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Continuous methods for convex programming and convex semidefinite programming

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Continuous Methods for Convex Programming and Convex Semidefinite Programming

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A thesis submitted in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

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August 2017

DECLARATION

I hereby declare that this thesis represents my own work which has been done after registration for the degree of PhD at Hong Kong Baptist University, and has not been previously included in a thesis or dissertation submitted to this or any other institution for a degree, diploma or other qualifications.

I have read the Universitys current research ethics guidelines, and accept responsibility for the conduct of the procedures in accordance with the Universitys Committee on the Use of Human & Animal Subjects in Teaching and Research (HASC). I have attempted to identify all the risks related to this research that may arise in conducting this research, obtained the relevant ethical and/or safety approval (where applicable), and acknowledged my obligations and the rights of the participants.

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Abstract

In this thesis, we study several interior point continuous trajectories for linearly constrained convex programming (CP) and convex semidefinite programming (SDP). The continuous trajectories are characterized as the solution trajectories of corresponding ordinary differential equation (ODE) systems. All our ODE systems are closely related to interior point methods.

First, we propose and analyze three continuous trajectories, which are the solutions of three ODE systems for linearly constrained convex programming. The three ODE systems are formulated based on an variant of the affine scaling direction, the central path, and the affine scaling direction in interior point methods. The resulting solutions of the first two ODE systems are called generalized affine scaling trajectory and generalized central path, respectively. Under some mild conditions, the properties of the continuous trajectories, the optimality and convergence of the continuous trajectories are all obtained. Furthermore, we show that for the example of Gilbert *et al*. [Math. Program., **103**, 63-94 (2005)], where the central path does not converge, our generalized central path converges to an optimal solution of the same example in the limit.

Then we analyze two primal dual continuous trajectories for convex programming. The two continuous trajectories are derived from the primal-dual path-following method and the primal-dual affine scaling method, respectively. Theoretical properties of the two interior point continuous trajectories are fully studied. The optimality and convergence of both interior point continuous trajectories are obtained for any interior feasible point under some mild conditions. In particular, with proper choice of some parameters, the convergence for both continuous trajectories does not require the strict complementarity or the analyticity of the objective function.

For convex semidefinite programming, four interior continuous trajectories defined by matrix differential equations are proposed and analyzed. Optimality and convergence of the continuous trajectories are also obtained under some mild conditions.

We also propose a strategy to guarantee the optimality of the affine scaling algorithm for convex SDP.

Keywords: Ordinary differential equation; Interior point method; Continuous trajectory; Affine scaling; Convex programming; Convex semidefinite programming.

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Chapter 1

Introduction

In this thesis, we focus on the studies of interior point continuous trajectories for linearly constrained convex programming and linearly constrained convex semidefinite programming. In optimization, interior point method is an important method because of the excellent performance and polynomial complexity in many interior point algorithms. The central path which is a continuous trajectory plays a vital role in the interior point method. Many primal dual path following interior point algorithms follow the central path. The central path can be defined by setting up the corresponding barrier function, or defined by the solution trajectory of corresponding ordinary differential equation (ODE) system. In fact, many discrete algorithms are iterative algorithms, and there exists a search direction at each iterate. Generally, a vector field can be defined by the search direction, and then many solution trajectories can be generated. For instance, the affine scaling continuous trajectory is generated in this way. The studies of the properties of such solution trajectory can give deep insight to the properties of the corresponding search direction. Moreover, such solution trajectory is closely related to the corresponding discrete algorithm since the solution trajectory can be regarded as being generated by the discrete algorithm using infinitesimal step size, and on the contrary, the discrete algorithm can be regarded as an implementation of the corresponding ODE system. Furthermore, more discrete algorithms may be generated by many other implementations of the corresponding ODE system. In the following, we will introduce related research for convex programming and convex semidefinite programming respectively.

1.1 Convex Programming

In this thesis, we will consider the linearly constrained convex programming in two forms. One of them is

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & Ax = b,\ x_i \geq 0,\ i = 1, ..., s, \end{array} \tag{P_1}$$

where $x \in \mathbb{R}^n$, $0 \leq s \leq n$, $f(x)$ is smooth and convex over the feasible set, A is an m by n matrix with full row rank. As a blanket assumption, we assume that the optimal value for problem (P_1) is finite and attainable, therefore we use min rather than inf in problem (P_1).

The other form is

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & Ax = b,\ x \geq 0, \end{array} \tag{P_2}$$

where $f(x)$ is a smooth convex function, $b \in R^m$, and A is an $m \times n$ matrix with full row rank, $m < n$.

Problems (P_1) and (P_2) are actually equivalent in the sense that problem (P_1) contains problem (P_2) as a special case and on the other hand, problem (P_1) can be converted to problem (P_2) by introducing the auxiliary variables (Let $x_i = \tilde{x}_i - \bar{x}_i$ with $\tilde{x}_i \geq 0$ and $\bar{x}_i \geq 0$ for $s + 1 \leq i \leq n$). Hence, for convenient, we will study the primal continuous trajectory for problem (P_1) in Chapter 2 and study the primal affine scaling continuous trajectory and primal-dual continuous trajectory for problem (P_2) in Chapters 3 and 4.

First we introduce the following notations

$$\begin{array}{rl} \mathbb{R}^n_{s+} = \{x \in \mathbb{R}^n | x_i \geq 0,\ 1 \leq i \leq s\}, & \mathbb{R}^n_{s++} = \{x \in \mathbb{R}^n | x_i > 0,\ 1 \leq i \leq s\}, \\ \mathcal{P}_1^+ = \{x \in \mathbb{R}^n_{s+} | Ax = b\}, & \mathcal{P}_1^{++} = \{x \in \mathbb{R}^n_{s++} | Ax = b\}, \\ R^n_+ = \{x \in R^n | x \geq 0\}, & R^n_{++} = \{x \in R^n | x > 0\}, \\ \mathcal{P}_2^+ = \{x \in R^n | Ax = b,\ x \geq 0\}, \text{and} & \mathcal{P}_2^{++} = \{x \in R^n | Ax = b,\ x > 0\}. \end{array}$$

In Chapter 2, for problem (P_1), we will study the (interior point) continuous solution trajectory of the following ordinary differential equation (ODE) systems

$$\frac{dx}{dt} = -DP_{AD}D\nabla f(x),\ x(t_0) = x^0 \in \mathcal{P}_1^{++} \quad \text{and} \tag{1.1}$$

$$\frac{dx}{dt} = -\left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1} DP_{AD}D\nabla f(x),\ x(t_0) = x^0 \in \mathcal{P}_1^{++}, \tag{1.2}$$

where

$$\gamma_1 > 0,\ \frac{1}{2} \le \gamma_2 < 1,\ t_0 > 0,$$
$$x \in \mathbb{R}^n_{s++},\ d \in \mathbb{R}^n,\ \{d_i\}_{i=1}^s = \{x_i^{\gamma_2}\}_{i=1}^s,\ d_i = 1 \text{ for } i = s+1, \ldots, n,$$
$$X = \text{diag }(x) \in \mathbb{R}^{n\times n},\ D = \text{diag }(d) \in \mathbb{R}^{n\times n},$$
$$P_{AD} = I_n - DA^T(AD^2A^T)^{-1}AD,$$

and I_n stands for the $n \times n$ identity matrix.

First we explain where ODE system (1.1) comes from when $s = n$. The right-hand side of ODE system (1.1) was used by Tseng *et al.* in [73], who proposed a first-order interior point method for linearly constrained smooth optimization which unifies and extends the first-order affine scaling method and the replicator dynamics method (see [7]) for standard quadratic programming. In [73], the objective function is not necessarily convex. It was proved in [73] that every accumulation point is a stationary point under nondegeneracy assumption and the sequence will converge in the quadratic case without nondegeneracy assumption if $\gamma_2 < 1$. In Chapter 2, we restrict the power γ_2 to be in $[\frac{1}{2}, 1)$. As a matter of fact, when the power γ_2 equals 1, the right-hand side of ODE system (1.1) is just the primal affine scaling direction for problem (P_1) when $s = n$. In 1967, Dikin first proposed the affine scaling algorithm in [12]. Since then many researchers studied the affine scaling algorithm in many different ways. In the case of linear programming, see [13, 60, 72, 74, 75]; for the continuous affine scaling trajectories, see [1, 5, 34, 40, 45]. In the case of convex quadratic programming and more general convex programming, see [22, 51, 66, 67, 73, 84]. However the solution curve of ODE system (1.1) is not the affine scaling trajectory if $\gamma_2 < 1$. Hence we call the solution curve of ODE system (1.1) as the generalized affine scaling trajectory. When $s = n$, the affine scaling trajectory and the generalized affine scaling trajectory are contained in the Cauchy trajectories for convex semidefinite programming in [35], but there were no strong convergence results for the Cauchy trajectories there.

Now we explain where ODE system (1.1) comes from when $s < n$. When $s = n$, the matrix X^{γ_2} is like a barrier to prevent the trajectory from going into the nonpositive region. Hence for $s < n$, it is natural that we replace $x_i^{\gamma_2}$ with 1 for $i = s + 1, \ldots, n$. As a result we can get the ODE system (1.1) for problem (P_1). In fact, when the power γ_2 equals 1, the right-hand side of ODE system (2.1) is also the search direction in primal affine scaling algorithm for problem (P_1) when $s < n$ which was studied in [22]. But in order to get the optimality of the affine scaling algorithm, the nondegeneracy assumption is also needed in [22].

Hence if we regard the right-hand side of ODE system (1.1) as an extension of the search direction in the first-order interior point method in [73] for problem (P_1) where $s \leq n$, then ODE system (1.1) can be regarded as the continuous model for the first-order interior point method in the sense that the step size is sufficiently small. For this continuous model, some stronger results can be obtained than the discrete algorithm in [73] for the convex case. Actually, the optimality of ODE system (1.1) can be obtained without nondegeneracy assumption, and the convergence of the solution trajectory can be obtained for any convex objective function. Furthermore, thanks to some potential function used in the study of ODE system (1.1), the optimality of the corresponding discrete algorithm can be obtained without nondegeneracy assumption.

ODE system (1.2) is closely related to the central path. The central path can be defined by setting up the corresponding barrier function. Among plenty of barrier functions, the logarithm barrier function is mostly used, and the barrier function usually only involves the constraints (for example, if in (P_1), x_n is free, then the barrier function usually does not contain x_n). However, this type barrier may fail to converge for problem (P_1) when $s < n$. In [19], Gilbert *et al*. constructed an example where the objective function is infinitely differentiable but the central path fails to converge with the logarithm barrier function. In [29], Iusem *et al*. proved that under rather general hypotheses the central path defined by a general barrier for a monotone variational inequality problem is well defined, bounded, continuous and converges to the analytic center of the solution set. We note that there is no conflict between the results of [19] and [29] since in [29], the hypotheses on the barrier function is quite general, and may not only involve the constraints. However, we note that [29] did not discuss how to find a good barrier function such that the corresponding central path can converge.

The first in depth study of the central path is due to McLinden [39]. Then the central path has been studied in more details for linear and convex quadratic programming, such as Adler and Monteiro [1], Güler [23], Kojima *et al*. [32], Megiddo and Shub [40], Megiddo [41], Monteiro and Tsuchiya [48], Vavasis and Ye [78]. There are also several papers in the literature studying the central path for nonlinear programming, such as Drummond and Svaiter [14], Iusem *et al*. [29], McLinden [39], Monteiro and Zhou [52]. For the central path where the log-barrier function is used for problem (P_1) with $s = n$, the convergence can be guaranteed under the strict complementarity condition [39], or the analyticity of $f(x)$ [52], or the condition that there exists a subspace W of $\mathbb{R}^n$ such that $Ker(\nabla^2 f(x)) = W$ [14]. However these conditions may not be easy to verify or to be satisfied in practice.

Now we explain where ODE system (1.2) comes from when $s = n$. The central path with the barrier function $-\beta_1 \sum\limits_{i=1}^{n} x_i^{\alpha_1}$ $(0 < \alpha_1 < 1,\ \beta_1 > 0)$ can be defined as the homotopy solution of the KKT system of an auxiliary optimization problem as follows

$$\begin{cases} \nabla f(x) - z - A^T y = 0, \\ Ax = b, \ x > 0, \\ X^{1-\alpha_1} z = \mu \alpha_1 \beta_1 e, \ z > 0, \end{cases} \tag{1.3}$$

where parameter $\mu > 0$, $e = (1, \cdots, 1)^T \in \mathbb{R}^n$, and $X^{1-\alpha_1}$ is the power matrix in the usual sense of matrix analysis. In [52], the existence, properties, and the convergence of the central path are studied for a more general convex programming. In a recent paper [11], the central path and the affine scaling trajectory are studied for both linear programming and semidefinite programming from a dynamical system perspective. Denote the solution of system (1.3) as $(x(\mu), y(\mu), z(\mu))$. If $(x(\mu), y(\mu), z(\mu))$ exists for any $\mu > 0$,, then we obtain a trajectory of $(x(\mu), y(\mu), z(\mu))$ in terms of μ. By taking the derivative with respect to μ in (1.3), we obtain

$$\frac{dx}{d\mu} = \frac{1}{\mu^2} \left[\gamma_1 I_n + \frac{1}{\mu} D P_{AD} D \nabla^2 f(x) \right]^{-1} D P_{AD} D \nabla f(x), \ x(\mu_0) = x_0 \in \mathcal{P}_1^{++},$$

where $D = X^{\gamma_2}$, and $\gamma_1 = (1 - \alpha_1)\alpha_1\beta_1$, $\gamma_2 = 1 - \frac{\alpha_1}{2}$. Let $t = \frac{1}{\mu}$ with $t_0 = \frac{1}{\mu_0}$ in the above equation (for a general case of the conversion, see Section 2.2 in [34]), we have

$$\frac{dx}{dt} = - \left[\gamma_1 I_n + t D P_{AD} D \nabla^2 f(x) \right]^{-1} D P_{AD} D \nabla f(x), \ x(t_0) = x_0 \in \mathcal{P}_1^{++}, \tag{1.4}$$

The right-hand side of ODE system (1.4) is just that of ODE system (1.2) with $\gamma_1 > 0$ and $\frac{1}{2} < \gamma_2 < 1$. By using barrier function $-\beta_2 \sum_{i=1}^{n} x_i \ln x_i$ ($\beta_2 > 0$), we can similarly obtain ODE system (1.4) with $\gamma_1 = \beta_2$ and $\gamma_2 = \frac{1}{2}$. This explains where ODE system (2.2) comes from. It should be also mentioned that the solution of ODE system (**??**) defines the primal central path only if x_0 is on the primal central path. In fact, finding a (x_0, y_0, z_0) on the central path is not an easy task. However, for our ODE system (1.2), x_0 is only required in $\mathcal{P}_1^{++}$.

Next we explain where ODE system (1.2) comes from when $s < n$. When $s = n$, the matrix X^{γ_2} is like a barrier to prevent the trajectory from going into the nonpositive region. Hence for $s < n$, it is natural that we replace $x_i^{\gamma_2}$ with 1 for $i = s+1, \ldots, n$. As a result we can get ODE system (1.2) for problem (P_1). In fact, ODE system (1.2) can also be generated by proposing some specific barrier function. If $\frac{1}{2} < \gamma_2 < 1$, the barrier function takes

$$-\beta_1 \sum_{i=1}^{k} (x_i^{\alpha_1} + p_i x_i) + \frac{1-\alpha_1}{2} \sum_{i=k+1}^{n} (x_i - \bar{x}_i^0)^2, \tag{1.5}$$

where p_i and $\bar{x}_i^0$ may depend on x^0, $\gamma_1 = (1-\alpha_1)\alpha_1\beta_1$, and $\gamma_2 = 1 - \frac{\alpha_1}{2}$. If $\gamma_2 = \frac{1}{2}$, the barrier function takes

$$\beta_2 \sum_{i=1}^{n} (x_i \ln x_i + p_i x_i) + \frac{\beta_2}{2} \sum_{i=k+1}^{n} (x_i - \bar{x}_i^0)^2, \tag{1.6}$$

where p_i and $\bar{x}_i^0$ may depend on x^0, $\gamma_1 = \beta_2$, and $\gamma_2 = \frac{1}{2}$. The barrier function (1.5) does not satisfy the conditions (H12) in [29], however after we obtain the existence, uniqueness, and boundedness of these paths, the convergence can be proved similarly by the method in [29] if x^0 is on the primal central path (but here we only require x^0 in ${\mathcal{P}^{++}}_1$. Since we aim at proposing a new method, hence we omit them. In order to distinguish the usual central path, we call the solution trajectories of ODE system (1.2) (parameterized by γ_1, γ_2 and x^0) as the generalized central paths.

In Chapter 3, we will study the first-order primal affine scaling continuous trajectory for problem (P_2), which is the solution curve of the following ODE system

$$\frac{dx}{dt} = -XP_{AX}X\nabla f(x), \; x(t_0) = x^0 \in \mathcal{P}_2^{++}, \; t \geq t_0 > 0. \tag{1.7}$$

In fact, the first-order primal affine scaling continuous trajectory has already been studied for linear programming in [1], but not yet for convex programming (P_2). Compared with [1], in the linear case, we do not require the boundedness of the optimal solution set, instead, we only need the existence of a finite optimal solution. It should be noted that the first-order primal affine scaling trajectory is contained in the Cauchy trajectories for convex semidefinite programming [35], but there is no strong convergence result for the Cauchy trajectories there. To our knowledge, our result here is the first one to obtain the strong convergence of the primal affine scaling continuous trajectory in the nonlinear case for problem (P_2).

In Chapter 4, for problem (P_2), we will study the interior point continuous trajectories that are closely related to some interior point methods, specifically, the primal-dual path-following method, and the primal-dual affine scaling method for problem (P_2). For linearly constrained convex programming (including linear programming), a number of interior point primal-dual methods have been proposed, for example, the primal-dual path-following algorithms by Kojima *et al.* [31], Monteiro and Adler [42, 43], Renegar [57], Roos [59], Todd and Ye [68], Gonzaga [21], Monteiro [47], Zhu [86], the primal-dual affine scaling algorithms by Monteiro *et al.* [44], and so on. The primal-dual path-following method is closely related to the central path, however, in primal-dual path-following method, the search direction is actually different from the tangent of the central path. In Chapter 4, we will study a continuous trajectory which is derived from the search direction in the primal-dual path-following method. The continuous trajectory corresponding to the search direction of the primal-dual affine scaling method will also be studied in Chapter 4. In the linear case, the primal-dual affine scaling continuous trajectory has been studied in [1]. In the nonlinear convex case, since the primal-dual affine scaling continuous trajectory is actually a weighted primal-dual central path (see [1] or Theorem **??** below), hence part of the results in Chapter 4 can be regarded as contained in [52].

1.2 Semidefinite Programming

In Chapters 5 and 6, we will consider linearly constrained semidefinite programming. Let $\mathcal{S}^n$ denote the vector space of real symmetric $n \times n$ matrices. The standard inner product on $\mathcal{S}^n$ is

$$A \bullet B = \mathrm{tr}(AB) = \sum_{i,j} A_{ij} B_{ij}, \quad \mathrm{tr}(\cdot) = \mathrm{trace}(\cdot).$$

By $X \succeq 0$ ($X \succ 0$), where $X \in \mathcal{S}^n$, we mean that X is positive semidefinite (positive definite). Consider the following convex semidefinite programming (**SDP**) problem

$$\begin{array}{rl} \min\limits_{X \in \mathcal{S}^n} & f(X) \\ s.t. & A_k \bullet X = b_k, \ k = 1, \ldots, m, \\ & X \succeq 0, \end{array} \tag{P_3}$$

where $f : \mathcal{S}^n \to R$ is convex, $b \in R^m$, and $A_k \in \mathcal{S}^n$, $k = 1, \ldots, m$. As a blanket assumption, we assume that the optimal value for problem (P_3) is finite and attainable, therefore, we use min rather than inf in problem (P_3).

