Note that Theorems 4.1 and 4.2 still hold in this part, and the FPDE for n-dimensional CGMYe model has the form

\[
\frac{\partial V(z, t)}{\partial t} + \frac{1}{2} \sum_{k=1}^{n} \eta_k^2 \frac{\partial^2 V(z, t)}{\partial z_k^2} + \sum_{k=1}^{n} b_k \frac{\partial V(z, t)}{\partial z_k} + \sum_{k=1}^{n} I_k = \left[ r + \sum_{k=1}^{n} C_k \Gamma(-Y_k) (M_k^{Y_k} + G_k^{Y_k}) \right] V(z, t),
\]

where \( z = Ax, b_j = r - q_j - v_j, v = \Psi(-i1_{n \times 1}) \) for \( 1_{n \times 1} = (1, \ldots, 1)' \), \( j = 1, \ldots, n \), and

\[
I_k = C_k \Gamma(-Y_k) \left[ e^{-G_k z_k} D_{z_k}^{Y_k} (e^{G_k z_k} V(z, t)) + e^{M_k z_k} D_{z_k}^{Y_k} (e^{-M_k z_k} V(z, t)) \right].
\]
4.4 Estimations and numerical simulations

In this section, we shall first estimate the parameter in the governing 2-D model, and then evaluated the European basket put option values based on the estimated parameters.

4.4.1 Parameter estimations

4.4.1.1 1-D estimation

In this section, we construct a time series of daily log price relatives \( \ln \left( \frac{S_t}{S_0} \right) \), and estimate the parameters \( C_j, G_j, M_j, Y_j, \) and \( \eta_j \) for each underlying asset from the adjusted return data. The data is chosen from the real stock markets, and we set the time step to be \( \Delta t := t_{m+1} - t_m = 1/(\text{number of trading days}) \). Note that the transition density of the 1-D CGMYe process \( \tilde{X}_t^j \) follows

\[
P(\tilde{X}_{t_{m+1}}^j = u|\tilde{X}_{t_m}^j = v) = \frac{1}{2\pi} \int_{-\infty + i\xi_j^i}^{\infty + i\xi_j^i} e^{i\xi_j(u-v) + \Delta t \Psi_j(\xi)} d\xi_j,
\]

which is an inverse Fourier transform of the characteristic function. Due to the lack of closed-form expression of the transition density, we shall use the fast Fourier transform (FFT) to evaluate at some prescribed points, and then the transition density is approximated by linear interpolation on these obtained values.

By writing the transition density as a function of \( C_j, G_j, M_j, Y_j, \eta_j \), we next apply the maximum likelihood estimation (MLE) as the parameter estimator. The log-likelihood function has the form

\[
\ln L(C_j, G_j, M_j, Y_j, \eta_j) = \sum_{i=1}^{n} \ln P(\tilde{X}_{t_{m+1}}^j = u|\tilde{X}_{t_m}^j = x).
\]

4.4.1.2 Algorithms for 2-D parameter estimations

The parameter for the 2-D process might be estimated by a 2-D MLE method. However, have being noted that there are 11 parameters need to be estimated, we use an
alternative and simplified approach in this part, that is to reduce the 2-D estimation problem to 1-D problems.

Due to the linear transform $A$, we assume the log-return of asset one $\ln S_1^t$ follows a 1-D CGMYe process and the log-return of asset two $\ln S_2^t$ depends on this process. That is

$$\ln S_1^t \sim \tilde{X}_1^t, \quad \text{and} \quad \ln S_2^t \sim \tilde{X}_2^t + \rho \tilde{X}_1^t,$$

where $\tilde{X}_1^t$ and $\tilde{X}_2^t$ are independent processes.

In the 2-D estimation, we first use the 1-D MLE to predict the parameters for $\tilde{X}_1^t$, which is described as above. Once the parameters for $\tilde{X}_1^t$ are obtained, the parameters of $\tilde{X}_2^t$ and $\rho$ can be estimated via the MLE based on the marginal transition density

$$P(Z_{t_{m+1}}^2 = u | Z_{t_m}^2 = v) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty+i\xi_2^t} e^{i\xi_1^t w + i\xi_2^t (v-u) + \Delta t \Psi(\xi)} d\xi_1 d\xi_2 dw,$$

where $\Psi(\xi)$ is given in Eq. (4.9). Note here the transition density can be evaluated via 2-D FFT and a numerical quadrature approach.