The following notations are used in our later discussions.

$$\begin{aligned} &\mathcal{S}^n_+ = \{X \in \mathcal{S}^n | X \succeq 0\}, \quad \mathcal{S}^n_{++} = \{X \in \mathcal{S}^n | X \succ 0\}, \\ &\mathcal{P}_3^+ = \{X \in \mathcal{S}^n | A_k \bullet X = b_k, \ k = 1, \ldots, m, \ X \succeq 0\}, \\ &\mathcal{P}_3^{++} = \{X \in \mathcal{S}^n | A_k \bullet X = b_k, \ k = 1, \ldots, m, \ X \succ 0\}. \end{aligned}$$

A comprehensive study of semidefinite programming can be found in [81]. There are many interior point algorithms for solving problem (P_3), for example, [2, 49, 50, 65, 69, 85] for linear $f(X)$, and [33, 54, 70, 71] for convex and quadratic $f(X)$. There are also some continuous methods for linear SDP, for example, a recurrent neural network for real-time SDP was proposed and studied in [30]. Many of the interior point algorithms for SDP are primal-dual path-following algorithms that are closely related to the central path [76]. In the linear case with $f(X) = \mathrm{tr}(CX)$ where $C \in \mathcal{S}^n$, the central path is the set of the solutions of the following system with the parameter $\mu > 0$ [81]

$$\begin{cases} A_k \bullet X = b_k, \ k = 1, \ldots, m, \\ \sum_{k=1}^m y_k A_k + Z = C, \\ XZ = \mu I, \ X \succeq 0, \ Z \succeq 0, \end{cases} \tag{1.8}$$

where I is the identity matrix. In [61], Shida and Shindoh studied the existence and convergence of the infeasible central path for the monotone semidefinite complementarity problem and showed that for the monotone semedefinite linear complementarity problem, the trajectory converges to the analytic center of the solution set provided that there exists a strictly complementary solution. Under the assumption of primal and dual strict feasibility, Goldfarb and Scheinberg [20] showed that the primal and dual central paths exist and converge to the analytic centers of the optimal faces of the primal and the dual problems, respectively. But later, Halická *et al.* [24] showed that the result is not correct in the absence of strict complementarity by a counterexample, where the central path converges to a different optimal solution, and they also gave a short proof that the central path always converges in SDP by using ideas from algebraic geometry. In [11], the central path in linear programming (LP) and SDP was also studied in ODE systems form. Furthermore, the study of limiting behavior of some infeasible weighted central paths for SDP can be found in [25, 37, 56]. There is also some research work on the central path for nonlinear SDP, for instance, [15, 16, 35]. For problem (P_3), López and Ramírez [35] showed the convergence of the central path where the logarithm barrier function is used under the analyticity of $f(X)$ by a similar method to [24], and other central paths defined with a large class of penalty and barrier functions were also studied there.

It should be noted that there have been some studies on other continuous trajectories. Sim and Zhao [62] studied the underlying paths in interior point methods for the monotone semidefinite linear complementarity problem. They showed that each off-central path is a well-defined analytic curve with parameter μ ranging over $(0, \infty)$ and any accumulation point of the off-central path is a solution. Furthermore they also studied the analyticity of the off-central path through a simple example. Then they investigated the asymptotic behavior of off-central paths for general semidefinite linear complementarity problems (using the dual HKM direction) under strict complementarity condition in [63]. The relationship between the interior point methods and the underlying paths is also discussed in [62].

In Chapter 5 we will study the interior point continuous trajectories for problem (P_3). In order to write down the equations explicitly, we need the following notations.

- Let svec map $\mathcal{S}^n$ to $R^{n(n+1)/2}$. If $U \in \mathcal{S}^n$, then svec(U) is defined by

$$\text{svec}(U) := (u_{11}, \sqrt{2}u_{21}, \ldots, \sqrt{2}u_{n1}, u_{22}, \sqrt{2}u_{32}, \ldots, \sqrt{2}u_{n2}, \ldots, u_{nn})^T.$$

- The symmetrized Kronecker product $\otimes_s$ is defined by

$$(G \otimes_s K)\text{svec}(H) = \frac{1}{2}\text{svec}(KHG^T + GHK^T),$$

where $G, K \in R^{n\times n}$ and $H \in \mathcal{S}^n$. The properties of operator $\otimes_s$ can be found in Appendix of [2] and [63].

- Let matrix $\mathcal{A}$ be defined as follows

$$\mathcal{A} = \begin{pmatrix} \text{svec}(A_1)^T \\ \vdots \\ \text{svec}(A_m)^T \end{pmatrix} \in R^{m\times n(n+1)/2}.$$

- For any $X \in \mathcal{S}^n$, let $\nabla^2 f(X)$ be the following matrix

$$\nabla^2 f(X) = \begin{pmatrix} \text{svec}(\frac{\partial(\frac{\partial f}{\partial X})_{11}}{\partial X})^T \\ \sqrt{2}\text{svec}(\frac{\partial(\frac{\partial f}{\partial X})_{21}}{\partial X})^T \\ \vdots \\ \sqrt{2}\text{svec}(\frac{\partial(\frac{\partial f}{\partial X})_{n1}}{\partial X})^T \\ \text{svec}(\frac{\partial(\frac{\partial f}{\partial X})_{22}}{\partial X})^T \\ \sqrt{2}\text{svec}(\frac{\partial(\frac{\partial f}{\partial X})_{32}}{\partial X})^T \\ \vdots \\ \sqrt{2}\text{svec}(\frac{\partial(\frac{\partial f}{\partial X})_{n2}}{\partial X})^T \\ \vdots \\ \text{svec}(\frac{\partial(\frac{\partial f}{\partial X})_{nn}}{\partial X})^T \end{pmatrix} \in R^{n(n+1)/2\times n(n+1)/2}.$$

Now we present the following four ordinary differential equation (ODE) systems

$$\mathrm{svec}(\dot{X}) = -\big(I - (X \otimes_s X)P_{\mathcal{A}X}\big)(X \otimes_s X)\mathrm{svec}(\frac{\partial f}{\partial X}), \tag{1.9}$$

$$\mathrm{svec}(\dot{X}) = -\Big(I + t\big(I - (X \otimes_s X)P_{\mathcal{A}X}\big)\big((X \otimes_s X)\nabla^2 f(X)\big)\Big)^{-1} \big(I - (X \otimes_s X)P_{\mathcal{A}X}\big)(X \otimes_s X)\mathrm{svec}(\frac{\partial f}{\partial X}), \tag{1.10}$$

$$\mathrm{svec}(\dot{X}) = -\big(I - (X \otimes_s X^{\frac{1}{2}})P_{\mathcal{A}X^{\frac{1}{2}}}\big)(X \otimes_s X^{\frac{1}{2}})\mathrm{svec}(\frac{\partial f}{\partial X}), \tag{1.11}$$

$$\mathrm{svec}(\dot{X}) = -\Big(I + t\big(I - (X \otimes_s X^{\frac{1}{2}})P_{\mathcal{A}X^{\frac{1}{2}}}\big)\big((X \otimes_s X^{\frac{1}{2}})\nabla^2 f(X)\big)\Big)^{-1} \big(I - (X \otimes_s X^{\frac{1}{2}})P_{\mathcal{A}X^{\frac{1}{2}}}\big)(X \otimes_s X^{\frac{1}{2}})\mathrm{svec}(\frac{\partial f}{\partial X}), \tag{1.12}$$

which have the same initial condition: $X(t_0) = X^0 \in \mathcal{P}_3^{++}$ and $t_0 > 0$, where

$$X \in \mathcal{S}^n_{++}, \quad X^{\frac{1}{2}} \in \mathcal{S}^n_{++} \text{ is the unique square root matrix of X,}$$
$$P_{\mathcal{A}X^\gamma} = \mathcal{A}^T(\mathcal{A}(X \otimes_s X^\gamma)\mathcal{A}^T)^{-1}\mathcal{A}, \ \gamma \in \{\frac{1}{2}, \ 1\},$$
$$I \text{ stands for the } \frac{n(n+1)}{2} \times \frac{n(n+1)}{2} \text{ identity matrix.}$$

The right-hand side of ODE (1.9) comes from the affine scaling direction for SDP in [53]. The right-hand side of ODE (1.10) comes from the central path. In fact, in the above central path system (1.8), if we replace the matrix C by $\frac{\partial f}{\partial X}$ and take the derivative with respect to μ, we can get

$$\mathrm{svec}(\frac{dX}{d\mu}) = \frac{1}{\mu^2}\Big(I + \frac{1}{\mu}(I - (X \otimes_s X)\mathcal{A}^T(\mathcal{A}(X \otimes_s X)\mathcal{A}^T)^{-1}\mathcal{A})((X \otimes_s X)\nabla^2 f(X))\Big)^{-1} (I - (X \otimes_s X)\mathcal{A}^T(\mathcal{A}(X \otimes_s X)\mathcal{A}^T)^{-1}\mathcal{A})(X \otimes_s X)\mathrm{svec}(\frac{\partial f}{\partial X}), \tag{1.13}$$

then we use the new variable t by setting $t = \frac{1}{\mu}$, and we have

$$\mathrm{svec}(\frac{dX}{dt}) = -\Big(I + t(I - (X \otimes_s X)\mathcal{A}^T(\mathcal{A}(X \otimes_s X)\mathcal{A}^T)^{-1}\mathcal{A})((X \otimes_s X)\nabla^2 f(X))\Big)^{-1} (I - (X \otimes_s X)\mathcal{A}^T(\mathcal{A}(X \otimes_s X)\mathcal{A}^T)^{-1}\mathcal{A})(X \otimes_s X)\mathrm{svec}(\frac{\partial f}{\partial X}), \tag{1.14}$$

which is exactly ODE system (1.10) except the initial points, where (1.8) or (1.14) requires the initial point must be on the central path but (1.10) only requires the initial point $X^0 \in \mathcal{P}_3^{++}$. ODEs (1.11) and (1.12) are some variants of ODEs (1.9) and (1.10), respectively. This kind of variants also exists in the linearly constrained smooth optimization [73]. In Chapter 5, we will show that the solutions of ODE systems (1.11) and (1.12) have stronger properties under weaker conditions than the solutions of ODE systems (1.9) and (1.10). It should be noted that the solutions of the four ODE systems define four interior point (verified in Chapter 5) continuous trajectories for problem (P_3).

In Chapter 6, we will study the affine scaling algorithm for problem (P_3). In [53], Muramatsu gave an example of linear SDP such that the affine scaling algorithm converges to a non-optimal point. In that example, for both the short and the long step version of the affine scaling algorithm, there exists a region of starting points such that the generated sequence converges to a non-optimal point. In the concluding remarks of [53], Muramatsu pointed that it may still be possible to prove the global convergence from well-chosen starting points, or by allowing variable step sizes. In Chapter 6, we focus on the second strategy – allowing variable step sizes, and propose a new step size rule which is similar to the Armijo-type rule [6]. Under this new rule of step size, we can prove that staring from any interior feasible point, every accumulation point of the affine scaling algorithm is an optimal solution of problem (P_3).

In Chapter 6, we will also consider a special case where X and A_i $(i = 1, ..., m)$ are all diagonal in (P_3) and with a slightly different but less restrictive step size rule. In this special case, problem (P_3) becomes linearly constrained convex programming which has been studied in [22] by affine scaling algorithm and in [73] through a first-order interior point method which includes the affine scaling algorithm as a special case. However, to insure the optimality, they both require the nondegeneracy assumptions, and the strong convergence of the affine scaling algorithm is still open.

1.3 Preliminaries

1.3.1 Some Lemmas

In this part, we introduce some technical lemmas which will be used in the following of the thesis.

Lemma 1.1. *If A, $B \in \mathbb{R}^{n\times n}$ are both symmetric and positive semidefinite, then all eigenvalues of AB are nonnegative.*

Proof. From Corollary 4.6.3 on page 99 in [79], the result is evident. □

Lemma 1.2. *([9]) Suppose f is differentiable (i.e., its gradient ∇f exists at each point in $\mathrm{dom} f$). Then f is convex if and only if $\mathrm{dom} f$ is convex and*

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \tag{1.15}$$

holds for all x, $y \in \mathrm{dom} f$.

Proof. See Section 3.1.3 in [9]. □

Lemma 1.3. *Let a be any positive constant and let $g(x) = x - a - a \cdot \ln \frac{x}{a}$. Then for any scalar $x > 0$, $g(x) \geq 0$ and $g(x) = 0$ if and only if $x = a$. Furthermore, $g(x) \to +\infty$ as $x \to 0^+$ or $x \to +\infty$.*

Proof. By taking the derivative of $g(x)$, we have

$$\frac{dg(x)}{dx} = \frac{x - a}{x}. \tag{1.16}$$

From (1.16), we can see $\frac{dg(x)}{dx} < 0$ if $0 < x < a$; and $\frac{dg(x)}{dx} > 0$ if $x > a$. So $g(x) > g(a) = 0$ if $x > 0$ and $x \neq a$.

As $x \to 0^+$, $x - a \to -a$ and $-a \cdot \ln \frac{x}{a} \to +\infty$, so we know $g(x) \to +\infty$. In addition, $\frac{dg(x)}{dx} > 0$ if $x > a$ and $\frac{dg(x)}{dx} \to 1$ as $x \to +\infty$. So it is not hard to see that $g(x) \to +\infty$ as $x \to +\infty$. The proof is complete. □

Lemma 1.4. *Let a be any positive constant and $1 < r < 2$. Then for any scalar $x > 0$, $g(x) = \frac{1}{2-r}(x^{2-r} - a^{2-r}) - \frac{a}{1-r}(\frac{1}{x^{r-1}} - \frac{1}{a^{r-1}}) \geq 0$ and $g(x) = 0$ if and only if $x = a$. Furthermore, $g(x) \to +\infty$ as $x \to 0^+$ or $x \to +\infty$.*

Proof. By taking the derivative of $g(x)$, we have

$$\frac{dg(x)}{dx} = \frac{x-a}{x^r}. \tag{1.17}$$

From (1.17), we can see that $\frac{dg(x)}{dx} < 0$ if $0 < x < a$; and $\frac{dg(x)}{dx} > 0$ if $x > a$. So $g(x) > g(a) = 0$ if $x > 0$ and $x \neq a$.

As $x \to 0^+$, $\frac{1}{2-r}(x^{2-r} - a^{2-r}) \to -\frac{a^{2-r}}{2-r}$ and $-\frac{a}{1-r}(\frac{1}{x^{r-1}} - \frac{1}{a^{r-1}}) \to +\infty$, so we know $g(x) \to +\infty$.

As $x \to +\infty$, $\frac{1}{2-r}(x^{2-r} - a^{2-r}) \to +\infty$ and $-\frac{a}{1-r}(\frac{1}{x^{r-1}} - \frac{1}{a^{r-1}}) \to \frac{a^{2-r}}{1-r}$, so we know that $g(x) \to +\infty$. The proof is complete. $\blacksquare$ $\square$

Lemma 1.5. *(Barbalat's Lemma [64]) If $f(t)$ is differentiable, $\lim_{t\to+\infty} f(t)$ exists, and $\dot{f}$ is uniformly continuous, then $\lim_{t\to+\infty} \dot{f}(t) = 0$.*

Lemma 1.6. *If $f(x)$ is convex and analytic, then for any two different optimal solutions x^1 and x^2 of problem (P_2), and any $x \in R^n$, $(x^2 - x^1)^T \nabla f(x) = 0$.*

Proof. Since $f(x)$ is convex, we have for any $\lambda \in [0, 1]$,

$$\begin{aligned} f(x^1 + \lambda\Delta x) &= f(\lambda x^2 + (1-\lambda)x^1) \\ &\leq \lambda f(x^2) + (1-\lambda) f(x^1) \\ &= f(x^1) = f(x^2), \end{aligned}$$

where $\Delta x = x^2 - x^1 \neq 0$. Moreover, x^1 and x^2 are two different optimal solutions for problem (P_2) and $\lambda x^2 + (1-\lambda)x^1 \in \mathcal{P}^+$ for any $\lambda \in [0, 1]$, hence

$$f(x^1 + \lambda\Delta x) = f(x^1) = f(x^2),$$

for any $\lambda \in [0, 1]$. Since $f(x)$ is analytic, then according to Corollary 1.2.5 in [55], we have for any $\lambda \in R$,

$$f(x^1 + \lambda\Delta x) = f(x^1) = f(x^2).$$

By Corollary 8.6.1 of Rockafellar [58], it follows that for any $x \in R^n$, $f(x+\lambda\Delta x)$ will be a constant function of $\lambda \in R$. Hence

$$\frac{df(x+\lambda\Delta x)}{d\lambda} = (x^2 - x^1)^T \nabla f(x+\lambda\Delta x) = 0,$$

for any $\lambda \in R$. Let $\lambda = 0$ in the above equality, we have for any $x \in R^n$, $(x^2 - x^1)^T \nabla f(x) = 0$. Thus the lemma is proved. □

1.3.2 Notation

For simplicity, in what follows, $\|\cdot\|$ or $\|\cdot\|_2$ denotes either the vector 2-norm or the matrix 2-norm, $\|\cdot\|_F$ denotes the Frobenius norm for the matrix. Let the j-th constituent of the vector x be x_j, vector whose elements are all equal to 1 be e, the vector whose elements are all zero except the ith element equal to 1 be e_i. C^k stands for the class of kth order continuously differentiable functions. Let x_J stand for the vector consist of the elements of $x \in R^n$ indexed by $j \in J$, for each index subset $J \subseteq \{1, \ldots, n\}$. Let I be the identity matrix, and the dimension of I is clear from the context. $diag(x)$ for the vector x denotes the diagonal matrix whose diagonal entries are that of x. $\text{rank}\,(Q)$ denotes the rank of matrix Q. For any $Q \in \mathcal{S}^n_+$, $\lambda_{\max}(Q)$ and $\lambda_{\min}(Q)$ denote the largest and smallest eigenvalues of Q, respectively.In the following, we may use some same notations in different chapter, and unless otherwise specified, the quotation of the same notation means the notation in the chapter where the quotation is.

1.4 Outline of the Thesis

The rest of the thesis is organized as follows. In Chapter 2, we propose and analyze the generalized affine scaling trajectory and generalized central path which are the solution of ODE systems (1.1) and (1.2) respectively for problem (P_1). Chapter 2 contains three sections. In the first section, we (i) introduce two potential functions for the two ODE systems (1.1) and (1.2), respectively; (ii) verify that each ODE system has a unique solution in $[t_0, +\infty)$; and (iii) introduce the weak convergence for system (1.1). In the second section, we prove that every accumulation point of the two continuous trajectories of ODE systems (1.1) and (1.2) is an optimal solution for problem (P_1). In the third section, we first show the strong convergence of the two continuous trajectories and then verify that each limiting point has the maximal number of the positive components in $\{x_1, \ldots, x_k\}$ among the optimal solutions. As a result, the weak convergence for the solution of ODE system (2.2) is obtained.

In Chapter 3, we study the primal affine scaling continuous trajectory for problem (P_2). Chapter 3 is composed of three sections. In the first section, we introduce the corresponding ordinary differential equation (ODE) system for the first-order primal affine scaling continuous trajectory, verify that the ODE system has a unique solution in $[t_0, +\infty)$, and show some properties of the primal affine scaling continuous trajectory. In the second section, we prove that every accumulation point of the primal affine scaling continuous trajectory is an optimal solution for problem (P_2). Finally, in the third section, we show the strong convergence of the first-order primal affine scaling continuous trajectory under the condition that $f(x)$ is analytic.

In Chapter 4, the weighted primal-dual path-following continuous trajectory and the extended primal-dual affine scaling continuous trajectory will be studied in two sections respectively. In each of these two sections, we divide our discussions into 3 subsections. In the first subsection, we introduce the corresponding ordinary differential equation (ODE) systems for each interior point continuous trajectory, verify that each ODE system has a unique solution in $[t_0, +\infty)$ and show some properties of each continuous trajectory. In the second subsection , we prove that every accumulation point of the solution of each ODE system is an optimal solution for problem (P_2) (and problem (D_2)). In the third subsection, the strong convergence of each system for any interior feasible point are given under some mild conditions.