### 4.4.1.3 MLE of the stock markets

For illustration, we choose the 5 single stocks from the NASDAQ Global Select (NasdaqGS) market and the New York Stock Exchange (NYSE), and the real data are chosen from January 2012 to December 2014. In the following examples, let the time step $\Delta t = 3/754$. Assume a stock follows the 1-D CGMYe process, the parameters $C, G, M, Y, \eta$ are then estimated via the method mentioned in Section 4.4.1.1. For the 2-D problem, we are only need to focus on the estimation of parameters for the second component, which can be done by the method described in Section 4.4.1.2.

The results of 1-D estimation are presented in Table 4.1 and a part of the results for the 2-D estimation are given in Tables 4.2 and 4.3. We choose AAPL and MSFT as the first component in Tables 4.2 and 4.3 respectively. Since the parameters of the
first component are already given in Table 4.1, we here only list the parameter values of the second components and the linear combination coefficient $\rho$.

Additionally, the results of the 1-D and 2-D estimations are plotted in Figure 4.1. Figure 4.1(a) illustrates the transition densities of the CGMYe processes and the Brownian motion, with the parameters being the MLE results. The histogram represents the daily changes of the stock prices from the market data. For a comparison purpose, we have scaled the histogram so that it has a total area one. Figure 4.1(b) gives the results of the 2-D MLE. This figure includes the marginal distribution of the 2-D CGMYe model for the second component, which is YHOO in this example, and the histogram is again scaled as in part(a).

From these figures, it is clear that, the CGMYe process is leptokurtic and heavy tailed, which makes it more accurate than the normal distribution assumed in the Black-Scholes model.

### Table 4.1: Results of 1-D estimation for stock markets

<table>
<thead>
<tr>
<th>Stock</th>
<th>C</th>
<th>G</th>
<th>M</th>
<th>Y</th>
<th>$\eta$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAPL</td>
<td>0.1464</td>
<td>11.6047</td>
<td>22.5197</td>
<td>1.3495</td>
<td>-0.0927</td>
<td>-9.10e-4</td>
<td>0.0169</td>
</tr>
<tr>
<td>IBM</td>
<td>1.0773</td>
<td>28.1160</td>
<td>37.5846</td>
<td>0.6444</td>
<td>-0.1145</td>
<td>-1.19e-4</td>
<td>0.0110</td>
</tr>
<tr>
<td>INTC</td>
<td>0.3893</td>
<td>34.9065</td>
<td>45.2965</td>
<td>1.1927</td>
<td>0.0000</td>
<td>-6.58e-4</td>
<td>0.0135</td>
</tr>
<tr>
<td>MSFT</td>
<td>0.0776</td>
<td>25.7883</td>
<td>23.5054</td>
<td>1.4631</td>
<td>0.0000</td>
<td>-8.44e-4</td>
<td>0.0137</td>
</tr>
<tr>
<td>YHOO</td>
<td>0.4166</td>
<td>21.7388</td>
<td>38.3684</td>
<td>1.1841</td>
<td>-0.1236</td>
<td>-1.50e-3</td>
<td>0.0171</td>
</tr>
</tbody>
</table>

#### 4.4.1.4 MLE of different market indices

In the following example, we choose the real data from 6 different market indices, i.e. the DAX, the FTSE, the HSI, the IXIC, the N225 and the S&P 500, through the same time period as the last example, that is from January 2012 to December 2014. The time step $\Delta t$ here is set to be 3/700. Then the 1-D parameter estimation
Figure 4.1: Comparison between CGMY_e process and the normal distribution
Table 4.2: Results of 2-D estimation on AAPL

<table>
<thead>
<tr>
<th>$S_2$</th>
<th>$S_1$: AAPL</th>
<th>$C$</th>
<th>$G$</th>
<th>$M$</th>
<th>$Y$</th>
<th>$\eta$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IBM</td>
<td>1.1609</td>
<td>29.7660</td>
<td>42.9640</td>
<td>0.6451</td>
<td>-0.0943</td>
<td>0.2017</td>
<td></td>
</tr>
<tr>
<td>INTC</td>
<td>0.4268</td>
<td>35.8290</td>
<td>48.5392</td>
<td>1.1702</td>
<td>0.0000</td>
<td>0.1557</td>
<td></td>
</tr>
<tr>
<td>MSFT</td>
<td>0.0846</td>
<td>27.8262</td>
<td>26.9715</td>
<td>1.4474</td>
<td>0.0000</td>
<td>0.1105</td>
<td></td>
</tr>
<tr>
<td>YHOO</td>
<td>0.4765</td>
<td>22.4163</td>
<td>40.0432</td>
<td>1.1384</td>
<td>-0.1273</td>
<td>0.1530</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: Results of 2-D estimation on MSFT

<table>
<thead>
<tr>
<th>$S_2$</th>
<th>$S_1$: MSFT</th>
<th>$C$</th>
<th>$G$</th>
<th>$M$</th>
<th>$Y$</th>
<th>$\eta$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAPL</td>
<td>0.1467</td>
<td>11.4544</td>
<td>21.9984</td>
<td>1.3287</td>
<td>-0.0949</td>
<td>0.2116</td>
<td></td>
</tr>
<tr>
<td>IBM</td>
<td>1.1749</td>
<td>31.4743</td>
<td>42.2061</td>
<td>0.6183</td>
<td>-0.0796</td>
<td>0.3558</td>
<td></td>
</tr>
<tr>
<td>INTC</td>
<td>0.4102</td>
<td>34.5509</td>
<td>45.1013</td>
<td>1.1206</td>
<td>0.0000</td>
<td>0.4613</td>
<td></td>
</tr>
<tr>
<td>YHOO</td>
<td>0.4424</td>
<td>21.6271</td>
<td>37.1532</td>
<td>1.1430</td>
<td>-0.1185</td>
<td>0.2575</td>
<td></td>
</tr>
</tbody>
</table>

and the 2-D estimation with S&P 500 index data as the first component are shown in Table 4.4 and 4.5 respectively.

4.4.2 Evaluating European basket put options by FFT

There are many kinds of multi-asset options due to the different payoffs. Here we take the most popular one, European basket option, as an example for illustration.

Considering a typical multi-asset option, basket option. This option has payoff that depend upon the average of two or more asset prices, and are frequently used in currency hedging. A commonly traded basket option is a vanilla call or put option on a linear combination of assets. Suppose $S_i^t, i = 1, ..., n$ are the prices of $n$ stocks at time $t$, and $a_i, i = 1, ..., n$ are constants. Then the stock basket can be simply
Table 4.4: Results of 1-D estimation for market indices

<table>
<thead>
<tr>
<th>Stock</th>
<th>CGMYe</th>
<th>Black-Scholes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>G</td>
</tr>
<tr>
<td>DAX</td>
<td>11227.1117</td>
<td>191.1351</td>
</tr>
<tr>
<td>FTSE</td>
<td>1521.1081</td>
<td>238.4759</td>
</tr>
<tr>
<td>HSI</td>
<td>1705.8587</td>
<td>208.9333</td>
</tr>
<tr>
<td>IXIC</td>
<td>273.3721</td>
<td>177.0295</td>
</tr>
<tr>
<td>N225</td>
<td>15.1694</td>
<td>82.2890</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>86.3049</td>
<td>183.4030</td>
</tr>
</tbody>
</table>

Table 4.5: Results of 2-D estimation on S&P 500 index

<table>
<thead>
<tr>
<th>$S_2$</th>
<th>$S_1$: S&amp;P 500 index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
</tr>
<tr>
<td>DAX</td>
<td>12184.9844</td>
</tr>
<tr>
<td>FTSE</td>
<td>1504.2677</td>
</tr>
<tr>
<td>HSI</td>
<td>1711.2522</td>
</tr>
<tr>
<td>IXIC</td>
<td>278.0389</td>
</tr>
<tr>
<td>N225</td>
<td>15.1694</td>
</tr>
</tbody>
</table>

written as

$$U_t = \sum_{i=1}^{n} a_i S^i_t,$$  \hspace{1cm} (4.31)

and the payoff of the option at expiry date $T$ is

$$\max(\delta(U_T - K), 0),$$  \hspace{1cm} (4.32)

where $K$ is the strike price, $\delta = 1$ for a call option, and $\delta = -1$ for a put option.