In Chapter 5, we consider four primal interior point continuous trajectories for problem (P_3). Chapter 5 includes three sections. In the first section, we (i) introduce four potential functions for the four ODE systems (1.9), (1.10), (1.11), and (1.12), respectively; (ii) verify that each ODE system has a unique solution in $[t_0, +\infty)$; and (iii) introduce the weak convergence for system (5.1). In the second section, we prove that every accumulation point of the solutions of the four ODE systems is an optimal solution for problem (P_3), and verify the weak convergence for system (5.2). In the third section, we first show the strong convergence of the solutions of the last two ODE systems under certain conditions, and verify that each limiting point has the maximal rank among the optimal solution set of problem (P_3), then we prove the convergence for the solutions of the first two ODE systems in the linear case.

In Chapter 6, we study the primal affine scaling algorithm for problem (P_3). There are four sections. In the first section, we study some properties of the affine scaling direction. In the second section, we propose an affine scaling algorithm with a new step size rule which is similar to the Armijo-type rule. In the third section, we prove that any accumulation point of the affine scaling algorithm with the new step size rule is an optimal solution from any starting interior feasible point without nondegeneracy assumptions. In the fourth section, we consider a special case of problem (P_3) where X and A_i's are diagonal. A slightly different step size rule, which is less restrictive, is proposed for the affine scaling algorithm. For any accumulation point of the resulting algorithm, the optimality is obtained as well.

At last, in Chapter 7, we give concluding comments and future research. A discrete algorithm is proposed based on ODE system (1.2) for linearly constrained convex quadratic programming. Preliminary numerical results are also reported.

Chapter 2

Two Primal Interior Point Continuous Trajectories for Convex Programming

In this chapter, we propose and analyze two continuous trajectories, which are the solutions of two ordinary differential equation (ODE) systems for problem (P_1). The two ODE systems are formulated based on a variant of the affine scaling direction and the central path in interior point methods. The resulting solutions of the corresponding two ODE systems are called generalized affine scaling trajectory and generalized central path, respectively. By only assuming the existence of a finite optimal solution, we show that, from any given interior feasible point, (i) two continuous trajectories are convergent; and (ii) the limit points are indeed the optimal solution(s) of the original optimization problem. Furthermore, we show that for the example of Gilbert *et al*. [19], where the central path does not converge, our generalized central path converges to an optimal solution of the same example in the limit.

We restate the two ordinary differential equation (ODE) systems as follows

$$\frac{dx}{dt} = -DP_{AD}D\nabla f(x),\ x(t_0) = x^0 \in \mathcal{P}_1^{++} \quad \text{and} \tag{2.1}$$

$$\frac{dx}{dt} = -\left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1} DP_{AD}D\nabla f(x),\ x(t_0) = x^0 \in \mathcal{P}_1^{++}, \tag{2.2}$$

where

$$\gamma_1 > 0,\ \frac{1}{2} \leq \gamma_2 < 1,\ t_0 > 0,$$
$$x \in \mathbb{R}^n_{s++},\ d \in \mathbb{R}^n,\ \{d_i\}_{i=1}^s = \{x_i^{\gamma_2}\}_{i=1}^s,\ d_i = 1 \text{ for } i = s+1, \ldots, n,$$
$$X = \text{diag }(x) \in \mathbb{R}^{n \times n},\ D = \text{diag }(d) \in \mathbb{R}^{n \times n},$$
$$P_{AD} = I_n - DA^T(AD^2A^T)^{-1}AD,$$

and I_n stands for the $n \times n$ identity matrix. For ODE system (2.2), we sometimes use its equivalent implicit form

$$\frac{dx}{dt} = -\frac{1}{\gamma_1} DP_{AD}D \left[\nabla f(x) + t\nabla^2 f(x)\frac{dx}{dt}\right],\ x(t_0) = x^0 \in \mathcal{P}_1^{++}. \tag{2.3}$$

For ODE system (2.1), we require $\nabla f(x) \in C^1$ on $\mathbb{R}^n_{s+}$, and for ODE system (2.2), we require $\nabla^2 f(x) \in C^1$ on $\mathbb{R}^n_{s+}$.

2.1 Fundamental Properties of The Continuous Trajectories

The following assumption is made throughout this chapter.

Assumption 2.1. *There exists a point $x^* \in \mathcal{P}_1^+$ such that $f(x^*)$ is the optimal value of problem (P_1).*

The next two theorems reveal some smoothness properties for the right-hand sides of ODE systems (2.1) and (2.2).

Theorem 2.1. *For any $\gamma_2 > 0$, $(AD^2A^T)^{-1} \in C^1$ on $\mathbb{R}^n_{s++}$.*

Proof. Since A has full row rank m, for any $x \in \mathbb{R}^n_{s++}$, it is not hard to see that $(AD^2A^T)^{-1}$ exists and $(AD^2A^T)^{-1}$ is continuous on $\mathbb{R}^n_{s++}$. Notice

$$(AD^2A^T)(AD^2A^T)^{-1} = I_m. \tag{2.4}$$

Take the partial derivative of x_i for any $i \in \{1, \ldots, s\}$ in (2.4), then we get

$$2\gamma_2 x_i^{2\gamma_2 - 1}(Ae_i e_i^T A^T)(AD^2A^T)^{-1} + (AD^2A^T)\frac{\partial (AD^2A^T)^{-1}}{\partial x_i} = 0,$$

consequently

$$\frac{\partial (AD^2A^T)^{-1}}{\partial x_i} = -2\gamma_2 x_i^{2\gamma_2 - 1}(AD^2A^T)^{-1}(Ae_i e_i^T A^T)(AD^2A^T)^{-1}. \tag{2.5}$$

For any $i \in \{s+1, \ldots, n\}$, it's evident that

$$\frac{\partial (AD^2A^T)^{-1}}{\partial x_i} = 0.$$

Thus the proof is complete. ∎ □

Theorem 2.2. *For any $\gamma_1 > 0$ and $\gamma_2 > 0$, if $\nabla f(x) \in C^1$, then $DP_{AD}D\nabla f(x) \in C^1$ on $\mathbb{R}^n_{s++}$; and if $\nabla^2 f(x) \in C^1$, then $(\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x))^{-1} \in C^1$ on $(t, x) \in (0, +\infty) \times \mathbb{R}^n_{s++}$.*

Proof. For any $\gamma_1 > 0$, $\gamma_2 > 0$ and $x \in \mathbb{R}^n_{s++}$, from Theorem 2.1, we know that

$$P_{AD} = I - DA^T(AD^2A^T)^{-1}AD \in C^1.$$

From this and $\nabla f(x) \in C^1$, it is easy to see

$$DP_{AD}D\nabla f(x) \in C^1.$$

If $\nabla^2 f(x) \in C^1$ and $(t, x) \in (0, +\infty) \times \mathbb{R}^n_{s++}$,

$$\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x) \in C^1.$$

Since $f(x)$ is convex, $\nabla^2 f(x)$ is positive semidefinite. Notice $P_{AD}^2 = P_{AD}$, it is easy to see that for any $x \in \mathbb{R}^n_{s++}$, $DP_{AD}D$ is positive semidefinite. So all eigenvalues of $tDP_{AD}D\nabla^2 f(x)$ are nonnegative from Lemma 1.1 for any $t \geq 0$. Then $(\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x))^{-1}$ always exists and continuous when $(t, x) \in (0, +\infty) \times \mathbb{R}^n_{s++}$. Notice

$$\left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]\left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1} = I_n.$$

Similar to the proof of $(AD^2A^T)^{-1} \in C^1$ on $\mathbb{R}^n_{s++}$ in Theorem 2.1, we know that for $\nabla^2 f(x) \in C^1$ and $(t, x) \in (0, +\infty) \times \mathbb{R}^n_{s++}$,

$$\left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1} \in C^1.$$

The proof is complete. ▮ □

Theorems 2.3 and 2.4 below guarantee the existence, uniqueness, and feasibility for the solutions of ODE systems (2.1) and (2.2).

Theorem 2.3. *For either ODE system (2.1) or ODE system (2.2), there exists a solution $x(t)$ which is unique on a maximal existence interval $[t_0, \alpha)$ or $[t_0, \beta)$, respectively, in addition, $x_i(t) > 0$ for $i = 1, \ldots, s$ on the existence interval for both ODE systems.*

Proof. By Theorem 2.2, $DP_{AD}D\nabla f(x)$ is locally Lipschitz continuous on $\mathbb{R}^n_{s++}$. Since $\mathbb{R}^n_{s++}$ is an open set, from Theorem IV.1.2 in [8], a solution $x(t)$ is existed and unique on the maximal existence interval $[t_0, \alpha)$, where $\alpha > t_0$ or $\alpha = +\infty$. Similarly a unique solution $x(t)$ for ODE system (2.2) is also existed on the maximal existence interval $[t_0, \beta)$, where $\beta > t_0$ or $\beta = +\infty$.

Because the right-hand sides of the two ODE systems (2.1) and (2.2) are defined on the open set $(0, +\infty) \times \mathbb{R}^n_{s++}$, the solutions of the two ODE systems are of course in the open set $\mathbb{R}^n_{s++}$, hence they both satisfy $x_i(t) > 0$ for $i = 1, \ldots, s$ on the existence intervals. The proof is complete. ▮ □

Later in this section, it will be shown that $\alpha = +\infty$ (Theorem 2.5) and $\beta = +\infty$ (Theorem 2.7). For simplicity, in the following, let x (or D) stand for $x(t)$ (or $D(t)$) and no confusion would occur.

Theorem 2.4. *(i) Let the solution of ODE system (2.1) on the maximal existence interval $[t_0, \alpha)$ be $x(t)$. Then $\forall t \in [t_0, \alpha)$, we have $Ax(t) = b$. (ii) Let the solution of ODE system (2.2) on the maximal existence interval $[t_0, \beta)$ be $x(t)$. Then $\forall t \in [t_0, \beta)$, we have $Ax(t) = b$.*

Proof. (i) First, for any $t \in [t_0, \alpha)$

$$x(t) = x^0 - \int_{t_0}^{t} (DP_{AD}D\nabla f(x)|_{t=\tau}) d\tau.$$

Notice

$$ADP_{AD} = AD - AD^2A^T(AD^2A^T)^{-1}AD \equiv 0,$$

we can get

$$Ax(t) = Ax^0 - \int_{t_0}^{t} (ADP_{AD}D\nabla f(x)|_{t=\tau}) d\tau = b.$$

(ii) For ODE system (2.2), we can use the implicit form (5.5) and verify $Ax(t) = b$ $\forall t \in [t_0, \beta)$ in the same way. Thus the theorem is proved. $\blacksquare$ $\square$

The next lemma lay the foundation for our two potential functions which will be introduced in (2.7)-(2.10).

Lemma 2.1. *For any fixed* $\gamma_2 \geq \frac{1}{2}$, *if* $0 < x_i \leq M$ *for* $i = 1, \ldots, s$ *and* $|x_i| \leq M$ *for* $s+1 \leq i \leq n$ *with* $M > 0$, *and* $\nabla f(x) \in C^1$ *on* $\mathbb{R}^n_{s+}$, *then every entry of* $DP_{AD}D\nabla f(x)$ *and* $D^{1-\frac{1}{\gamma_2}}P_{AD}D\nabla f(x)$ *is bounded, and the bound depends only on* A, M, n, *and* f.

Proof. From Lemma 3 and the Remark in Sun [66], we know that if $d > 0$ every entry of $(AD^2A^T)^{-1}AD^2$ is bounded, and the bound depends only on A and n. So if $0 < x_i \leq M$ for $i = 1, \ldots, s$, every entry of

$$DP_{AD}D = D^2 - D^2A^T\left(AD^2A^T\right)^{-1}AD^2 \tag{2.6}$$

is bounded, and the bound depends only on A, M, and n. Since $\nabla f(x) \in C^1$ on $\mathbb{R}^n_{s+}$, if $0 < x_i \leq M$ for $i = 1, \ldots, s$ and $|x_i| \leq M$ for $s+1 \leq i \leq n$, it is easy to see that every entry of $\nabla f(x)$ is bounded. Furthermore the bound depends only on M, n, and f.

Hence, we know that every entry of $DP_{AD}D\nabla f(x)$ is bounded. Furthermore the bound depends only on A, M, n, and f. Notice that

$$D^{1-\frac{1}{\gamma_2}}P_{AD}D\nabla f(x) = D^{2-\frac{1}{\gamma_2}}\nabla f(x) - D^{2-\frac{1}{\gamma_2}}A^T(AD^2A^T)^{-1}AD^2\nabla f(x),$$

and $\gamma_2 \geq \frac{1}{2}$, similarly we can verify that every entry of $D^{1-\frac{1}{\gamma_2}} P_{AD} D \nabla f(x)$ is bounded, and the bound depends only on A, M, n, and f. Thus the proof is complete. ∎ □

Now we introduce two potential functions for the two ODE systems (2.1) and (2.2), respectively. With the help of these two potential functions, the boundedness of the optimal solution set is no longer needed in the convergence proofs for the solutions of ODE systems (2.1) and (2.2). Instead, only the weaker Assumption 1 is needed. In 1983, Losert and Akin [36] introduced a kind of potential function for both the discrete and continuous dynamical systems in a classical model of population genetics. Their potential function can be extended for our purpose. In order to define our potential functions, we first introduce some notations: for any $y \in \mathbb{R}^n_{s+}$, $B(y) = \{i \mid y_i > 0,\ i = 1, \ldots, s\}$ and $N(y) = \{i \mid y_i = 0,\ i = 1, \ldots, s\}$. Obviously, for any $y \in \mathbb{R}^n_{s+}$, $B(y) \cap N(y) = \emptyset$ and $B(y) \cup N(y) = \{1, \ldots, s\}$. Then the potential function $I(x, y)$ for ODE system (2.1) can be defined as

$$I(x,y) = \sum_{i=s+1}^{n} \frac{1}{2}(x_i - y_i)^2 +$$

$$\begin{cases} \displaystyle\sum_{i=1}^{k}(x_i - y_i) - \sum_{i \in B(y)} y_i \cdot \ln \frac{x_i}{y_i} & \text{if } \gamma_2 = \frac{1}{2},\ B(y) \subseteq B(x), \quad (2.7) \\ \sum_{i=1}^{k} \frac{x_i^{2-2\gamma_2} - (y_i)^{2-2\gamma_2}}{2-2\gamma_2} & \\ - \sum_{i \in B(y)} \frac{y_i}{1-2\gamma_2} \left(\frac{1}{x_i^{2\gamma_2 - 1}} - \frac{1}{y_i^{2\gamma_2 - 1}} \right) & \text{if } \frac{1}{2} < \gamma_2 < 1,\ B(y) \subseteq B(x), \quad (2.8) \\ +\infty & \text{if } B(y) \nsubseteq B(x), \quad (2.9) \end{cases}$$

where $x \in R^n_{k+}$ is the variable, $y \in \mathbb{R}^n_{k+}$ is a parameter.

The potential function $V(t, x, y)$ for ODE system (2.2) can be defined as follows:

$$V(t,x,y) = \gamma_1 I(x,y) + t\left[f(y) - f(x) + (x-y)^T \nabla f(x)\right]. \tag{2.10}$$

where $t > 0$, $x \in \mathbb{R}^n_{s+}$ are the variables, and $y \in \mathbb{R}^n_{s+}$ is a parameter.

A direct application of function $I(x, y)$ in (2.7)-(2.9) is the following Theorems 2.5 and 2.6.

Theorem 2.5. *Let the solution of ODE system (2.1) on the maximal existence interval* $[t_0, \alpha)$ *be* $x(t)$. *Then* $\alpha = +\infty$.

Proof. Assume $\alpha \neq +\infty$. We can choose the x^* in Assumption 1, and define the function $I_1(x)$ as follows

$$I_1(x) = I(x, x^*) \quad \forall x \in \mathbb{R}^n_{s+}. \tag{2.11}$$

From Theorem 2.3, it is easy to see that when $t \in [t_0, \alpha)$, $x(t) \in \mathbb{R}^{s++}_n$, so $I_1(x(t)) \equiv I_1(x)$ is well defined.

From Theorem 2.4, (2.7), and (2.1), we have

$$\begin{aligned}
\frac{dI_1(x(t))}{dt} &= (x^* - x)^T D^{-2} D P_{AD} D \nabla f(x) \\
&= (x^* - x)^T D^{-2} (D^2 - D^2 A^T (AD^2 A^T)^{-1} AD^2) \nabla f(x) \\
&= (x^* - x)^T \nabla f(x) - (x^* - x)^T A^T (AD^2 A^T)^{-1} AD^2 \nabla f(x) \\
&= (x^* - x)^T \nabla f(x) - (b - b)^T (AD^2 A^T)^{-1} AD^2 \nabla f(x) \\
&= (x^* - x)^T \nabla f(x).
\end{aligned}$$

Because of (1.15) and $f(x^*) \leq f(x(t)) \ \forall t \in [t_0, \alpha)$, we have

$$(x^* - x)^T \nabla f(x) \leq f(x^*) - f(x) \leq 0 \quad \forall t \in [t_0, \alpha).$$

So we know that

$$\frac{dI_1(x(t))}{dt} \leq 0 \quad \forall t \in [t_0, \alpha).$$

Therefore $I_1(x(t)) \leq I_1(x(t_0)) \ \forall t \in [t_0, \alpha)$. Notice that

$$I_1(x) = \sum_{i=s+1}^{n} \frac{1}{2}(x_i - x_i^*)^2 + \sum_{i \in B(x^*)} (x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}) + \sum_{i \in N(x^*)} (x_i - 0),$$

from Lemma 1.3, we know that if there exists some $i \in \{1, \ldots, n\}$ such that $|x_i| \to +\infty$, then we must have $I_1(x) \to +\infty$. But $I_1(x(t)) \leq I_1(x(t_0)) \ \forall t \in [t_0, \alpha)$. So we know that there exists an $M > 0$ which depends only on x^0 and x^*, such that $|x_i(t)| \leq M \ \forall t \in [t_0, \alpha)$ for $i = 1, \ldots, n$.

From Lemma 2.1 and (2.1), we know that there exists an $L > 0$ such that for every $i \in \{1, \ldots, s\}$, we have

$$\left|\frac{dx_i}{dt}\right| \leq Lx_i \quad \forall t \in [t_0, \alpha), \tag{2.12}$$

and for every $i \in \{s+1, \ldots, n\}$, we have

$$\left|\frac{dx_i}{dt}\right| \leq L \quad \forall t \in [t_0, \alpha), \tag{2.13}$$

furthermore this L depends only on A, n, x^0, x^*, and f.

For every $i \in \{1, \ldots, n\}$, from inequality (2.12), (2.13) and $0 < |x_i(t)| \leq M$ $\forall t \in [t_0, \alpha)$, we know that (without loss of generality we assume $M \geq 1$)

$$\left|\frac{dx_i}{dt}\right| \leq LM \quad \forall t \in [t_0, \alpha), \tag{2.14}$$

furthermore, $x(t)$ is continuous on $[t_0, \alpha)$, and it is not hard to see that $\lim_{t \to \alpha^-} x(t)$ exists. We denote this limit as $x(\alpha)$. Evidently $x(\alpha) \in \mathbb{R}^n_{k+}$. According to the Extension Theorem in **S**2.5, [3], we know that the solution $x(t)$ will go to the boundary of the open set $(0, +\infty) \times \mathbb{R}^n_{s++}$. But because of the hypothesis, $\alpha \neq +\infty$, so there must exist at least one $i \in \{1, \ldots, s\}$ such that $x_i(\alpha) = 0$. From inequality (2.12), we know that if $t \in [t_0, \alpha)$,

$$\frac{dx_i}{x_i} \geq -Ldt.$$

Integrating the inequality above, we have for every $t \in [t_0, \alpha)$

$$\ln x_i(t) - \ln x_i(t_0) \geq -L(t - t_0).$$

Since $x_i(t) \to x_i(\alpha) = 0$ as $t \to \alpha^-$, $\ln x_i(t) - \ln x_i(t_0) \to -\infty$ as $t \to \alpha^-$, but $-L(t - t_0) \geq -L(\alpha - t_0)$. This is a contradiction. Thus $\alpha = +\infty$ and the proof is complete. ∎ □

From Theorem 2.5, we can define the limit set for the solution of ODE system (2.1). The limit set of the solution of ODE system (2.1) $\{x(t)\}$ can be defined as follows

$$\Omega^1(x^0) = \left\{ p \in \mathbb{R}^n \mid \exists \ \{t_k\}_{k=0}^{+\infty} \text{ with } \lim_{k \to +\infty} t_k = +\infty \ \text{ such that } \lim_{k \to +\infty} x(t_k) = p \right\}.$$

Theorem 2.6. *The limit set $\Omega^1(x^0)$ is nonempty, compact, and connected. Furthermore $\Omega^1(x^0)$ is contained in $\mathcal{P}_1^+$.*

Proof. From Theorems 2.3, 2.4, and 2.5, we know that the limit set $\Omega^1(x^0)$ is contained in $\mathcal{P}_1^+$. From the proof of Theorem 2.5, we know that the solution $x(t)$ is contained in a bounded closed set. So similar to the proof of Theorem 1.1 on page 390 in [10] (the proof in [10] is for $n = 2$, but it can be easily extended to the general case), it can be verified that $\Omega^1(x^0)$ is nonempty, compact, and connected. ∎ □

A direct application of $V(t, x, y)$ in (2.10) is the following Theorems 2.7 and 2.8. First, the following Lemmas 2.2 and 2.3 are needed.