In the following example, we set $\delta = -1$, $n = 2$, and $a_1 = 0.8$, $a_2 = 0.2$ for $S^1 = S_{INTC}$, $S^2 = S_{IBM}$. Moreover, let $T = 1$, $r = 0.05$, $K = a_1 S_{INTC} + a_2 S_{IBM}$.

The results of European basket put option values are plotted in figure 4.2.
Figure 4.2: European basket put option evaluation
Chapter 5

Conclusion and future studies

5.1 Conclusion of the thesis

Option pricing models with Lévy processes are considered to be superior to the classical Black-Scholes model since they incorporate extra market behaviors. Unlike the Black-Scholes model, however, there is much less known about the pricing formulas for these models.

In this thesis, we concentrate on two typical 1-D option pricing models under the general exponential Lévy processes, namely the finite moment log-stable (FMLS) model and the the Carr-Geman-Madan-Yor-eta (CGMYe) model, and we also propose a multi-asset CGMYe model. Both framework and numerical simulations are studied in the present work.

In the analytical studies, we derive the FPDEs and the fundamental solutions of European option prices for FMLS model and CGMYe model by using the complex Fourier transform method in Section 2.1 and 3.1. Comparing the two one-dimensional models, FMLS model has only left fractional derivative in its FPDE, while CGMYe model follows a more general process, and hence has both left and right fractional derivative in its equation.

As an extension of the European options, we also focus on the framework of American option pricing problems for both 1-D FMLS and CGMYe models in Section 2.2 and 3.2. We show that the American option price has an integral representation, and give the Doob-Meyer decomposition formulas of the American option values as
a sum of the corresponding European option values and an early exercise premiums. We also derive the integral equations of the optimal-exercise boundaries. Further properties for American option prices, such as the near-expiry behaviors and the analytical approximation for both long and short maturities, are also developed.

Moreover, we extend the 1-D CGMYe model to a multi-variate case by using affine transformation method. Similar framework, such as the derivation of the fundamental solution and the FPDE for European put options, are provided for the multi-asset CGMYe model in Section 4.1. The general case of American options are also mentioned in Section 4.2.

Numerical simulations are given right behind the framework of each model. We use Gauss-Jacobi spectral method to simulate the fractional derivatives and evaluate the option values for FMLS model in Section 2.4. Numerical evidence suggests that the Gauss-Jacobi spectral method yields more consistent approximations than the finite difference method does. Besides studying the effect of the order of fractional derivative to option values, we demonstrate the flexibility of the proposed method with a problem featuring a stochastic interest rate and an American option pricing.

For 1-D CGMYe model, we use the Gauss-Chebyshev quadrature to evaluate the integrals in the decomposition formula for European option prices, and apply the collocation method and the secant method to solve the integral equation of the optimal-exercise boundary for American options, the results are shown in Section 3.3. Our numerical examples show the influence of different choice of $C$ and $Y$ on the European options; and also compare the the put option values between CGMYe model and Black-Scholes model for the optimal-exercise boundaries. The numerical evidence suggests that the CGMYe model is more reliable and closer to the reality than the traditional Black-Scholes model.

For multi-asset CGMYe model, we use the statistical estimation method for the parameter estimations, and compare the transition densities between CGMYe process
and the normal distribution. The CGMYe process is leptokurtic and heavy tailed, which makes it more accurate than the normal distribution assumed in the Black-Scholes model. With these estimated parameters, we evaluate a 2-asset European basket put option through FFT method. The numerical estimations and simulations for the multi-variate CGMYe model are shown in Section 3.3.

5.2 Description of future works

In the last section of the thesis, we shall describe our future studies that based on our current work.

5.2.1 Fast algorithm of numerical simulations for American options

A very important part of our future works is on numerical methods.

Although there are several numerical methods are proposed to the American option pricing problem under the CGMY model, see Almendral and Oosterlee [2], these methods are time-consuming due to the fractional order. In the future work, we shall continue to study the FPDE of CGMYe models and find an efficient numerical scheme for the American pricing problems.