Lemma 2.2. *For any $(t, x) \in (0, +\infty) \times \mathbb{R}^n_{s++}$, the formula*

$$-\left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1} DP_{AD}D\nabla f(x) \tag{2.15}$$

in ODE system (2.2) has the following equivalent forms:

$$-DP_{AD}\left[\gamma_1 I_n + tP_{AD}D\nabla^2 f(x)DP_{AD}\right]^{-1} P_{AD}D\nabla f(x) \tag{2.16}$$

and

$$-DP_{AD}D\left[\gamma_1 I_n + tD^{-1}P_{AD}D\nabla^2 f(x)DP_{AD}D\right]^{-1} D^{-1}P_{AD}D\nabla f(x). \tag{2.17}$$

Proof. Since $(t, x) \in (0, +\infty) \times \mathbb{R}^n_{s++}$, we know that the inverse of D is always exists. So the proof of the equivalence of form (2.16) and form (2.17) is straightforward. Now we only need to prove the equivalence of form (2.15) and form (2.16).

First because D is invertible and $P_{AD} = P_{AD}^2$, we have

$$\begin{aligned}(2.15) &= -D\left[\gamma_1 I_n + tP_{AD}D\nabla^2 f(x)D\right]^{-1} P_{AD}D\nabla f(x)\\ &= -D\left[\gamma_1 I_n + tP_{AD}^2 D\nabla^2 f(x)D\right]^{-1} P_{AD}^2 D\nabla f(x).\end{aligned}$$

Let $P_\epsilon = P_{AD} + \epsilon I_n$, so when $\epsilon > 0$, P_ϵ is symmetric and positive definite. For any $\epsilon \geq 0$, we define

$$F(\epsilon) = -D\left[\gamma_1 I_n + tP_\epsilon^2 D\nabla^2 f(x)D\right]^{-1} P_\epsilon^2 D\nabla f(x),$$

$$G(\epsilon) = -DP_\epsilon \left[\gamma_1 I_n + tP_\epsilon D\nabla^2 f(x) DP_\epsilon\right]^{-1} P_\epsilon D\nabla f(x).$$

It is easy to see that when $\epsilon > 0$,

$$F(\epsilon) = G(\epsilon).$$

Let adj A denote the adjoint matrix of A and $|A|$ denote the determinant of A. Then

$$F(\epsilon) = -D\frac{\text{adj } [\gamma_1 I_n + tP_\epsilon^2 D\nabla^2 f(x)D]}{|[\gamma_1 I_n + tP_\epsilon^2 D\nabla^2 f(x)D]|} P_\epsilon^2 D\nabla f(x)$$

and

$$G(\epsilon) = -DP_\epsilon\frac{\text{adj } [\gamma_1 I_n + tP_\epsilon D\nabla^2 f(x)DP_\epsilon]}{|[\gamma_1 I_n + tP_\epsilon D\nabla^2 f(x)DP_\epsilon]|} P_\epsilon D\nabla f(x).$$

Since P_ϵ is symmetric and positive semidefinite for any $\epsilon \geq 0$, so is P_ϵ^2. From Lemma 1.1, $|[\gamma_1 I_n + tP_\epsilon^2 D\nabla^2 f(x)D]| \geq \gamma_1^n$ for any $\epsilon \geq 0$. In addition, since $P_\epsilon D\nabla^2 f(x)DP_\epsilon$ is symmetric and positive semidefinite, then $|[\gamma_1 I_n + tP_\epsilon D\nabla^2 f(x)DP_\epsilon]| \geq \gamma_1^n$ for any $\epsilon \geq 0$. Furthermore, for any $\epsilon \geq 0$, $i,\ j = 1, \ldots, n$, $F_{ij}(\epsilon)$ and $G_{ij}(\epsilon)$ are both rational functions and of course are continuous. So when we let $\epsilon \to 0^+$, from $F(\epsilon) = G(\epsilon)$ and the continuity of $F(\epsilon)$ and $G(\epsilon)$, we know that

$$F(0) = G(0).$$

But $P_{\epsilon=0} = P_{AD}$, thus the proof is complete. ▮ □

Lemma 2.3. *For any fixed $\gamma_1 > 0$ and $\gamma_2 \geq \frac{1}{2}$, if $0 < x_i \leq M$ for $i = 1, \ldots, s$ and $|x_i| \leq M$ for $s+1 \leq i \leq n$ with $M > 0$, $\nabla^2 f(x) \in C^1$ on $\mathbb{R}^n_{s+}$, and $0 \leq t \leq \beta$ with $t_0 < \beta < +\infty$, where β is defined in Theorem 2.3, then*

(i) every entry of $\left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1} DP_{AD}D\nabla f(x)$ is bounded, and the bound depends only on A, M, n, β, and f;

(ii) every entry of $D^{-\frac{1}{\gamma_2}} \left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1} DP_{AD}D\nabla f(x)$ is bounded, and the bound depends only on A, M, n, β, and f.

Proof. (i) From Lemma 2.1, we know that when $0 < x_i \leq M$ for $i = 1, \ldots, s$ and $|x_i| \leq M$ for $s+1 \leq i \leq n$, every entry of $DP_{AD}D\nabla f(x)$ and $D^{1-\frac{1}{\gamma_2}}P_{AD}D\nabla f(x)$ is bounded, and the bound depends only on A, M, n, and f.

Since $\nabla^2 f(x) \in C^1$ on $\mathbb{R}^n_{s+}$, when $0 < x_i \leq M$ for $i = 1, \ldots, s$ and $|x_i| \leq M$ for $s+1 \leq i \leq n$, it is easy to see that every entry of $\nabla^2 f(x)$ is bounded, and the bound depends only on M and f. So we know that every entry of $DP_{AD}D\nabla^2 f(x)$ is bounded, and the bound depends only on A, M, n, and f.

Therefore when $0 < x_i \leq M$ for $i = 1, \ldots, s$ and $|x_i| \leq M$ for $s+1 \leq i \leq n$ and $0 \leq t \leq \beta$, for matrix $\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)$, every entry of it is bounded, and every entry of its adjoint matrix is also bounded. Furthermore its determinant is not less than γ_1^n because of the eigenvalues of $tDP_{AD}D\nabla^2 f(x)$ are all nonnegative from Lemma 1.1. Hence every entry of

$$\left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1}$$

is bounded, and the bound depends only on A, M, n, β, and f.

Therefore we know that if $0 < x_i \leq M$ for $i = 1, \ldots, s$ and $|x_i| \leq M$ for $s+1 \leq i \leq n$ and $0 \leq t \leq \beta$, every entry of

$$\left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1} DP_{AD}D\nabla f(x)$$

is bounded, and the bound depends only on A, M, n, β, and f.

(ii) Let

$$\tilde{d} = -\left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1} DP_{AD}D\nabla f(x),$$

then every entry of $\tilde{d}$ is bounded, and we can write $\tilde{d}$ in the following form

$$\tilde{d} = -\frac{1}{\gamma_1}DP_{AD}D(\nabla f(x) + t\nabla^2 f(x)\tilde{d}),$$

then we have

$$D^{-\frac{1}{\gamma_2}}\tilde{d} = -\frac{1}{\gamma_1}D^{1-\frac{1}{\gamma_2}}P_{AD}D(\nabla f(x) + t\nabla^2 f(x)\tilde{d}).$$

From Lemma 2.1, we know that every entry of $D^{1-\frac{1}{\gamma_2}} P_{AD} D \nabla f(x)$ is bounded, and the bound depends only on A, M, n, and f. Furthermore it is easy to see that when $0 < x_i \le M$ for $i = 1, \ldots, s$ and $|x_i| \le M$ for $s+1 \le i \le n$ and $0 \le t \le \beta$, every entry of $\nabla f(x) + t \nabla^2 f(x) \tilde{d}$ is bounded, and the bound depends only on A, M, n, β, and f. So we know that every entry of

$$D^{-\frac{1}{\gamma_2}} \tilde{d} = D^{-\frac{1}{\gamma_2}} \left[\gamma_1 I_n + t D P_{AD} D \nabla^2 f(x) \right]^{-1} D P_{AD} D \nabla f(x)$$

is bounded, and the bound depends only on A, M, n, β, and f. Thus the proof is complete. ∎ □

Theorem 2.7. *Let the solution of ODE system (2.2) on the maximal existence interval* $[t_0, \beta)$ *be* $x(t)$*. Then* $\beta = +\infty$*.*

Proof. Similar to the proof of Theorem 2.5, we can define function $I_2(t, x)$ as follows

$$I_2(t, x) = V(t, x, x^*) \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}^n_{s+},$$

where $x^* \in \mathcal{P}_1^+$ according to Assumption 1.

From Theorem 2.3, $x(t) \in \mathbb{R}^n_{s++}$, so $I_2(t, x(t)) \equiv I_2(t, x)$ is well defined. From the implicit form (5.5), we have

$$\begin{aligned} \frac{dI_2(t, x(t))}{dt} &= (x^* - x)^T \left[\nabla f(x) + t \nabla^2 f(x) \frac{dx}{dt} \right] + t (x - x^*)^T \nabla^2 f(x) \frac{dx}{dt} \\ &\quad + f(x^*) - f(x) + (x - x^*)^T \nabla f(x) \\ &= (x^* - x)^T \nabla f(x) + f(x^*) - f(x) + (x - x^*)^T \nabla f(x) \\ &= f(x^*) - f(x) \\ &\le 0. \end{aligned}$$

From Lemma 1.2, we have

$$f(x^*) - f(x) + (x - x^*)^T \nabla f(x) \ge 0. \qquad (2.18)$$

From Lemmas 1.3, 1.4 and inequality (2.18), we know that for any $(t, x) \in (0, +\infty) \times \mathbb{R}^n_{s++}$, if there exists an i so that $|x_i| \to +\infty$, then $I_2(t, x) \to +\infty$. From this fact and Lemma 2.3, similar to the proof of Theorem 2.5, we can show that the solution $x(t)$ is bounded and $\beta = +\infty$. Thus the proof is complete. ∎ □

From Theorem 2.7, we can also define the limit set for the solution of ODE system (2.2). The limit set of the solution of ODE system (2.2) $x(t)$ can be defined as follows

$$\Omega^2(x^0) = \left\{ p \in \mathbb{R}^n \mid \exists \{t_k\}_{k=0}^{+\infty} \text{ with } \lim_{k \to +\infty} t_k = +\infty \text{ such that } \lim_{k \to +\infty} x(t_k) = p \right\}.$$

Theorem 2.8. *The limit set $\Omega^2(x^0)$ is nonempty, compact, and connected. Furthermore $\Omega^2(x^0)$ is contained in $\mathcal{P}_1^+$.*

Proof. The proof is similar to the one for Theorem 2.6. ∎ □

In the rest of this section, we will show the weak convergence for system (2.1). First, we need to reveal some fundamental results for the solutions of ODE systems (2.1) and (2.2).

Theorem 2.9. *Let the solution of ODE system (2.1) be $x(t)$, then $f(x(t))$ is a non-increasing function on $[t_0, +\infty)$. Furthermore, if $x^0 \in \mathcal{P}_1^{++}$ is an optimal solution for problem (P_1), then $x(t) \equiv x^0$ on $[t_0, +\infty)$; otherwise $f(x(t))$ is a strictly decreasing function on $[t_0, +\infty)$.*

Proof. Notice for $t \geq t_0$,

$$\frac{df(x(t))}{dt} = -\nabla f(x)^T D P_{AD} D \nabla f(x) = -\|P_{AD} D \nabla f(x)\|^2 \leq 0,$$

we know that $f(x(t))$ is a noincreasing function on $[t_0, +\infty)$.

For problem (P_1), the KKT conditions are

$$\begin{cases} Ax = b, \ x \in \mathbb{R}^n_{s+}, \\ Xz = 0, \ z \in \mathbb{R}^n_{s+}, \\ A^T y + z = \nabla f(x), \\ z_i = 0, \quad \text{for } s+1 \leq i \leq n, \end{cases} \tag{2.19}$$

where $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

If $x \in \mathcal{P}_1^{++}$ is an optimal solution or problem (P_1), which implies that there must exist a (y, z) such that (x, y, z) satisfies system (2.19), then

$$z = 0 \quad \text{and} \quad A^T y = \nabla f(x),$$

thus, it is easy to see that

$$P_{AD} D \nabla f(x) = P_{AD} D A^T y = 0.$$

Hence if $x^0 \in \mathcal{P}_1^{++}$ is an optimal solution, we know that the right-hand side of ODE system (2.1) equals zero at $x = x_0$, i.e.,

$$D P_{AD} D \nabla f(x)|_{x=x_0} = 0,$$

therefore $x(t) \equiv x^0$ for $t \geq t_0$ is a solution of ODE system (2.1). Because of the uniqueness of the solution, we know that $x(t) \equiv x^0$ on $[t_0, +\infty)$.

If $x^0 \in \mathcal{P}_1^{++}$ is not an optimal solution, we show that $f(x(t))$ is a strictly decreasing function on $[t_0, +\infty)$. If not, then there must exist t_1 and t_2 with $t_0 \leq t_1 < t_2$ such that $f(x(t_1)) = f(x(t_2))$. Since $\frac{df(x(t))}{dt} \leq 0$, we know that when $t_1 \leq t \leq t_2$, $\frac{df(x(t))}{dt} \equiv 0$. From $\frac{df(x(t))}{dt} = -\|P_{AD} D \nabla f(x)\|^2 = -\|D^{-1} \frac{dx}{dt}\|^2 = 0$, Theorems 2.3 and 2.5, it is easy to see $\frac{dx}{dt} = 0$ on $[t_1, t_2]$, then $x(t) = x(t_1)$ on $[t_1, t_2]$, furthermore $x(t) \equiv x(t_1)$ on $[t_0, +\infty)$ will be a solution of the ODE

$$\frac{dx}{dt} = -D P_{AD} D \nabla f(x)$$

that passes through the point $(t_1, x(t_1))$. However according to the uniqueness of the solution, we know that the solution of ODE system (2.1) is actually $x(t) \equiv x(t_1) = x^0$ on $[t_0, +\infty)$. But from the inequality

$$\frac{dI_1(x(t))}{dt} = (x^* - x)^T \nabla f(x) \leq f(x^*) - f(x),$$

we know that at x^0, $\frac{dI_1(x(t))}{dt}|_{t=t_0} < 0$ because x^0 is not an optimal solution. This is a contradiction. Hence $f(x(t))$ is a strictly decreasing function on $[t_0, +\infty)$. ∎ □

Theorem 2.10. *Let the solution of ODE system (2.2) be $x(t)$. Then $f(x(t))$ is a nonincreasing function on $[t_0, +\infty)$. Furthermore, if $x^0 \in \mathcal{P}_1^{++}$ is an optimal solution for problem (P_1), then $x(t) \equiv x^0$ on $[t_0, +\infty)$; otherwise $f(x(t))$ is a strictly decreasing function on $[t_0, +\infty)$.*

Proof. Notice the fact that when $x \in \mathbb{R}^n_{s++}$, $DP_{AD}D$ and $\nabla^2 f(x)$ are both symmetric and positive semidefinite, similar to the proof in Theorem 2.9, we can prove this theorem with the equivalent form (2.16) in Lemma 2.2. ∎ □

Lemma 2.4. *For any fixed $\gamma_2 \geq \frac{1}{2}$, if $0 < x_i \leq M$ for $i = 1, \ldots, s$ and $|x_i| \leq M$ for $s+1 \leq i \leq n$ with $M > 0$ and $\nabla f(x) \in C^1$ on $\mathbb{R}^n_{s+}$, then for every $i \in \{1, \ldots, n\}$, every entry of*

$$\frac{\partial DP_{AD}D\nabla f(x)}{\partial x_i}$$

is bounded, and the bound depends only on A, M, n, and f.

Proof. Let $Q = (AD^2A^T)^{-1}AD^2$. From equality (2.6), we know that for $i = 1, \ldots, s$

$$\frac{\partial DP_{AD}D}{\partial x_i} = \frac{\partial D^2}{\partial x_i} - \frac{\partial D^2A^T}{\partial x_i}Q - D^2A^T\frac{\partial (AD^2A^T)^{-1}}{\partial x_i}AD^2 - Q^T\frac{\partial AD^2}{\partial x_i},$$

and for $i = s+1, \ldots, n$

$$\frac{\partial DP_{AD}D}{\partial x_i} = 0.$$

From the proof of Lemma 2.1, we know that when $x \in \mathbb{R}^n_{s++}$, every entry of Q and Q^T is bounded, and the bound depends only on A and n.

From equality (2.5) in Theorem 2.1, for $i = 1, \ldots, s$ we have

$$D^2A^T\frac{\partial (AD^2A^T)^{-1}}{\partial x_i}AD^2 = -2\gamma_2 x_i^{2\gamma_2-1}Q^T(Ae_ie_i^TA^T)Q.$$

Therefore when $0 < x_i \leq M$ for $i = 1, \ldots, s$ and $|x_i| \leq M$ for $s+1 \leq i \leq n$, every entry of

$$\frac{\partial DP_{AD}D}{\partial x_i}$$

is bounded, and the bound depends only on A, M, and n. Furthermore we know that $\nabla f(x) \in C^1$ on $\mathbb{R}^n_{s+}$ and $DP_{AD}D$ is bounded from the proof of Lemma 2.1, so it is evident that for every $i \in \{1, \dots, n\}$, every entry of

$$\frac{\partial DP_{AD}D\nabla f(x)}{\partial x_i}$$

is bounded, and the bound depends only on A, M, n, and f. $\blacksquare$ $\square$

Theorem 2.11. *Let the solution of ODE system (2.1) be $x(t)$. Then*

$$\lim_{t \to +\infty} DP_{AD}D\nabla f(x) = 0.$$

Proof. From the proof of Theorem 2.5, we know that there exists an $M > 0$ such that the solution of ODE system (2.1) $x(t)$ is contained in the bounded closed set $\{x \in \mathbb{R}^n | 0 \le x_i \le M,\ for\ i = 1, \dots, s,\ |x_i| \le M\ for\ i = s+1, \dots, n\}$. Since $\nabla f(x) \in C^1$ on $\mathbb{R}^n_{s+}$, this along with Theorem 2.3 and Lemma 2.4 indicate that there exists a constant L_1 such that for every $i \in \{1, \dots, n\}$, every entry of

$$\frac{\partial \nabla f(x)^T DP_{AD}D\nabla f(x)}{\partial x_i} \tag{2.20}$$

is bounded by L_1, and L_1 depends only on A, n, x^0, x^*, and f.

From Theorem 2.9, we know that $f(x(t))$ is a nonincreasing function and $f(x(t)) \ge f(x^*)$ on $[t_0, +\infty)$. Thus $f(x(t))$ has a finite limit as $t \to +\infty$. From (3.4) and (3.7), we can obtain

$$\begin{aligned} & \left| \frac{df(x(t))}{dt}\ |_{t=t_1} - \frac{df(x(t))}{dt}\ |_{t=t_2} \right| \\ = & \left| \int_0^1 \frac{\partial \nabla f(x)^T DP_{AD}D\nabla f(x)}{\partial x} |_{x=x(t_2)+\tau(x(t_1)-x(t_2))} (x(t_1) - x(t_2)) d\tau \right| \\ \le & \sqrt{n} L_1 \cdot \|x(t_1) - x(t_2)\| = \sqrt{n} L_1 \cdot \| \int_{t_1}^{t_2} \frac{dx}{d\tau} d\tau \| \le nL_1LM|t_1 - t_2|, \end{aligned}$$

where the last inequality is obtained from inequality (3.4).

Thus $\frac{df(x(t))}{dt}$ is uniformly continuous. From Barbalat's Lemma, we know that

$$\lim_{t \to +\infty} \frac{df(x(t))}{dt} = - \lim_{t \to +\infty} \|P_{AD}D\nabla f(x)\|^2 = 0.$$

Therefore it is easy to see that

$$\lim_{t \to +\infty} DP_{AD}D\nabla f(x) = 0.$$

Thus the proof is complete. ▮ □

2.2 Optimality of The Cluster Point(s)

In this part, we will show that every accumulation point of the solutions of the two ODE systems (2.1) and (2.2) is actually an optimal solution for problem (P_1).

Theorem 2.12. *For any $x^{(1)} \in \Omega^1(x^0)$ and $x^{(2)} \in \Omega^2(x^0)$, $x^{(1)}$ and $x^{(2)}$ are both optimal solutions for problem (P_1).*

Proof. Because when $i \in N(x^*)$, we have $x_i^* = 0$. From Lemmas 1.3 and 1.4, it is easy to see that $I_1(x) \geq 0$ for all $x \in \mathbb{R}^n_{s++}$. So for all $t \in [t_0, +\infty)$, $I_1(x(t))$ is bounded below. This along with the fact that $\frac{dI_1(t)}{dt} \leq 0$ imply that $I_1(x(t))$ has a finite limit as $t \to +\infty$.

From the proof of Theorem 2.5, it is easy to see solution $x(t)$ is contained in the bounded closed set $\{x \in \mathbb{R}^n | 0 \leq x_i \leq M, \ for \ i = 1, \ldots, s, \ |x_i| \leq M \ for \ i = s+1, \ldots, n\}$ for some $M > 0$. Since $\frac{dI_1(x(t))}{dt} = (x^* - x)^T \nabla f(x)$ (see the proof of Theorem 2.5) is continuously differentiable, so when x is in that bounded closed set, there must exist a constant $L_2 > 0$ such that

$$\left| \frac{dI_1(x(t))}{dt} |_{t=t_1} - \frac{dI_1(x(t))}{dt} |_{t=t_2} \right| \leq L_2 \|x(t_1) - x(t_2)\| = L_2 \| \int_{t_1}^{t_2} \frac{dx}{dt} dt \|.$$

Using inequality (3.4), we have

$$\left| \frac{dI_1(x(t))}{dt} |_{t=t_1} - \frac{dI_1(x(t))}{dt} |_{t=t_2} \right| \leq L_2 \| \int_{t_1}^{t_2} \frac{dx}{dt} dt \| \leq \sqrt{n} L_2 L M |t_1 - t_2|,$$

thus $\frac{dI_1(x(t))}{dt}$ is uniformly continuous. From Barbalat's Lemma, we know that

$$\lim_{t \to +\infty} \frac{dI_1(x(t))}{dt} = \lim_{t \to +\infty} (x^* - x)^T \nabla f(x) = 0. \tag{2.21}$$

For any $x^{(1)} \in \Omega^1(x^0)$, from the definition of $\Omega^1(x^0)$, we know that there exists a sequence $\{t_k\}_0^{+\infty}$ with $t_k \to +\infty$ as $k \to +\infty$ such that $x(t_k) \to x^{(1)}$ as $k \to +\infty$. Then since $(x^* - x)^T \nabla f(x)$ is continuous at $x^{(1)}$, from equality (2.21), we have

$$0 = \lim_{t \to +\infty} (x^* - x)^T \nabla f(x) = \lim_{k \to +\infty} (x^* - x(t_k))^T \nabla f(t_k) = (x^* - x^{(1)})^T \nabla f(x^{(1)}).$$

From Lemma 1.2, let $y = x^*$, $x = x^{(1)}$ in (1.15), we have

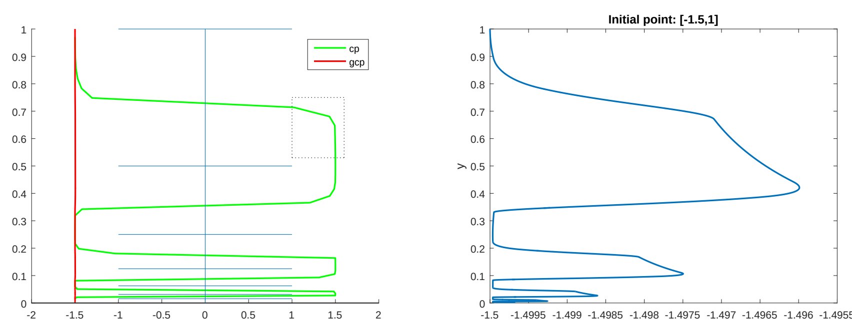
$$f(x^{(1)}) \leq f(x^*) + (x^{(1)} - x^*)^T \nabla f(x^{(1)}) = f(x^*).$$

But from Assumption 2.1, we know that x^* is an optimal solution for problem (P_1), that is, $f(x^{(1)}) \geq f(x^*)$. So we must have $f(x^{(1)}) = f(x^*)$, that is, $x^{(1)}$ is an optimal solution.

As for $x^{(2)} \in \Omega^2(x^0)$, from the proof of Theorem 2.7, we have

$$\frac{dI_2(t, x(t))}{dt} = f(x^*) - f(x) \leq 0,$$

this along with inequality (2.18) indicate that $I_2(t, x) \geq 0$ for all $(t, x) \in (0, +\infty) \times \mathbb{R}^n_{s++}$. So we can similarly prove that $x^{(2)}$ is an optimal solution for problem (P_1) too. Thus the theorem is proved. $\square$



(a) Trajectories of cp and gcp (b) Trajectory of gcp

Figure 2.1: The central path and generalized central path.

Now we use an example of class C^2 in [19] to show the limiting behaviors of the central path and our generalized central paths. The examples in [19] have the

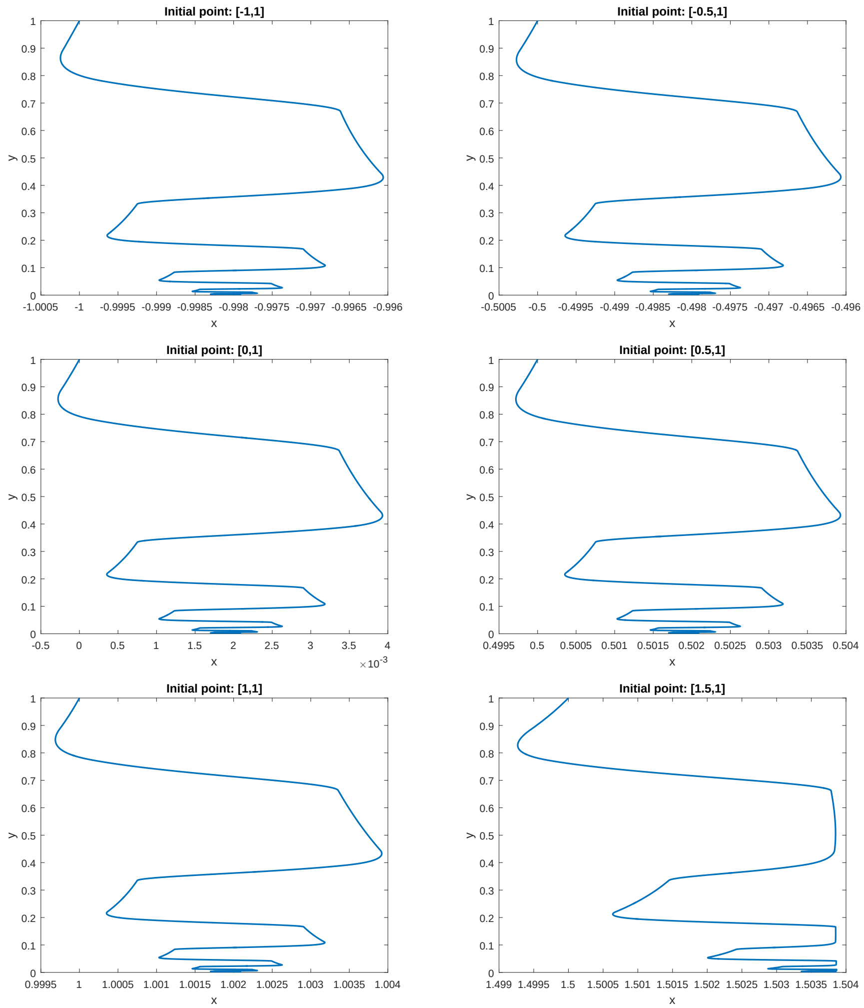


Figure 2.2: The generalized paths with different initial points.

following form

$$\begin{aligned} \min \quad & F(x,y) \\ \text{s.t.} \quad & y > 0, \end{aligned}$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$ are variables and the solutions are on the x-axis $\{(x,0)|x \in \mathbb{R}\}$. We choose an example of class C^2 which is described in Section 5.2 of [19]. The parameter ϵ_k needs to satisfy $0 < |\epsilon_k| \leq \min\{c_k/4, |\epsilon_{k-1}|\}$ (see (15) in [19]), we choose $|\epsilon_k| = \min\{3c_k/32, |\epsilon_{k-1}|\}$. In this example, for $z = (x,y)^T$ and $y_k = 2^{-k}$ (k = 0, 1, ...),

$$f_k = a_k(y - y_k) + b_k + c_k g_k(z) + \epsilon_k x,$$

where $a_k = y_k^2$, $b_k = \frac{y_k^3}{3}$, $c_k = \frac{1}{4}\min\{r_k, r_{k+1}\}$, and $\epsilon_k = (-1)^k|\epsilon_k|$. For $k = 0, 1, ...,$ $g_k(z) = (\frac{g_k^0(z)}{2})^3$, where

$$g_k^0(z) = \frac{1}{2}\left(\sqrt{(x+1)^2 + (y-y_k)^2} + \sqrt{(x-1)^2 + (y-y^k)^2 - 2}\right),$$

and for $k = 1, 2, ...,$

$$r_k = \frac{1}{2}\min\{b_k - (b_{k-1} + a_{k-1}(y_k - y_{k-1})), b_{k-1} - (b_k + a_k(y_{k-1} - y_k))\}.$$

In Section 5 of [19], many interesting properties for this example are given. For $y = y_k = 2^{-k}$ $(k \geq 0)$, $F(\cdot, y_k) = f_k(\cdot, y_k)$, and an easy calculation yields

$$\arg\min_{x\in\mathbb{R}} F(x, y_k) = \{(-1)^{k+1}(1 + \tau_k)\},$$

where $0 < \tau_k = (8|\epsilon_k|/3c_k)^{\frac{1}{2}} \leq 0.5$. Hence the central path will be a zig-zag path. We plot the central path (see Fig. 2.1) and our generalized central paths with different initial points (see Fig. 2.1 and Fig. 2.2). For the generalized central paths (we let $\gamma_1 = 1$, $\gamma_2 = 0.75$, and $t_0 = 1$), since the big matrix $[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)]$ in the ODE system (2.2) is invertible everywhere, we can use an ode-solver ode23s in Matlab to plot them. For the central path (the path named cp in Fig. 2.1), a Matlab code provided by Prof. Karas of [19] is used . The path named gcp in Fig. 2.1 is the generalized central path with the initial point $(-1.5, 1)$. Fig. 2.1 and Fig. 2.2 show that the central path is a zig-zag path with large loop and the generalized central paths seem to converge. Even though the generalized central paths swing back and forth, but the amplitude or width is quite small and becomes smaller and smaller. In the next section, we will prove that both the generalized affine scaling trajectory and the generalized central path actually converge. We think the reason that the central path does not converge in this example is that the barrier function only contains y, hence does not have any restriction on x. However, for the barrier function $-\frac{1}{\alpha_1}y^{\alpha_1}$ $(0 < \alpha_1 < 1)$, if we add another item $\frac{1-\alpha_1}{2}(x - \bar{x}_0)^2$ to the barrier function, then the corresponding ODE system can be described by our ODE system (2.2). The added item $\frac{1-\alpha_1}{2}(x - \bar{x}_0)^2$ can be regarded as a restriction on x, hence the new path may converge.

2.3 Convergence of The Continuous Trajectories

Now, it comes to the key results of this chapter. Theorems 2.13 and 2.14 below show that the solutions of the two ODE systems (2.1) and (2.2) converge as $t \to +\infty$.

Theorem 2.13. *The limit set $\Omega^1(x^0)$ contains a single point only.*

Proof. From Theorem 2.6, we know that $\Omega^1(x^0)$ is not empty. So we can choose a point $\bar{x} \in \Omega^1(x^0)$, and evidently $\bar{x} \in \mathcal{P}_1^+$. From (2.7)-(2.9), for any $x \in \mathbb{R}^n_{s+}$, we can define $V_1(x)$ as follows

$$V_1(x) = I(x, \bar{x}) = \sum_{i=s+1}^{n} \frac{1}{2}(x_i - \bar{x}_i)^2 + \begin{cases} \sum_{i \in N(\bar{x})} x_i + \sum_{i \in B(\bar{x})} (x_i - \bar{x}_i - \bar{x}_i \cdot \ln \frac{x_i}{\bar{x}_i}) & \text{if } \gamma_2 = \frac{1}{2},\ B(x) \subseteq B(\bar{x}), \\ \sum_{i \in N(\bar{x})} \frac{x_i^{2-2\gamma_2}}{2-2\gamma_2} + \sum_{i \in B(\bar{x})} \Big[\frac{x_i^{2-2\gamma_2} - (\bar{x}_i)^{2-2\gamma_2}}{2-2\gamma_2} & \\ -\frac{\bar{x}_i}{1-2\gamma_2} \left(\frac{1}{x_i^{2\gamma_2 - 1}} - \frac{1}{\bar{x}_i^{2\gamma_2 - 1}} \right) \Big] & \text{if } \frac{1}{2} < \gamma_2 < 1,\ B(x) \subseteq B(\bar{x}), \\ +\infty & \text{if } B(x) \not\subseteq B(\bar{x}). \end{cases} \tag{2.22}$$

From Lemmas 1.3, 1.4, and (2.22), it is straightforward to see that $V_1(x) \geq 0$ for any $x \in \{x \in \mathbb{R}^n_{s+} \mid x_i > 0 \text{ if } i \in B(\bar{x})\}$ and $V_1(x) = 0 \iff x = \bar{x}$.

Let the solution of ODE system (2.1) be $x(t)$. Similar to the proof in Theorem 2.5, we know that $V_1(x(t))$ is well defined on $[t_0, +\infty)$, and we also have

$$\frac{dV_1(x(t))}{dt} = (\bar{x} - x)^T \nabla f(x).$$

From Theorem 2.12 we have $f(x(t)) \geq f(\bar{x})$ for any $t \in [t_0, +\infty)$. This together with Lemma 1.2 imply that for any $t \in [t_0, +\infty)$

$$\frac{dV_1(x(t))}{dt} = (\bar{x} - x)^T \nabla f(x) \leq f(\bar{x}) - f(x) \leq 0. \tag{2.23}$$

$V_1(x) \geq 0$ and (2.23) ensure that $\lim_{t \to +\infty} V_1(x(t))$ exists. From the continuity of $V_1(x)$ at $\bar{x}$ and $\bar{x} \in \Omega^1(x^0)$, we must have

$$\lim_{t \to +\infty} V_1(x(t)) = V_1(\bar{x}) = I(\bar{x}, \bar{x}) = 0. \tag{2.24}$$

If $\bar{x}$ is not the only point of $\Omega^1(x^0)$, there must exist another point $\hat{x} \in \Omega^1(x^0)$ with $\hat{x} \neq \bar{x}$. Since $\hat{x} \in \Omega^1(x^0)$, there exists a sequence $\{t_k\}_{k=0}^{+\infty}$ such that $t_k \to +\infty$ and $x(t_k) \to \hat{x}$ as $k \to +\infty$. From Lemmas 1.3 and 1.4, we know that if there exists some $i \in B(\bar{x})$ such that $\hat{x}_i = 0$, then we must have $V_1(x(t_k)) \to +\infty$ as $k \to +\infty$. But from inequality (2.23), we know that for any $t \in [t_0, +\infty)$, $V_1(x(t)) \leq V_1(x(t_0))$, so we know that if $i \in B(\bar{x})$, we must have $\hat{x}_i > 0$. Therefore, $V_1(x)$ is continuous at $\hat{x}$ from (2.22).

From the continuity of $V_1(x)$ at $\hat{x}$ and (2.24), we have

$$V_1(\hat{x}) = V_1(\lim_{k\to+\infty} x(t_k)) = \lim_{k\to+\infty} V_1(x(t_k)) = 0.$$

But this is a contradiction with that $V_1(x) = 0 \iff x = \bar{x} \neq \hat{x}$. Therefore, the limit set $\Omega^1(x^0)$ is a singleton. □

Theorem 2.14. *The limit set $\Omega^2(x^0)$ only contains a single point.*

Proof. From Theorem 2.8, we know that $\Omega^2(x^0)$ is not empty. So we can choose a point $\check{x} \in \Omega^2(x^0)$, and evidently $\check{x} \in \mathcal{P}_1^+$. From (2.10), we can define $V_2(t, x)$ as follows

$$V_2(t, x) = V(t, x, \check{x}) \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}^n_{s+}.$$

Let the solution of ODE system (2.2) be $x(t)$, similar to the proof in Theorem 2.13, we can have

$$\frac{dV_2(t, x(t))}{dt} = f(\check{x}) - f(x(t)) \leq 0. \tag{2.25}$$

From Lemma 1.2, it is straightforward to see that $V_2(t, x) \geq 0$ for any $t \geq t_0 > 0$ and $x \in \mathbb{R}^n_{s+}$. In addition, $V_2(t, x) = 0 \iff x = \check{x}$. Noticing from Lemma 1.2 that $f(\check{x}) - f(x) + (x - \check{x})^T \nabla f(x) \geq 0$ for any $x \in \mathbb{R}^n_{s+}$, similar to the proof in Theorem 2.13, if we can prove

$$\lim_{t\to+\infty} V_2(t, x(t)) = 0,$$

then the limit set $\Omega^2(x^0)$ only contains a single point.