The main idea is to adopt the fast alternating-direction implicit finite difference method to solve the target FPDE. This method was introduced by Wang and Wang (2011) in [97], it has computational work count of $O(N \log N)$ per time step for 1-D model and $O(N \log^2 N)$ for 2-D model, and a memory requirement of $O(N)$, while retaining the same accuracy as a traditional finite difference method with Gaussian elimination. Numerical examples demonstrate that, this method significantly reduces the CPU time from more than 2 months to 1.5 hours, and it requires only less than one thousandth of memory the standard method does. It is worth acknowledging
that, beyond the computational advantages, the values of fractional orders need not to be the same, which lead to the solution of simple operator-stable densities with orthogonal eigenvectors and no multi-dimensional skewness.

Here, we present the analytical studies of this fast numerical method for 2-D CGMYe model. Recall the FPDE of European option values under the multi-variate CGMYe model

\[
\frac{\partial V(z,t)}{\partial t} + \frac{\eta_1^2}{2} \frac{\partial^2 V(z,t)}{\partial z_1^2} + \frac{\eta_2^2}{2} \frac{\partial^2 V(z,t)}{\partial z_2^2} + b_1 \frac{\partial V(z,t)}{\partial z_1} + b_2 \frac{\partial V(z,t)}{\partial z_2} + I + II
\]

\[
= \left[ r + C_1 \Gamma(-Y_1)(M_1^{Y_1} + G_1^{Y_1}) + C_2 \Gamma(-Y_2)(M_2^{Y_2} + G_2^{Y_2}) \right] V(z,t),
\]

where

\[
I = C_1 \Gamma(-Y_1) \left[ e^{-G_1 z_1} D_{-\infty}^{Y_1} (e^{G_1 z_1} V(z,t)) + e^{M_1 z_1} D_{\infty}^{Y_1} (e^{-M_1 z_1} V(z,t)) \right],
\]

\[
II = C_2 \Gamma(-Y_2) \left[ e^{-G_2 z_2} D_{-\infty}^{Y_2} (e^{G_2 z_2} V(z,t)) + e^{M_2 z_2} D_{\infty}^{Y_2} (e^{-M_2 z_2} V(z,t)) \right].
\]

If we further truncate the unbounded domain to a rectangular domain \(\Omega\) with \(\Omega = (z_{1,L}, z_{1,R}) \times (z_{2,L}, z_{2,R})\), then the fractional derivatives \(-\infty D_{z_j}^{Y_j} (e^{G_j z_j} V(z,t))\) and \(z_j D_{\infty}^{Y_j} (e^{-M_j z_j} V(z,t))\), \(j = 1, 2\) can be defined in the (computational feasible)
Gr"unwald-Letnikov form

\[ -\infty \mathcal{D}_{z_1}^Y_1 (e^{G_1 z_1 V(z,t)}) := \frac{\partial^Y_1 U_{1,L}(z,t)}{\partial_{z_1}^Y_1} = \lim_{h_1 \to 0^+} \frac{1}{h_1} \sum_{k=0}^{\lfloor (z_1 - z_{1,L})/h_1 \rfloor} g^{(Y_1)}_k U_{1,L}(z_1 - kh_1, z_2, t), \]

\[ z_1 \mathcal{D}_{\infty}^Y_1 (e^{-M_1 z_1 V(z,t)}) := \frac{\partial^Y_1 U_{1,R}(z,t)}{\partial_{z_1}^Y_1} = \lim_{h_1 \to 0^+} \frac{1}{h_1} \sum_{k=0}^{\lfloor (z_1, R - z_{1})/h_1 \rfloor} g^{(Y_1)}_k U_{1,R}(z_1 + kh_1, z_2, t), \]

\[ -\infty \mathcal{D}_{z_2}^Y_2 (e^{G_2 z_2 V(z,t)}) := \frac{\partial^Y_2 U_{2,L}(z,t)}{\partial_{z_2}^Y_2} = \lim_{h_2 \to 0^+} \frac{1}{h_2} \sum_{k=0}^{\lfloor (z_2 - z_{2,L})/h_2 \rfloor} g^{(Y_2)}_k U_{2,L}(z_1, z_2 - kh_2, t), \]

\[ z_2 \mathcal{D}_{\infty}^Y_2 (e^{-M_2 z_2 V(z,t)}) := \frac{\partial^Y_2 U_{2,R}(z,t)}{\partial_{z_2}^Y_2} = \lim_{h_2 \to 0^+} \frac{1}{h_2} \sum_{k=0}^{\lfloor (z_2, R - z_{2})/h_2 \rfloor} g^{(Y_2)}_k U_{2,R}(z_1, z_2 + kh_2, t), \]

where

\[ U_{j,L}(z,t) = e^{G_j z_j V(z,t)}, \quad \text{and} \quad U_{j,R}(z,t) = e^{-M_j z_j V(z,t)}, \]

for \( j = 1, 2 \), \( \lfloor z_j \rfloor \) represents the floor of \( z_j \), and

\[ g^{(Y_j)}_k = (-1)^k (Y_j k), \quad (5.1) \]

with \( (Y_j k) \) being the fractional binomial coefficients.

5.2.1.1 Overview of the implicit finite difference method

Meerschaert and Tadjeran [66, 67] showed that a fully implicit finite difference scheme with a direct truncation of the series in \((5.1)\) turns out to be unstable. Instead, they proposed to use a shifted Gr"unwald approach and proved the convergence and unconditional stability of the corresponding finite difference method.
Assume $N_j$ and $M$ be positive integers, and set $h_j = (z_{j,R} - z_{j,L})/(N_j + 1)$ and $\Delta t = T/M$ be the sizes of spatial grid and time step respectively. Here we define a spatial and temporal partition $z_{1,p} = z_{1,L} + ph_1$ for $p = 0, 1, \ldots, N_1+1$, $z_{2,q} = z_{2,L} + qh_2$ for $q = 0, 1, \ldots, N_2 + 1$, and $t^m = m\Delta t$ for $m = 0, 1, \ldots, M$. By applying the shifted Grünwald approach, we further let

\[
V_{p,q}^m = V(z_{1,p}, z_{2,q}, t^m),
\]

\[
U_{1,L,p,q}^m = U_{1,L}(z_{1,p}, z_{2,q}, t^m) = e^{G_{1z_1,p}}V_{p,q}^m,
\]

\[
U_{1,R,p,q}^m = U_{1,R}(z_{1,p}, z_{2,q}, t^m) = e^{-M_{1z_1,p}}V_{p,q}^m,
\]

\[
U_{2,L,p,q}^m = U_{2,L}(z_{1,p}, z_{2,q}, t^m) = e^{G_{2z_2,q}}V_{p,q}^m,
\]

\[
U_{2,R,p,q}^m = U_{2,R}(z_{1,p}, z_{2,q}, t^m) = e^{-M_{2z_2,q}}V_{p,q}^m,
\]

\[
d_{p,q}^{+,m} = d_+(z_{1,p}, z_{2,q}, t^m) = C_1\Gamma(-Y_1)e^{-G_{1z_1,p}}
\]

\[
d_{p,q}^{-,m} = d_-(z_{1,p}, z_{2,q}, t^m) = C_1\Gamma(-Y_1)e^{M_{1z_1,p}}
\]

\[
e_{p,q}^{+,m} = e_+(z_{1,p}, z_{2,q}, t^m) = C_2\Gamma(-Y_2)e^{-G_{2z_2,q}}
\]

\[
e_{p,q}^{-,m} = e_-(z_{1,p}, z_{2,q}, t^m) = C_2\Gamma(-Y_2)e^{M_{2z_2,q}}
\]

\[
W = r + C_1\Gamma(-Y_1)(M_{1Y_1} + G_{1Y_1}) + C_2\Gamma(-Y_2)(M_{2Y_2} + G_{2Y_2}).
\]