Now let's prove that $\lim_{t\to+\infty} V_2(t,x(t)) = 0$. For any $T > t_0$ and $x(T) \in \mathbb{R}^n_{s++}$ (guaranteed by Theorems 2.3 and 2.7), we can define $V_3(t,x)$ in (2.10) as follows:

$$V_3(t,x) = V(t,x,x(T)).$$

Then we have

$$\frac{dV_3(t,x(t))}{dt} = f(x(T)) - f(x(t)),$$

and

$$\frac{d(V_2(t,x(t)) - V_3(t,x(t)))}{dt} = f(\check{x}) - f(x(T)) \le 0.$$

But $V_3(T,x(T)) = V(T,x(T),x(T)) = 0$, so we have

$$V_2(T,x(T)) - V_3(T,x(T)) = V_2(T,x(T)) \le V_2(t_0,x(t_0)) - V_3(t_0,x(t_0)).$$

If $\gamma_2 = \frac{1}{2}$,

$$\begin{aligned} V_2(t_0,x(t_0)) - V_3(t_0,x(t_0)) &= \frac{\gamma_1}{2}\sum_{i=k+1}^{n}\left[(x_i^0 - \check{x}_i)^2 - (x_i^0 - x(T)_i)^2\right] \\ &+ t_0\left[f(\check{x}) - f(x(T)) + (x(T)-\check{x})^T\nabla f(x^0)\right] \\ &+ \sum_{i=1}^{k}\gamma_1(x(T)_i - \check{x}_i) + \sum_{i\in N(\check{x})}\gamma_1 x(T)_i \ln\frac{x_i^0}{x(T)_i} \\ &- \sum_{i\in B(\check{x})}\gamma_1\left(\check{x}_i\ln\frac{x_i^0}{\check{x}_i} - x(T)_i\ln\frac{x_i^0}{x(T)_i}\right). \end{aligned}$$

For $i \in B(\check{x})$, $\check{x}_i > 0$, and for $i \in N(\check{x})$, $\check{x}_i = 0$. Notice that

$$\lim_{a\to 0} a\ln\frac{x_i^0}{a} = 0,$$

for any $x_i^0 > 0$, $i = 1,\ldots,s$. Therefore $V_2(t_0,x(t_0)) - V_3(t_0,x(t_0))$ can be sufficiently small if $\|x(T) - \check{x}\|$ is sufficiently small. However $\check{x}$ is a limit point, hence we can choose T big enough such that $\|x(T) - \check{x}\|$ is sufficiently small. So for any $\epsilon > 0$, we can choose the T big enough such that

$$V_2(t_0,x(t_0)) - V_3(t_0,x(t_0)) = V(t_0,x(t_0),\check{x}) - V(t_0,x(t_0),x(T)) \le \epsilon.$$

If $\frac{1}{2} < \gamma_2 < 1$, we can get this similarly. Then we get $V_2(T, x(T)) \leq \epsilon$. But $V_2(t, x(t))$ is nonincreasing from (2.25), hence we can get

$$\lim_{t \to +\infty} V_2(t, x(t)) = 0.$$

Thus the theorem is proved. ▮ □

Theorem 2.15. *Let the solution of ODE system (2.2) be $x(t)$. Then*

$$\lim_{t \to +\infty} \left[\gamma_1 I_n + tDP_{AD}D\nabla^2 f(x)\right]^{-1} DP_{AD}D\nabla f(x) = 0.$$

Proof. From the equivalent form (2.16), we know that if we can prove

$$\lim_{t \to +\infty} \|P_{AD}D\nabla f(x)\| = 0, \tag{2.26}$$

then the theorem holds.

We prove this by contradiction. If (5.19) is not true, then there exists a constant $c_0 > 0$ such that for any $T > t_0$, there always exists a $t > T$ such that $\|P_{AD}D\nabla f(x)\| > c_0$.

Let's consider the following cluster of trajectories: each trajectory is defined by the solution of ODE system (2.1) with initial point $x(t)$ at initial time t_0, where $x(t)$ denotes the solution of ODE system (2.2) at time t. We use $\tilde{x}(\tau, t)$ to denote this trajectory, then we have

$$\frac{d\tilde{x}(\tau, t)}{d\tau} = -\tilde{D}P_{A\tilde{D}}\tilde{D}\nabla f(\tilde{x}),\ \tilde{x}(\tau = t_0, t) = x(t),$$

where $\tilde{D} = D|_{x=\tilde{x}(\tau,t)}$ and $x(t)$ is the solution of ODE system (2.2) at time t. Using potential functions $I(x, x^*)$ in (2.7)-(2.9) and $V(t, x, x^*)$ in (2.10) with $x^* \in \mathcal{P}_1^+$ according to Assumption 1, and the fact that $I(x, x^*) \leq \frac{1}{\gamma_1}V(t, x, x^*)$ for any $t \geq 0$ from (2.18), we know for any $\tau \geq t_0$ and $t \geq t_0$,

$$I(\tilde{x}(\tau, t), x^*) \leq I(\tilde{x}(t_0, t), x^*) = I(x(t), x^*) \leq \frac{1}{\gamma_1}V(t, x(t), x^*) \leq \frac{1}{\gamma_1}V(t_0, x^0, x^*).$$

Then, it is not hard to see that every trajectory $\tilde{x}(\tau, t)$ with $t \in [t_0, +\infty)$ is contained in a bounded set $\{x \in \mathbb{R}^n | 0 \leq x_i \leq M,\ for\ i = 1, \ldots, s,\ |x_i| \leq M\ for\ i = s+1, \ldots, n\}$, where M depends only on t_0, γ_2, x^0, x^*, and f.

Similar to the proof in Theorem 2.11, we can have that there exists a constant L^0 which depends only on A, n, x^0, x^*, and f such that for any $t \in [t_0, +\infty)$ and $t_0 \leq \tau_1 \leq \tau_2$,

$$\left| \frac{df(\tilde{x}(\tau, t))}{d\tau} |_{\tau=\tau_1} - \frac{df(\tilde{x}(\tau, t))}{d\tau} |_{\tau=\tau_2} \right| \leq L^0 |\tau_1 - \tau_2|. \tag{2.27}$$

From Theorem 2.12, we have $\lim_{\tau \to +\infty} f(\tilde{x}(\tau, t)) = f(x^*)$ for any $t \in [t_0, +\infty)$, then we have for any $t \in [t_0, +\infty)$,

$$\int_{t_0}^{+\infty} -\frac{df(\tilde{x}(\tau, t))}{d\tau} d\tau = f(\tilde{x}(t_0, t)) - f(x^*) = f(x(t)) - f(x^*). \tag{2.28}$$

Since $\lim_{t \to +\infty} f(x(t)) = f(x^*)$ from Theorem 2.12, we can choose T_1 such that

$$f(x(T_1)) - f(x^*) < \frac{{c_0}^4}{4L^0}. \tag{2.29}$$

However, from the earlier hypothesis, we know that there exists a $t = T_2 > T_1$ such that $\|P_{AD} D \nabla f(x)\|^2 |_{x=x(T_2)} > c_0^2$. Noticing $\frac{df(\tilde{x}(\tau,t))}{d\tau} = -\|P_{A\tilde{D}} \tilde{D} \nabla f(\tilde{x})\|^2$, from (2.27), we can have that if $t = T_2$, for $\tau_2 = \tau \geq \tau_1 = t_0$

$$\|P_{A\tilde{D}} \tilde{D} \nabla f(\tilde{x})\|^2 |_{\tilde{x}=\tilde{x}(\tau, T_2)} - \|P_{A\tilde{D}} \tilde{D} \nabla f(\tilde{x})\|^2 |_{\tilde{x}=\tilde{x}(t_0, T_2)} \geq -L^0(\tau - t_0).$$

But

$$\|P_{A\tilde{D}} \tilde{D} \nabla f(\tilde{x})\|^2 |_{\tilde{x}=\tilde{x}(t_0, T_2)} = \|P_{A\tilde{D}} \tilde{D} \nabla f(\tilde{x})\|^2 |_{\tilde{x}=x(T_2)} \geq c_0^2,$$

then

$$\|P_{AD} D \nabla f(x)\|^2 |_{x=\tilde{x}(\tau, T_2)} \geq \max(c_0^2 - L^0(\tau - t_0), 0).$$

Thus

$$\int_{t_0}^{+\infty} -\frac{df(\tilde{x}(\tau, T_2))}{d\tau} d\tau = \int_{t_0}^{+\infty} \|P_{AD} D \nabla f(x)\|^2 |_{x=\tilde{x}(\tau, T_2)} d\tau \geq \frac{{c_0}^4}{2L^0},$$

but from (2.28), (2.29), and the fact that $f(x(T_2)) \leq f(x(T_1))$, we have

$$\int_{t_0}^{+\infty} -\frac{df(\tilde{x}(\tau, T_2))}{d\tau} d\tau = f(x(T_2)) - f(x^*) \leq f(x(T_1)) - f(x^*) < \frac{{c_0}^4}{4L^0}.$$

This inequality contradicts with the previous inequality. Thus the proof is complete.

$\blacksquare$ $\square$

Next, we will reveal that the limit points of the two solutions of ODE systems (2.1) and (2.2) have the maximal number of the positive components in $\{x_1, \ldots, x_k\}$ among the optimal solution set.

Theorem 2.16. *The limit points of the two solutions of ODE systems (2.1) and (2.2) both have the maximal number of the positive components in $\{x_1, \ldots, x_s\}$ among the optimal solution set.*

Proof. We consider the function $I_1(x(t))$ in Theorem 2.5 and the function $I_2(t, x(t))$ in Theorem 2.7. Let the solution of ODE systems (2.1) and (2.2)be $x(t)$ and $\bar{x}(t)$ respectively. From the proofs of Theorem 2.5 and Theorem 2.7, we have for any $t \in [t_0, +\infty)$,

$$I_1(x(t)) \leq I_1(x(t_0)),$$

and

$$I_2(t, \bar{x}(t)) \leq I_2(t_0, \bar{x}(t_0)).$$

Then from Lemmas 1.3 and 1.4, it is easy to see that for each $i \in B(x^*)$ (without loss of generality, we can assume that the optimal solution x^* in Assumption 2.1 has the maximal number of the positive components in $\{x_1, \ldots, x_s\}$ among the optimal solution set), the ith component of each solution of the two ODE systems $x_i(t)$ or $\bar{x}_i(t)$ is bounded below by some positive constant c_i. So each limit point will have the maximal number of the positive components in $\{x_1, \ldots, x_s\}$ among the optimal solution set. Thus the proof is complete. $\blacksquare$ $\square$

2.4 A Preliminary Solution Scheme and Numerical Results

In this section, we give a preliminary solution scheme which is actually the explicit Euler scheme for ODE system (2.2). For simplicity, we only consider the problem (P_1) with $s = n$ and $f(x)$ is quadratic. In the following, we state the new algorithm without any proof, and give some numerical results. The presentation only provides some indications on the possible future development on the numerical aspect and implementation of the ODE system (2.2). The discussions consist of the following three parts: (i) a method for finding an initial interior point in $\mathcal{P}_1^{++}$, (ii) an algorithm for finding the limit point of ODE system (2.2), and (iii) some preliminary numerical results.

When $s = n$, finding an interior feasible point in $\mathcal{P}_1^{++}$ can be regarded as minimizing a convex quadratic function with nonnegative constraints. We can use the algorithm in [73] with $\gamma_2 = \frac{1}{2}$ to solve it. However, there are two concerns: one is that the convergence of the algorithm may be slow; and the other is whether the sequence will converge to a point with the property that all the components are positive. The way to address the first concern is that when the error $\|b - Ax_k\|$ is small, a projection of the residual can be adopted. That is, at the iteration k, let

$$\Delta x_k = A^T (AA^T)^{-1} (b - Ax_k),$$

then the point $x_k + \Delta x_k$ will satisfy the equality constraint, and if the point is positive, then we get an interior feasible point. As for the second concern, the iterative method does not have this property as we know, but we have this property in the continuous situation according to Theorem 2.16 when $s = n$.

Now we introduce a new algorithm for finding the limit point of ODE system (2.2). The algorithm is basically an explicit Euler scheme for ODE system (2.2), where the right-hand side is replaced by (2.16) due to Lemma 2.2. Our algorithm is only for convex quadratic programming problems, i.e., $f(x) = \frac{1}{2}x^T Qx + c^T x$ in problem (P_1) with $s = n$.

Algorithm 2.4.1

Step 0: Initialize $\theta \in (0,1)$, $x_0 \in \mathcal{P}_1^{++}$, $t_0 > 0$, and $k = 0$.

Step 1: Calculate d_k from (2.16), that is

$$d_k = -DP_{AD}\left[\gamma_1 I_n + tP_{AD}DQDP_{AD}\right]^{-1} P_{AD}D(Qx + c),$$

where $x = x_k$ and $t = t_k$.

Step 2: If the norm of d_k is very small, stop; otherwise, let

$$\bar{h}_1 = \arg\min_h \{f(x_k + hd_k) | h \geq 0\}, \quad \text{and} \quad \bar{h}_2 = \sup_h \{x_k + hd_k > 0\}.$$

Then take $h_k = \min(\bar{h}_1, \theta\bar{h}_2)$, $x_{k+1} = x_k + h_k d_k$, $t_{k+1} = t_k + h_k$, $k = k + 1$; go to Step 1.

Our convex quadratic test problems are constructed as follows. The matrix Q is generated by

$$Q = H^T H, \ H = randn(r1, n),$$

where $randn$ is the random function in $Matlab$. Then matrix A is generated by $A = randn(m, n)$, and an optimal solution x^* is also generated randomly with 20% of components on the boundary, i.e., $x_i^* = 0$. Finally, vectors b and c are obtained from KKT condition (1.3).

We next report some numerical results of the new algorithm. All our tests are conducted in Matlab (version 2014b) platform on a PC. In addition to our new algorithm, a Matlab function *quadprog* is also tested as the benchmark. For our new algorithm, we set $\gamma_2 = 0.75$ (which seems to achieve the best performance), $\gamma_1 = 1$, $\theta = 0.9$, and $t_0 = 1$. Since the optimal solution in our test is known, two stopping criteria are adopted: (i) the relative error in the objective function value, i.e., $\frac{|f(x_k)-f^*|}{|f^*|+1} < \epsilon_1$, where f^* is the optimal objective function value, and (ii) the norm of the direction d_k, i.e., $\|d_k\|_\infty < \epsilon_2$. In our tests, we set two values for ϵ_1 as 10^{-10} and 10^{-12}; two values for ϵ_2 as 10^{-6} and 10^{-8}. The stopping criterion for Matlab function *quadprog* just follows the default setting. The CPU time (in seconds) reported for our algorithm includes the time for finding the initial interior feasible point. Each number reported in the following tables is the average of 20 runs.

Remarks on the numerical results: (i) The results presented in Tables 2.1-2.5 are only for our randomly generated test problems, these results are for indication only. (ii) Our algorithm is just a simple explicit Euler scheme for ODE system (2.2), more efficient solution schemes may be developed. (iii) The convergence and convergence rate for our algorithm are not discussed here since these are beyond the scope of this paper.

Table 2.1: Numerical results for $Rank(Q) = 0.5n$, $m = 0.1n$ and $\epsilon_1 = 10^{-10}$

	CPU Time		$\frac{\|f(x_k)-f^*\|}{\|f^*\|+1}$	$\|Ax-b\|_2$		Iter. Number	
n	our alg.	quadprog	quadprog	our alg.	quadprog	our alg.	quadprog
2500	8.3	65.5	1.20e-10	1.20e-12	1.27e-12	19.05	12.60
5000	37.5	456.8	2.63e-10	3.37e-12	3.41e-12	20.10	14.05
10000	181.3	3259.5	1.67e-10	9.52e-12	8.29e-12	20.55	14.90
20000	1056.4	23538.0	1.95e-10	2.71e-11	2.34e-11	21.35	16.35

Note: The values of $\frac{|f(x_k)-f^*|}{|f^*|+1}$ for our algorithm with above n's are not listed since they are all less than $\epsilon_1 = 10^{-10}$.

Table 2.2: Numerical results for $Rank(Q) = 0.5n$, $m = 0.2n$ and $\epsilon_1 = 10^{-10}$

	CPU Time		$\frac{\lvert f(x_k)-f^*\rvert}{\lvert f^*\rvert+1}$	$\lVert Ax-b\rVert_2$		Iter. Number	
n	our alg.	quadprog	quadprog	our alg.	quadprog	our alg.	quadprog
2500	10.8	110.5	4.00e-10	1.66e-12	1.37e-12	21.90	13.00
5000	51.3	787.8	6.24e-10	4.64e-12	3.87e-12	23.25	14.05
10000	289.7	5615.3	5.49e-10	1.32e-11	1.08e-11	24.10	14.95
20000	1688.6	42773.6	9.00e-10	3.76e-11	3.00e-11	25.20	16.25

Note: The values of $\frac{|f(x_k)-f^*|}{|f^*|+1}$ for our algorithm with above n's are not listed since they are all less than $\epsilon_1 = 10^{-10}$.

Table 2.3: Numerical results for $Rank(Q) = 0.5n$ and $\epsilon_1 = 10^{-12}$

	CPU Time		$\lVert Ax-b\rVert_2$		Iter. Number	
n	$m = 0.1n$	$m = 0.2n$	$m = 0.1n$	$m = 0.2n$	$m = 0.1n$	$m = 0.2n$
2500	9.5	12.3	1.20e-12	1.66e-12	21.80	24.85
5000	43.0	53.6	3.39e-12	4.65e-12	23.00	26.35
10000	208.8	331.2	9.50e-12	1.32e-11	23.65	14.90
20000	1245.5	1999.3	2.71e-11	3.74e-11	25.15	29.90

Note: (i) The results for Matlab function *quadprog* are not listed in Table 2.3 since they are the same as the ones in Tables 2.1 and 2.2 (due to the same default setting). (ii) The values of $\frac{|f(x_k)-f^*|}{|f^*|+1}$ for our algorithm with above n's are not listed since they are all less than $\epsilon_1 = 10^{-12}$.

Table 2.4: Numerical results for $Rank(Q) = 0.5n$ and $m = 0.1n$

	CPU Time		$\frac{\lvert f(x_k)-f^*\rvert}{\lvert f^*\rvert+1}$		$\lVert Ax-b\rVert_2$		Iter. Number	
	ϵ_2		ϵ_2		ϵ_2		ϵ_2	
n	10^{-6}	10^{-8}	10^{-6}	10^{-8}	10^{-6}	10^{-8}	10^{-6}	10^{-8}
2500	8.4	9.2	8.86e-11	3.00e-12	1.18e-12	1.18e-12	18.95	20.80
5000	40.3	43.8	2.91e-11	8.52e-13	3.38e-12	3.39e-12	21.20	23.20
10000	203.1	218.2	5.98e-12	4.47e-13	9.52e-12	9.55e-12	22.55	24.35
20000	1248.2	1338.9	1.43e-12	6.91e-14	2.72e-11	2.72e-11	24.75	26.60

Note: (i) The results for Matlab function *quadprog* are not listed in Table 2.4 since they are the same as the ones in Tables 2.1 and 2.2 (due to the same default setting). (ii) The final results of $\|d_k\|_\infty$ for our algorithm with above n's are not listed since they are all less than respective ϵ_2 values.

Table 2.5: Numerical results for $Rank(Q) = 0.5n$ and $m = 0.2n$

	CPU Time		$\frac{\lvert f(x_k)-f^*\rvert}{\lvert f^*\rvert+1}$		$\lVert Ax-b\rVert_2$		Iter. Number	
	ϵ_2		ϵ_2		ϵ_2		ϵ_2	
n	10^{-6}	10^{-8}	10^{-6}	10^{-8}	10^{-6}	10^{-8}	10^{-6}	10^{-8}
2500	10.9	11.8	8.89e-11	6.81e-12	1.65e-12	1.65e-12	21.75	23.65
5000	53.6	58.5	3.17e-11	7.02e-13	4.64e-12	4.66e-12	24.10	26.45
10000	331.2	351.8	4.92e-12	2.19e-13	1.32e-11	1.32e-11	27.00	28.80
20000	2021.9	2189.6	1.22e-12	3.35e-14	3.75e-11	3.76e-11	29.55	32.00

Note: (i) The results for Matlab function *quadprog* are not listed in Table 2.5 since they are the same as the ones in Tables 2.1 and 2.2 (due to the same default setting). (ii) The final results of $\|d_k\|_\infty$ for our algorithm with above n's are not listed since they are all less than respective ϵ_2 values.

Chapter 3

First-order Primal Affine Scaling Continuous Trajectory for Convex Programming

A first-order primal affine scaling continuous trajectory for problem (P_2) is studied in this chapter. By assuming the existence of an optimal solution in the linear case or the boundedness of the optimal solution set in the general case, we show that starting from any interior feasible point, (i) every accumulation point is indeed an optimal solution; and (ii) if the objective function is analytic, the primal affine scaling continuous trajectory converges to a point which is actually in the relative interior of the optimal solution set.

The first-order primal affine scaling direction for problem (P_2) can be given by

$$-XP_{AX}X\nabla f(x),$$

where $x \in R^n_{++}$, $X = \text{diag}\,(x) \in R^{n\times n}$, $P_{AX} = I_n - XA^T(AX^2A^T)^{-1}AX$, and I_n (or I) stands for the $n \times n$ identity matrix. As a result, the first-order primal affine scaling continuous trajectory for problem (P_2) is the solution curve of the following ODE system

$$\frac{dx}{dt} = -XP_{AX}X\nabla f(x),\ x(t_0) = x^0 \in \mathcal{P}_2^{++},\ t \geq t_0 > 0. \tag{3.1}$$

3.1 Fundamental Properties of The Continuous Trajectory

The following assumptions are made throughout this chapter.

Assumption 3.1. *If $f(x) = c^T x$, we assume that there exists a point $x^* \in \mathcal{P}_2^+$ such that $c^T x^*$ is the optimal value of problem (P_2). Otherwise, we assume that the optimal solution set of problem (P_2) is non-empty and bounded.*

Assumption 3.2. *The set $\mathcal{P}_2^{++}$ is not empty.*

First we state two simple technical lemmas without proof.

Lemma 3.1. $(AX^2A^T)^{-1} \in C^1$ *on* R^n_{++}.

Lemma 3.2. $XP_{AX}X\nabla f(x) \in C^1$ *on* R^n_{++}.

Lemma 3.2 reveals the smoothness property for the right-hand side of ODE system (3.1). Theorem 3.1 and Theorem 3.2 below guarantee the existence, uniqueness, and feasibility for the solution of ODE system (3.1).

Theorem 3.1. *For ODE system (3.1), there exists a solution $x(t)$ which is unique on a maximal existence interval $[t_0, \alpha)$, in addition, $x(t) > 0$ on this existence interval.*

Proof. By Lemma 3.2, $XP_{AX}X\nabla f(x)$ is locally Lipschitz continuous on R^n_{++}. Since R^n_{++} is an open set, from Theorem IV.1.2 in [8], a solution $x(t)$ is existed and unique on the maximal existence interval $[t_0, \alpha)$, for some $\alpha > t_0$ or $\alpha = +\infty$.

Because the right-hand side of ODE system (3.1) is defined on the open set $(0, +\infty) \times R^n_{++}$, the solution of ODE system (3.1) is of course in the open set R^n_{++}, so $x(t)$ is positive on the existence interval. The proof is complete. □

Later in this section, it will be shown that $\alpha = +\infty$ (Theorem 3.4). For simplicity, in the following, let x (or X) stand for $x(t)$ (or $X(t)$).

Theorem 3.2. *Let the solution of ODE system (3.1) on the maximal existence interval* $[t_0, \alpha)$ *be* $x(t)$*. Then* $Ax(t) = b\ \forall t \in [t_0, \alpha)$*.*

Proof. For any $t \in [t_0, \alpha)$

$$x(t) = x^0 - \int_{t_0}^{t} (XP_{AX}X\nabla f(x)|_{t=\tau})d\tau.$$

Notice

$$AXP_{AX} = AX - AX^2A^T(AX^2A^T)^{-1}AX \equiv 0,$$

we can get

$$Ax(t) = Ax^0 - \int_{t_0}^{t} (AXP_{AX}X\nabla f(x)|_{t=\tau})d\tau = b.$$

Thus the theorem is proved. □

Next we show that the solution curve is contained in a bounded set.

Theorem 3.3. *The unique solution* $x(t)$ *of ODE system (3.1) is contained in a bounded set in* R^n_+*.*

Proof. If $f(x) = c^Tx$, from Theorem 3.1, $x(T) > 0$ for any $T \in [t_0, \alpha)$, then we can define

$$V_1(t) = \sum_{i=1}^{n} \frac{(x(T)_i - x_i^*)}{x(t)_i}, \quad t \in [t_0, \alpha),$$

where x^* is from Assumption 3.1. From Theorem 3.2, we have

$$\begin{aligned}
\frac{dV_1(t)}{dt} &= \sum_{i=1}^{n} -\frac{(x(T)_i - x_i^*)}{x(t)_i^2} \cdot \frac{dx_i}{dt} \\
&= -(x(T) - x^*)^T X(t)^{-2}\frac{dx}{dt} \\
&= (x(T) - x^*)^T X^{-2} XP_{AX}Xc \\
&= (x(T) - x^*)^T X^{-2}(X^2 - X^2A^T(AX^2A^T)^{-1}AX^2)c \\
&= (x(T) - x^*)^T c - (b - b)^T(AX^2A^T)^{-1}AX^2c \\
&= c^Tx(T) - c^Tx^* \\
&\geq 0,
\end{aligned}$$

then

$$V_1(t_0) \le V_1(T) = \sum_{i=1}^{n} 1 - \sum_{i=1}^{n} \frac{x_i^*}{x(T)_i} \le n. \tag{3.2}$$

Therefore for any $T \in [t_0, \alpha)$, we have

$$\|x(T)\| \le e^T x(T) \le \max_i(x_i^0) \sum_{i=1}^{n} \frac{x(T)_i}{x_i^0} \le \max_i(x_i^0) \left(n + \sum_{i=1}^{n} \frac{x_i^*}{x_i^0} \right).$$

The last inequality is from (3.2), which indicates that $x(T)$ is bounded, and the bound depends only on x^0, x^*, and n.

Otherwise, notice for $t \in [t_0, \alpha)$,

$$\frac{df(x(t))}{dt} = -\nabla f(x)^T X P_{AX} X \nabla f(x) = -\|P_{AX} X \nabla f(x)\|^2 \le 0,$$

we know that $f(x(t))$ is a nonincreasing function on $t \in [t_0, \alpha)$. Hence $x(t)$ will be contained in the level set $\{x \mid x \in \mathcal{P}_2^+,\ f(x) \le f(x^0)\}$. Under Assumption 3.1, from Theorem 24 on page 93 in [18], the level set will be bounded as well. Thus the proof is complete. $\square$

After we get the boundedness of the solution curve, we can extend the existence interval of the solution to infinity.

Theorem 3.4. *Let the solution of ODE system (3.1) with the maximal existence interval* $[t_0, \alpha)$ *be* $x(t)$*. Then* $\alpha = +\infty$*.*

Proof. Assume $\alpha \ne +\infty$. From Theorem 3.3, we know that $0 < x(t) \le Me\ \forall t \in [t_0, \alpha)$ for an $M > 0$. Furthermore, P_{AX} is symmetric and idempotent, which leads to $\|P_{AX}\| \le 1$. Therefore the vector $P_{AX} X \nabla f(x)$ is bounded. Then we know that there exists an $L > 0$ such that for every $i \in \{1, \dots, n\}$, we have

$$\left| \frac{dx_i}{dt} \right| \le L x_i \quad \forall t \in [t_0, \alpha), \tag{3.3}$$

and this L depends only on M, and f.

For every $i \in \{1, \ldots, n\}$, from inequality (3.3) and $0 < x(t) \leq Me \; \forall t \in [t_0, \alpha)$, we know that

$$\left|\frac{dx_i}{dt}\right| \leq LM \quad \forall t \in [t_0, \alpha), \tag{3.4}$$

furthermore, $x(t)$ is continuous on $[t_0, \alpha)$, and it is not hard to see that $\lim_{t \to \alpha^-} x(t)$ exists. We denote this limit as $x(\alpha)$. Evidently $x(\alpha) \geq 0$. According to the Extension Theorem in **S**2.5, [3], we know that the solution $x(t)$ will go to the boundary of the open set $(0, +\infty) \times R^n_{++}$. But because of the hypothesis, $\alpha \neq +\infty$, so there must exist at least one $i \in \{1, \ldots, n\}$ such that $x_i(\alpha) = 0$. From inequality (3.3), we know that if $t \in [t_0, \alpha)$,

$$\frac{dx_i}{x_i} \geq -Ldt.$$

Integrating the inequality above, we have for every $t \in [t_0, \alpha)$

$$\ln x_i(t) - \ln x_i(t_0) \geq -L(t - t_0).$$

Since $x_i(t) \to x_i(\alpha) = 0$ as $t \to \alpha^-$, $\ln x_i(t) - \ln x_i(t_0) \to -\infty$ as $t \to \alpha^-$, but $-L(t - t_0) \geq -L(\alpha - t_0)$. This is a contradiction. Therefore $\alpha = +\infty$, and the proof is complete. □

From Theorem 3.4, we can define the limit set for the solution of ODE system (3.1). The limit set of the solution of system (3.1) $\{x(t)\}$ can be defined as follows

$$\Omega^1(x^0) = \left\{ x \in R^n \mid \exists \; \{t_k\}_{k=0}^{+\infty} \text{ with } \lim_{k \to +\infty} t_k = +\infty \; \text{ such that } \lim_{k \to +\infty} x(t_k) = x \right\}.$$

Theorem 3.5. *The limit set $\Omega^1(x^0)$ is nonempty, compact, and connected. Furthermore $\Omega^1(x^0)$ is contained in $\mathcal{P}_2^+$.*

Proof. From Theorems 3.1, 3.2, and 3.4, we know that the limit set $\Omega^1(x^0)$ is contained in $\mathcal{P}_2^+$. From Theorem 3.3, we know that the solution $x(t)$ is contained in a bounded closed set. So similar to the proof of Theorem 1.1 on page 390 in [10] (the proof in [10] is for $n = 2$, but it can be easily exntended to the general case), it can be verified that $\Omega^1(x^0)$ is nonempty, compact, and connected. □

Now we also introduce a kind of potential function for ODE system (3.1). The potential function $I(x, y)$ for ODE system (3.1) can be defined as

$$\tilde{I}(x, y) = \sum_{i=1}^{n} (\ln x_i) + \sum_{i=1}^{n} \frac{y_i}{x_i}, \tag{3.5}$$

where $x \in R^n_{++}$ is the variable, $y \in R^n_+$ is a parameter.

In the rest of this section, we will show that for the solution of system (3.1), $XP_{AX}X\nabla f(x) \to 0$ as $t \to +\infty$. But first, we reveal some fundamental results.

Theorem 3.6. *Let the solution of ODE system (3.1) be $x(t)$. Then $f(x(t))$ is a nonincreasing function on $[t_0, +\infty)$. Furthermore, if $x^0 \in P^{++}$ is an optimal solution for problem (P_2), then $x(t) \equiv x^0$ on $[t_0, +\infty)$; otherwise $f(x(t))$ is a strictly decreasing function on $[t_0, +\infty)$.*

Proof. Since for $t \geq t_0$,

$$\frac{df(x(t))}{dt} = -\|P_{AX}X\nabla f(x)\|^2 \leq 0,$$

we know that $f(x(t))$ is a noincreasing function on $[t_0, +\infty)$.

For problem (P_2), the KKT conditions are

$$\begin{cases} Ax = b, \ x \geq 0, \\ Xz = 0, \ z \geq 0, \\ A^T y + z = \nabla f(x), \end{cases} \tag{3.6}$$

where $z \in R^n$ and $y \in R^m$.

If $x \in \mathcal{P}_2^{++}$ is an optimal solution, which implies that there must exist an (y, z) such that (x, y, z) satisfies system (3.6), then

$$z = 0 \quad \text{and} \quad A^T y = \nabla f(x),$$

thus, it is easy to see that

$$P_{AX}X\nabla f(x) = P_{AX}XA^T y = 0.$$

So if $x^0 \in \mathcal{P}_2^{++}$ is an optimal solution, we know that the right-hand side of ODE system (3.1) equals zero at $x = x_0$, i.e.,

$$XP_{AX}X\nabla f(x)|_{x=x_0} = 0,$$

therefore $x(t) \equiv x^0$ for $t \geq t_0$ is a solution of ODE system (3.1). Because of the uniqueness of the solution, we know that $x(t) \equiv x^0$ on $[t_0, +\infty)$.

If $x^0 \in \mathcal{P}_2^{++}$ is not an optimal solution, we will show $f(x(t))$ is a strictly decreasing function on $[t_0, +\infty)$. If not, then there must exist t_1 and t_2 with $t_0 \leq t_1 < t_2$ such that $f(x(t_1)) = f(x(t_2))$. Since $\frac{df(x(t))}{dt} \leq 0$, we know that when $t_1 \leq t \leq t_2$, $\frac{df(x(t))}{dt} \equiv 0$. From $\frac{df(x(t))}{dt} = -\|P_{AD}D\nabla f(x)\|^2 = -\|D^{-1}\frac{dx}{dt}\|^2 = 0$, Theorem 3.1 and Theorem 3.4, it is easy to see $\frac{dx}{dt} = 0$ on $[t_1, t_2]$, then $x(t) \equiv x(t_1)$ on $[t_0, +\infty)$ will be a solution of the ODE

$$\frac{dx}{dt} = -XP_{AX}X\nabla f(x)$$

that passes through the point $(t_1, x(t_1))$. But according to the uniqueness of the solution, we know that the solution of ODE system (3.1) is actually $x(t) \equiv x(t_1) = x^0$ on $[t_0, +\infty)$. Hence $\frac{dx}{dt}|_{t=t_0} = 0$, which implies

$$(I - A^T(AX_0^2A^T)^{-1}AX_0^2)\nabla f(x^0) = 0,$$

where $X_0 = X(t_0)$. Let $y = (AX_0^2A^T)^{-1}AX_0^2\nabla f(x^0)$ and $z = 0$, then (x^0, y, z) will satisfy the KKT system (3.6). Therefore x^0 must be an optimal solution. However this is a contradiction. Thus $f(x(t))$ is a strictly decreasing function on $[t_0, +\infty)$. □

Lemma 3.3. *If $0 < x \leq Me$ with $M > 0$, then for every $i \in \{1, \ldots, n\}$, every entry of*

$$\frac{\partial XP_{AX}X\nabla f(x)}{\partial x_i}$$

is bounded, and the bound depends only on A, M, n, and f.

Proof. Let $H = (AX^2A^T)^{-1}AX^2$. From Lemma 3 and the Remark in Sun [66], we know that if $x > 0$ every entry of $(AX^2A^T)^{-1}AX^2$ is bounded, and the bound depends only on A and n. Notice

$$\frac{\partial XP_{AX}X}{\partial x_i} = \frac{\partial X^2}{\partial x_i} - \frac{\partial X^2A^T}{\partial x_i}H - X^2A^T\frac{\partial (AX^2A^T)^{-1}}{\partial x_i}AX^2 - H^T\frac{\partial AX^2}{\partial x_i},$$

and

$$X^2A^T\frac{\partial (AX^2A^T)^{-1}}{\partial x_i}AX^2 = -2x_iH^T(Ae_ie_i^TA^T)H.$$

Therefore when $0 < x \leq Me$, every entry of

$$\frac{\partial XP_{AX}X}{\partial x_i}$$

is bounded, and the bound depends only on A, M, and n. Then it is evident that for every $i \in \{1, \ldots, n\}$, every entry of

$$\frac{\partial XP_{AX}X\nabla f(x)}{\partial x_i}$$

is bounded, and the bound depends only on A, M, n, and f. □

Theorem 3.7. *Let the solution of ODE system (3.1) be* $x(t)$. *Then*

$$\lim_{t\to+\infty} DP_{AD}D\nabla f(x) = 0.$$

Proof. From Theorem 3.3, we know that $x(t)$ is contained in the bounded closed set $\{x \in R^n | 0 \leq x \leq Me\}$ for an $M > 0$. This along with Lemma 3.3 indicates that there exists a constant L_1 such that for every $i \in \{1, \ldots, n\}$, every entry of

$$\frac{\partial \nabla f(x)^TXP_{AX}X\nabla f(x)}{\partial x_i} \tag{3.7}$$

is bounded by L_1, and L_1 depends only on A, M, n, and f.

From Theorem 3.6, we know that $f(x(t))$ is a nonincreasing function and $f(x(t)) \geq f(x^*)$ on $[t_0, +\infty)$. Thus $f(x(t))$ has a finite limit as $t \to +\infty$. From (3.4), we have

$$\begin{aligned}
& \left|\frac{df(x(t))}{dt}\,|_{t=t_1} - \frac{df(x(t))}{dt}\,|_{t=t_2}\right| \\
= & \left|\int_0^1 \frac{\partial \nabla f(x)^TXP_{AX}X\nabla f(x)}{\partial x}|_{x=x(t_2)+\tau(x(t_1)-x(t_2))}(x(t_1)-x(t_2))d\tau\right| \\
\leq & \sqrt{n}L_1 \cdot \|x(t_1) - x(t_2)\| \\
= & \sqrt{n}L_1 \cdot \|\int_{t_1}^{t_2}\frac{dx}{d\tau}d\tau\| \\
\leq & nL_1LM|t_1 - t_2|,
\end{aligned}$$

where the last inequality is obtained from inequality (3.4).

Thus, $\frac{df(x(t))}{dt}$ is uniformly continuous. From Barbalat's Lemma, we know that

$$\lim_{t\to+\infty}\frac{df(x(t))}{dt} = -\lim_{t\to+\infty}\|P_{AX}X\nabla f(x)\|^2 = 0,$$

which indicates

$$\lim_{t\to+\infty} XP_{AX}X\nabla f(x) = 0.$$

Thus the proof is complete. □

3.2 Optimality of The Cluster Point(s)

Theorem 3.8. *For any $x^{(1)} \in \Omega^1(x^0)$, $x^{(1)}$ is an optimal solution for problem (P_2).*

Proof. We prove this by contradiction. Assume $x^{(1)}$ is not an optimal solution, then from Theorem 3.6, we know $f(x^0) > f(x^{(1)}) = \lim_{k\to+\infty} f(x(t_k)) > f(x^*)$, where $\lim_{k\to+\infty} x(t_k) = x^{(1)}$. Let's define

$$y^{(1)} = \frac{f(x^{(1)}) - f(x^*)}{2(f(x^0) - f(x^*))}x^0 + \left[1 - \frac{f(x^{(1)}) - f(x^*)}{2(f(x^0) - f(x^*))}\right]x^*,$$

then $y^{(1)} \in \mathcal{P}_2^{++}$. Since $y^{(1)}$ is a convex combination of x^0 and x^*, obviously

$$f(y^{(1)}) \le \frac{f(x^{(1)}) - f(x^*)}{2(f(x^0) - f(x^*))}f(x^0) + \left[1 - \frac{f(x^{(1)}) - f(x^*)}{2(f(x^0) - f(x^*))}\right]f(x^*) = \frac{f(x^{(1)}) + f(x^*)}{2}.$$

Then we can define

$$V_2(t) = \tilde{I}(x(t), y^{(1)}) = \sum_{i=1}^{n}(\ln x_i) + \sum_{i=1}^{n}\frac{y_i^{(1)}}{x_i},$$

where $t \in [t_0, +\infty)$ and $x(t)$ is the unique solution of ODE system (3.1). Then from

Theorem 3.2 and Lemma 1.2, we have $\forall t \geq t_0$

$$\begin{aligned}
\frac{dV_2(t)}{dt} &= (x - y^{(1)})^T X^{-2} \frac{dx}{dt} \\
&= (y^{(1)} - x)^T X^{-2} X P_{AX} X \nabla f(x) \\
&\leq f(y^{(1)}) - f(x) \\
&\leq f(y^{(1)}) - f(x^{(1)}) \\
&\leq \frac{f(x^*) - f(x^{(1)})}{2} \\
&< 0,
\end{aligned}$$

therefore $V_2(t) \to -\infty$ as $t \to +\infty$. But

$$V_2(t) = \sum_{i=1}^{n}((\ln x_i) + \frac{y_i^{(1)}}{x_i}) \geq \sum_{i=1}^{n}(\ln y_i^{(1)} + 1) > -\infty.$$

Hence the hypothesis is not true, and $x^{(1)}$ is an optimal solution for problem (P_2). □

3.3 Convergence of The Continuous Trajectory

Now, it comes to the key result of this chapter. Theorem 3.9 below shows that if $f(x)$ is analytic, the solution of ODE system (3.1) converges to a point which is in the relative interior of the optimal solution set as $t \to +\infty$.