Then the implicit finite difference method can be written as

\[
\frac{V_{p,q}^m - V_{p,q}^{m-1}}{\Delta t} + \frac{\eta_2^2 V_{p+1,q}^m + V_{p-1,q}^m - 2V_{p,q}^m}{h_1^2} + \frac{\eta_2^2 V_{p,q+1}^m + V_{p,q-1}^m - 2V_{p,q}^m}{h_2^2}
\]

\[
+ b_1 \frac{V_{p+1,q}^m - V_{p-1,q}^m}{2h_1} + b_2 \frac{V_{p,q+1}^m - V_{p,q-1}^m}{2h_2}
\]

\[
+ \frac{d_{p,q}^{+,m}}{h_1} \sum_{k=0}^p g_k Y_1 U_{1,L,p-k+1,q}^m + \frac{d_{p,q}^{-,m}}{h_1} \sum_{k=0}^{N_1-p+1} g_k Y_1 U_{1,R,p+k-1,q}^m
\]

\[
+ \frac{e_{p,q}^{+,m}}{h_2} \sum_{l=0}^q g_l Y_2 U_{2,L,p,q-l+1}^m + \frac{e_{p,q}^{-,m}}{h_2} \sum_{l=0}^{N_2-q+1} g_l Y_2 U_{2,R,p,q+l-1}^m
\]

\[
= W V_{p,q}^m, \quad 1 \leq p \leq N_1, \quad 1 \leq q \leq N_2, \quad 1 \leq m \leq M. \quad (5.2)
\]

Moreover, let $N = N_1N_2$, $I$ be the $N$-by-$N$ identity matrix, and $V^m$ be an $N$-dimensional vector given by

\[
V^m = [V_{1,1,1}^m, \ldots, V_{N_1,1}^m, V_{1,2,1}^m, \ldots, V_{N_1,2}^m, \ldots, V_{1,N_2}^m, \ldots, V_{N_1,N_2}^m]^t. \quad (5.3)
\]
The finite difference scheme [5,2] can be hence transformed to the following matrix form
\[
(I + \Delta t B^m)V^m = V^{m-1}.
\] (5.4)

The N-by-N matrix \( B^m \) is expressed as an \( N_2 \)-by-\( N_2 \) block matrix with each \( N_1 \)-by-\( N_1 \) block \( B^m_{p,q} \) given by
\[
(B^m_{p,q})_{p,p} = \left( r_{p,q}^+ e^{G_{11,p,q}} + r_{p,q}^- e^{-M_{11,p,q}} \right) g_1^m Y_1
+ \left( s_{p,q}^+ e^{G_{22,p,q}} + s_{p,q}^- e^{-M_{22,p,q}} \right) g_2^m Y_2
- \frac{\eta_1^2}{h_1^2} - \frac{\eta_2^2}{h_2^2} - W,
\]
\[
(B^m_{p,q})_{p,p-1} = r_{p,q}^+ e^{G_{11,p,q}} g_1^m Y_1
+ r_{p,q}^- e^{-M_{11,p,q}} g_0^m Y_1
+ \frac{\eta_1^2}{2h_1^2} - \frac{b_1}{2h_1},
\]
\[
(B^m_{p,q})_{p,p+1} = r_{p,q}^+ e^{G_{11,p,q}} g_0^m Y_1
+ r_{p,q}^- e^{-M_{11,p,q}} g_2^m Y_1
+ \frac{\eta_1^2}{2h_1^2} + \frac{b_1}{2h_1},
\]
\[
(B^m_{p,q})_{p,k} = r_{p,q}^+ e^{G_{11,p,q}} g_{p-k+1}^m Y_1,
\]
\[
(B^m_{p,q})_{p,k} = r_{p,q}^- e^{-M_{11,p,q}} g_{p-k+1}^m Y_1,
\]
\[
(B^m_{q+1,p,q})_{p,k} = \delta_{p,k} \left( s_{p,q}^+ e^{G_{22,p,q}} g_2^m Y_2
+ s_{p,q}^- e^{-M_{22,p,q}} g_0^m Y_2
+ \frac{\eta_2^2}{2h_2^2} - \frac{b_2}{2h_2} \right),
\]
\[
(B^m_{q,q+1})_{p,k} = \delta_{p,k} \left( s_{p,q}^+ e^{G_{22,p,q}} g_2^m Y_2
+ s_{p,q}^- e^{-M_{22,p,q}} g_0^m Y_2
+ \frac{\eta_2^2}{2h_2^2} + \frac{b_2}{2h_2} \right),
\]
\[
(B^m_{l,q})_{p,k} = \delta_{p,k} s_{p,q}^+ e^{G_{22,q-l+1,p,q}} g_{q-l+1}^m Y_2,
\]
\[
(B^m_{q,q})_{p,k} = \delta_{p,k} s_{p,q}^- e^{-M_{22,q-l+1,p,q}} g_{q-l+1}^m Y_2,
\]
where
\[
\delta_{p,k} = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k, \end{cases}
\]
and \( r_{p,q}^+; r_{p,q}^-; s_{p,q}^+; s_{p,q}^- \) are defined as
\[
r_{p,q}^+ = \frac{d_{p,q}^+}{h_1^2}, \quad r_{p,q}^- = \frac{d_{p,q}^-}{h_1^2}, \quad s_{p,q}^+ = \frac{e_{p,q}^+}{h_2^2}, \quad \text{and} \quad s_{p,q}^- = \frac{e_{p,q}^-}{h_2^2}.
\]

**5.2.1.2 The fast alternating-direction implicit (ADI) finite difference method**

The fast ADI method developed by Wang and Wang in [97] has a computational work of \( O(N \log^2 N) \) per time step along with the improved memory requirement of \( O(N) \).
This fast ADI finite difference method can be described as: Let \( m = 1, 2, \ldots, M \), then at time point \( t^m \),

- Step 1: for each fixed \( z_{2,q} \), solve the following equation in \( z_1 \)-direction for an intermediate solution \( V^{m,*}_{p,q} \)

\[
V^{m,*}_{p,q} = \Delta t W V^m_{p,q} + \Delta t \frac{\eta_1^2}{2} \frac{V^m_{p+1,q} + V^m_{p-1,q} - 2V^m_{p,q}}{h_1^2} + \Delta t b_V V^m_{p+1,q} - V^m_{p-1,q} \]

\[
+ \Delta t r^{+,m}_{p,q} \sum_{k=0}^{p} g_k^1 U^m_{1,L,p-k+1,q} + \Delta t r^{-,m}_{p,q} \sum_{k=0}^{N_1-p+1} g_k^1 U^m_{1,R,p+k-1,q} \]

\[
= V^{m-1}_{p,q} \quad 1 \leq p \leq N_1, \quad 1 \leq q \leq N_2, \quad (5.5)
\]

- Step 2: for each fixed \( z_{1,p} \), solve the following equation in \( z_2 \)-direction for an intermediate solution \( V^m_{p,q} \)

\[
V^m_{p,q} = \Delta t W V^m_{p,q} + \Delta t \frac{\eta_2^2}{2} \frac{V^m_{p,q+1} + V^m_{p,q-1} - 2V^m_{p,q}}{h_2^2} + \Delta t b_V V^m_{p,q+1} - V^m_{p,q-1} \]

\[
+ \Delta t s^{+,m}_{p,q} \sum_{l=0}^{q} g_l^2 U^m_{2,L,p,q-l+1} + \Delta t s^{-,m}_{p,q} \sum_{l=0}^{N_2-q+1} g_l^2 U^m_{2,R,p,q+l-1} \]

\[
= V^{m-*}_{p,q} \quad 1 \leq q \leq N_2, \quad 1 \leq p \leq N_1. \quad (5.6)
\]

If we further let \( I_{N_1} \) be the \( N_1 \)-by-\( N_1 \) identity matrix, and

\[
V^m_q = [V^m_{1,q}, V^m_{2,q}, \ldots, V^m_{N_1,q}]', \quad \text{and} \quad V^{m,*}_q = [V^{m,*}_{1,q}, V^{m,*}_{2,q}, \ldots, V^{m,*}_{N_1,q}]'.
\]

The Eq. (5.5) can be rewritten as fully decoupled 1-D systems

\[
(I_{N_1} + \Delta t B^m_{q,z_1}) V^{m,*}_j = V^{m-1}_j, \quad 1 \leq q \leq N_2, \quad (5.7)
\]

and the matrix form of Eq. (5.6) can be complicated by counting index \( p \) first. Note that, the subsystem of Eq. (5.6) is of the same form with its 1-D analogue, which can be solved efficiently though the fast method introduced in [95].

The numerical examples need to be added based on the algorithm described above. Comparing with the current numerical method we used, \( \text{i.e.} \) solving the quintic integrals directly, this method will clearly simply the computation and reduce the complexity of our problems.
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