Theorem 3.9. *If the objective function $f(x)$ in problem (P_2) is analytic, then the limit set $\Omega^1(x^0)$ contains a single point only which is in the relative interior of the optimal solution set of problem (P_2).*

Proof. From Theorem 3.5, we know that $\Omega^1(x^0)$ is not empty. So we can choose a point $\bar{x} \in \Omega^1(x^0)$, and evidently $\bar{x} \in \mathcal{P}_2^+$. Without loss of generality, we assume an optimal solution x^* has the maximal number of positive components in the optimal solution set for problem (P_2), which is actually in the relative interior of the optimal solution set since $f(x)$ is analytic. We denote this number as k. If $k = 0$, the proof is complete, and we assume $1 \leq k \leq n$ below. Let's define

$$V_3(t) = \sum_{i=1}^{n} \frac{\bar{x}_i - x_i^*}{x(t)_i}, \quad t \in [t_0, +\infty).$$

Then from Lemma 1.6 and Theorem 3.8, we have

$$\frac{dV_3(t)}{dt} = (\bar{x} - x^*)^T X^{-2} X P_{AX} X \nabla f(x) = (\bar{x} - x^*)^T \nabla f(x) = 0,$$

so the function $V_3(t)$ is a constant. Since $\bar{x} \in \Omega^1(x^0)$, there exists a sequence $\{t_k\}$ with $t_k \to +\infty$ as $k \to +\infty$ such that $x(t_k) \to \bar{x}$ as $k \to +\infty$. Thus as $k \to +\infty$, for any index i, there are four situations:

(i) if $\bar{x}_i = x_i^* = 0$, then $\frac{\bar{x}_i - x_i^*}{x(t_k)_i} \equiv 0$;

(ii) if $\bar{x}_i = 0$, $x_i^* > 0$, then $\frac{\bar{x}_i - x_i^*}{x(t_k)_i} \to -\infty$;

(iii) if $\bar{x}_i > 0$, $x_i^* = 0$, then $\frac{\bar{x}_i - x_i^*}{x(t_k)_i} \to 1$;

(iv) if $\bar{x}_i > 0$, $x_i^* > 0$, then $\frac{\bar{x}_i - x_i^*}{x(t_k)_i} \to 1 - \frac{x_i^*}{\bar{x}_i}$.

Therefore, for any index i such that $x_i^* > 0$, $\bar{x}_i$ must be positive because $V_3(t)$ is a constant. Since x^* has the maximal number of positive components in the optimal solution set for problem (P_2) and $\bar{x}$ is also an optimal solution, we know $\bar{x}$ must also have k positive components and hence must be in the relative interior of the optimal solution set.

If $\bar{x}$ is not the only point in $\Omega^1(x^0)$, there must exist another point $\tilde{x} \in \Omega^1(x^0)$ with $\tilde{x} \neq \bar{x}$. Obviously, $\tilde{x}$ also has k positive components. Without loss of generality, we assume that the first k components of $\bar{x}$ and $\tilde{x}$ are positive. Then we define

$$V_4(t) = \sum_{i=1}^{n} \frac{\bar{x}_i - \tilde{x}_i}{x(t)_i} = \sum_{i=1}^{k} \frac{\bar{x}_i - \tilde{x}_i}{x(t)_i}, \quad t \in [t_0, +\infty).$$

Similar to $V_3(t)$, we can get that $V_4(t)$ is also a constant. Therefore if we let $x(t_k) \to \bar{x}$ and $x(t_l) \to \tilde{x}$ as $k,\ l \to +\infty$, respectively, we can get

$$\sum_{i=1}^{k}(1 - \frac{\tilde{x}_i}{\bar{x}_i}) = \sum_{i=1}^{k}(\frac{\bar{x}_i}{\tilde{x}_i} - 1),$$

which indicates that for $1 \leq i \leq k$, $\bar{x}_i = \tilde{x}_i$. So $\bar{x} = \tilde{x}$. Therefore, the limit set $\Omega^1(x^0)$ is a singleton. $\square$

Chapter 4

Two Primal-Dual Interior Point Continuous Trajectories for Convex Programming

In this chapter, we analyze two primal-dual interior point continuous trajectories for convex programming for problem (P_2). The two continuous trajectories are derived from the primal-dual path-following method and the primal-dual affine scaling method respectively. Theoretical properties of the two interior point continuous trajectories are fully studied. The optimality and convergence of both interior point continuous trajectories are obtained for any interior feasible point under some mild conditions. In particular, with proper choice of some parameters, the convergence for all both interior point continuous trajectories does not require the strict complementarity or the analyticity of the objective function. We assume $f(x) \in C^3$ on R^n throughout this chapter.

The Wolfe dual problem [47] associated to (P_2) is

$$\begin{array}{ll} \max & L(y,z) = f(x) - \nabla f(x)^T x + b^T y \\ \text{s.t.} & -\nabla f(x) + A^T y + z = 0,\ z \geq 0, \end{array} \qquad (D_2)$$

and the following notations will be used

$$\begin{aligned}
\mathcal{D}_2^+ &= \{(x,y,z) \in R^n \times R^m \times R^n | -\nabla f(x) + A^T y + z = 0,\ z \geq 0\}, \\
\mathcal{D}_2^{++} &= \{(x,y,z) \in R^n \times R^m \times R^n | -\nabla f(x) + A^T y + z = 0,\ z > 0\}, \\
\mathcal{F}_2 &= \{(x,y,z) \in R^n \times R^m \times R^n | x \in \mathcal{P}_2^+,\ (x,y,z) \in \mathcal{D}_2^+\}, \\
\mathcal{F}_2^0 &= \{(x,y,z) \in R^n \times R^m \times R^n | x \in \mathcal{P}_2^{++},\ (x,y,z) \in \mathcal{D}_2^{++}\}.
\end{aligned}$$

4.1 The Weighted Primal-Dual Path-Following Continuous Trajectory

In this section, we study a continuous trajectory that is closely related to the primal-dual interior point method. The search direction in most of primal-dual interior point algorithms is the solution of the following system:

$$\begin{pmatrix} -\nabla^2 f(x) & A^T & I \\ A & 0 & 0 \\ Z & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -XZe + \sigma\mu e \end{pmatrix}, \tag{4.1}$$

where $\sigma \in [0,1]$ and $\mu = x^T z/n$. The search directions of the primal-dual path-following algorithms and primal-dual potential reduction algorithms are generally the solution of system (4.1). The difference between them is the choice of the stepsize at each iteration. The stepsize in primal-dual path-following algorithms is determined by the neighborhood of the central path, while the stepsize in primal-dual potential reduction algorithms is determined by the potential function. If $\sigma = 0$, the search direction is usually called the primal-dual affine scaling direction, and for the corresponding continuous trajectory we call it the primal-dual affine scaling continuous trajectory which will be studied in Section 3. If $\sigma = 1$ and $(x,y,z) \in \mathcal{F}_2^0$, the search direction is actually the Newton direction of the following equation

$$\begin{pmatrix} A^T y + z \\ Ax \\ XZe \end{pmatrix} = \begin{pmatrix} \nabla f(x) \\ b \\ \mu e \end{pmatrix},$$

where $\mu = x^T z/n$. Evidently, the solution of the above equation is a point on the central path with the dual gap $\mu = \frac{x^T z}{n}$, not the optimal solution if $\mu > 0$. Therefore in this section we study the search direction with $\sigma \in (0, 1)$. We first extend system (4.1) to a more general form to get the following ODE system

$$\begin{cases} -\nabla^2 f(x)\frac{dx}{dt} + A^T\frac{dy}{dt} + \frac{dz}{dt} = 0, \\ A\frac{dx}{dt} = 0, \\ \gamma_1 X^{\gamma_1 - 1} Z^{\gamma_2}\frac{dx}{dt} + \gamma_2 X^{\gamma_1} Z^{\gamma_2 - 1}\frac{dz}{dt} = -(X^{\gamma_1} Z^{\gamma_2} e - \sigma\mu w), \\ (x(t_0), y(t_0), z(t_0) = (x^0, y^0, z^0) \in \mathcal{F}_2^0, \end{cases} \tag{4.2}$$

where

$$t_0 > 0,\ \sigma \in (0,1),\ \mu = \frac{e^T X^{\gamma_1} Z^{\gamma_2} e}{n},\ \gamma_1 > 0,\ \gamma_2 > 0,\ w \in R^n_{++},\ \sum_{i=1}^n w_i = n,$$
$$x \in R^n_{++},\ X = \text{diag }(x) \in R^{n\times n},\ z \in R^n_{++},\ Z = \text{diag }(z) \in R^{n\times n}.$$

We call this ODE system the weighted primal-dual path-following ODE system. The unique (Theorem 4.1) solution of ODE system (4.2) defines the weighted primal-dual path-following continuous trajectory for problem (P_2). The condition that the weighted vector w satisfies $\sum_{i=1}^n w_i = n$ is to guarantee $\mu = \frac{e^T X^{\gamma_1} Z^{\gamma_2} e}{e^T w} = \frac{e^T X^{\gamma_1} Z^{\gamma_2} e}{n}$. From the above ODE system, we can get two equivalent explicit forms, one is

$$\begin{cases} \frac{dx}{dt} = -\frac{1}{\gamma_1}\left[I_n + \frac{\gamma_2}{\gamma_1}(XZ^{-1})^{\frac{1}{2}} P_{A(XZ^{-1})^{\frac{1}{2}}} (XZ^{-1})^{\frac{1}{2}} \nabla^2 f(x)\right]^{-1} [X^{1-\gamma_1} Z^{-\gamma_2} \\ \qquad - Z^{-1} X A^T (AZ^{-1} X A^T)^{-1} A X^{1-\gamma_1} Z^{-\gamma_2}](X^{\gamma_1} Z^{\gamma_2} e - \sigma\mu w), \\ \frac{dy}{dt} = \frac{1}{\gamma_2}(AZ^{-1} X A^T)^{-1} A X^{1-\gamma_1} Z^{-\gamma_2}(X^{\gamma_1} Z^{\gamma_2} e - \sigma\mu w) \\ \qquad + (AZ^{-1} X A^T)^{-1} A X Z^{-1} \nabla^2 f(x)\frac{dx}{dt}, \\ \frac{dz}{dt} = \nabla^2 f(x)\frac{dx}{dt} - A^T\frac{dy}{dt}, \quad (x(t_0), y(t_0), z(t_0)) = (x^0, y^0, z^0) \in \mathcal{F}_2^0, \end{cases} \tag{4.3}$$

where $P_{A(XZ^{-1})^{\frac{1}{2}}} = I_n - (XZ^{-1})^{\frac{1}{2}} A^T (AZ^{-1}XA^T)^{-1} A (XZ^{-1})^{\frac{1}{2}}$, I_n is the identity matrix of order n. The other explicit form is

$$\begin{cases} \frac{dx}{dt} = -G\left[X^{1-\gamma_1}Z^{1-\gamma_2} - \gamma_2 XA^T(AGXA^T)^{-1}AGX^{1-\gamma_1}Z^{1-\gamma_2}\right] \\ \qquad (X^{\gamma_1}Z^{\gamma_2}e - \sigma\mu w), \\ \frac{dy}{dt} = \left[(AGXA^T)^{-1}AGX^{1-\gamma_1}Z^{1-\gamma_2}\right](X^{\gamma_1}Z^{\gamma_2}e - \sigma\mu w), \\ \frac{dz}{dt} = \nabla^2 f(x)\frac{dx}{dt} - A^T\frac{dy}{dt}, \quad (x(t_0), y(t_0), z(t_0)) = (x^0, y^0, z^0) \in \mathcal{F}_2^0, \end{cases}$$

where $G = (\gamma_1 Z + \gamma_2 X \nabla^2 f(x))^{-1}$. The following assumption is made throughout Section 2.

Assumption 4.1. *The set $\mathcal{F}_2^0$ is nonempty.*

4.1.1 Fundamental Properties of The Weighted Primal-Dual Path-Following Continuous Trajectory

First we need the following lemma to obtain the existence and uniqueness of the solution of the weighted primal-dual path-following ODE system (4.3).

Lemma 4.1. $(AZ^{-1}XA^T)^{-1} \in C^1$ *on $R^n_{++} \times R^n_{++}$ with respect to x and z.*

Theorem 4.1 and Theorem 4.2 below guarantee the existence, uniqueness, and feasibility for the solution of ODE system (4.3).

Theorem 4.1. *For the weighted primal-dual path-following ODE system (4.3), there exists a unique solution $(x(t), y(t), z(t))$ with the maximal existence interval $[t_0, \beta)$, in addition, $x(t) > 0$, $z(t) > 0$ on the existence interval.*

Proof. Since $(XZ^{-1})^{\frac{1}{2}} P_{A(XZ^{-1})^{\frac{1}{2}}} (XZ^{-1})^{\frac{1}{2}}$ and $\nabla^2 f(x)$ are both symmetric and positive semidefinite, from Lemma 1.1 we know

$$\left[I + \frac{\gamma_2}{\gamma_1}(XZ^{-1})^{\frac{1}{2}} P_{A(XZ^{-1})^{\frac{1}{2}}} (XZ^{-1})^{\frac{1}{2}} \nabla^2 f(x)\right]$$

is always invertible for any $x > 0$ and $z > 0$. Furthermore, from $\nabla^2 f(x) \in C^1$ on R^n_+, $Z^{-1} \in C^1$ on R^n_{++} with respect to z, and Lemma 4.1, it is not hard to see that the right-hand sides of (4.3) are all locally Lipschitz continuous on $R^n_{++} \times R^m \times R^n_{++}$. Since $R^n_{++} \times R^m \times R^n_{++}$ is an open set, from Theorem IV.1.2 in [8], there exists a unique solution $(x(t), y(t), z(t))$ of ODE system (4.3) on the maximal existence interval $[t_0, \beta)$, for some $\beta > t_0$ or $\beta = +\infty$.

Because the right-hand sides of ODE system (4.3) are defined on the open set $(0, +\infty) \times R^n_{++} \times R^m \times R^n_{++}$, the solution of ODE system (4.3) is of course in the open set $R^n_{++} \times R^m \times R^n_{++}$, so $x(t)$ and $z(t)$ are both positive on the existence interval. Thus the proof is complete. □

Later in this section, it will be shown that $\beta = +\infty$ (Theorem 4.4). For simplicity, in the following, let x, y, and z (or X, Z) stand for $x(t)$, $y(t)$, and $z(t)$ (or $X(t)$, $Z(t)$) respectively.

Theorem 4.2. *Let the solution of the weighted primal-dual path-following ODE system (4.3) on the maximal existence interval* $[t_0, \beta)$ *be* $(x(t), y(t), z(t))$*. Then* $Ax(t) = b$ *and* $A^T y(t) + z(t) = \nabla f(x(t))$ $\forall t \in [t_0, \beta)$.

Proof. Notice $A^T y^0 + z^0 = \nabla f(x^0)$ and the first equation in (4.2), then

$$A^T \frac{dy}{dt} + \frac{dz}{dt} \equiv \nabla^2 f(x) \frac{dx}{dt},$$

we have for any $t \in [t_0, \beta)$,

$$\begin{aligned} A^T y(t) + z(t) &= A^T(y^0 + \int_{t_0}^t \frac{dy}{dt}|_{t=\tau} d\tau) + z^0 + \int_{t_0}^t \frac{dz}{dt}|_{t=\tau} d\tau \\ &= A^T y^0 + z^0 + \int_{t_0}^t (A^T \frac{dy}{dt} + \frac{dz}{dt})|_{t=\tau} d\tau \\ &= A^T y^0 + z^0 + \int_{t_0}^t \nabla^2 f(x) \frac{dx}{dt}|_{t=\tau} d\tau \\ &= \nabla f(x^0) + \nabla f(x(t)) - \nabla f(x^0) = \nabla f(x(t)). \end{aligned}$$

From the second equation in (4.2) and $Ax^0 = b$, we can get $Ax(t) = b$ similarly. □

Next we show that the solution curve is contained in a bounded set.

Theorem 4.3. *The unique solution $(x(t), y(t), z(t))$ of the weighted primal-dual path-following ODE system (4.3) is contained in a bounded set in $R^n_+ \times R^m \times R^n_+$, and the bound of $x(t)$ and $z(t)$ depends on x^0, z^0, γ_1, and γ_2 only, the bound of $y(t)$ depends on x^0, z^0, γ_1, γ_2, A, and f only.*

Proof. First we assume that $x(t)^T z(t)$ is bounded by $M > 0$. By Theorem 4.2 and the convexity of $f(x)$, for any $t \in [t_0, \beta)$

$$\begin{aligned}
(x(t) - x^0)^T (z(t) - z^0) &= (x - x^0)^T (-A^T y + A^T y^0 + \nabla f(x) - \nabla f(x^0)) \\
&= (Ax - Ax^0)^T (y^0 - y) + (x - x^0)^T (\nabla f(x) - \nabla f(x^0)) \\
&= (x - x^0)^T (\nabla f(x) - \nabla f(x^0)) \geq 0.
\end{aligned}$$

Then we get

$$(x^0)^T z(t) + x(t)^T z^0 \leq (x^0)^T z^0 + x(t)^T z(t) \leq (x^0)^T z^0 + M.$$

Since $x(t) > 0$ and $z(t) > 0$ (Theorem 4.1), we know for any $t \in [t_0, \beta)$ and any $1 \leq i \leq n$,

$$x(t)_i \leq \frac{(x^0)^T z^0 + M}{z_i^0}, \ \ z(t)_i \leq \frac{(x^0)^T z^0 + M}{x_i^0},$$

therefore $x(t)$ and $z(t)$ are contained in a bounded set in R^n_+, and the bound depends on x^0, z^0 and M only.

For $y(t)$, since $A^T y(t) + z(t) = \nabla f(x(t))$ and the matrix A is of full row rank which implies AA^T is invertible, then we have

$$y(t) = (AA^T)^{-1} A (\nabla f(x(t)) - z(t)).$$

Thus $y(t)$ must be contained in a bounded set in R^m and the bound depends on x^0, z^0, M, A, and f only.

Next we show that there exists a bound $M > 0$ which depends only on x^0, z^0, γ_1

and γ_2. First from the third equation in (4.2), we have

$$\begin{aligned}\frac{de^T X(t)^{\gamma_1} Z(t)^{\gamma_2} e}{dt} &= \gamma_1 e^T X^{\gamma_1 - 1} Z^{\gamma_2} \frac{dx}{dt} + \gamma_2 e^T X^{\gamma_1} Z^{\gamma_2 - 1} \frac{dz}{dt} \\ &= -e^T (X^{\gamma_1} Z^{\gamma_2} e - \sigma\mu w) = -e^T X^{\gamma_1} Z^{\gamma_2} e + \sigma n \mu \\ &= -(1-\sigma) e^T X^{\gamma_1} Z^{\gamma_2} e,\end{aligned}$$

hence

$$\mu = \mu_0 e^{-(1-\sigma)(t-t_0)}, \tag{4.4}$$

where $\mu_0 = e^T X(t_0)^{\gamma_1} Z(t_0)^{\gamma_2} e/n$.

If $\gamma_1 > \gamma_2$, for any $1 \leq i \leq n$,

$$x(t)_i z(t)_i = x_i z_i^{\frac{\gamma_2}{\gamma_1}} z_i^{(1-\frac{\gamma_2}{\gamma_1})} = (x_i^{\gamma_1} z_i^{\gamma_2})^{\frac{1}{\gamma_1}} z_i^{(1-\frac{\gamma_2}{\gamma_1})} \leq (n\mu)^{\frac{1}{\gamma_1}} z_i^{(1-\frac{\gamma_2}{\gamma_1})},$$

furthermore,

$$z(t)_i \leq \frac{(x^0)^T z^0 + x(t)^T z(t)}{x_i^0} \leq \frac{(x^0)^T z^0 + x(t)^T z(t)}{\min(x^0)},$$

along with (4.4), we get

$$x(t)_i z(t)_i \leq (n\mu_0)^{\frac{1}{\gamma_1}} e^{-(1-\sigma)(t-t_0)/\gamma_1} \left[\frac{(x^0)^T z^0 + x(t)^T z(t)}{\min(x^0)}\right]^{(1-\frac{\gamma_2}{\gamma_1})},$$

and

$$x(t)^T z(t) \leq n(n\mu_0)^{\frac{1}{\gamma_1}} e^{-(1-\sigma)(t-t_0)/\gamma_1} \left[\frac{(x^0)^T z^0 + x(t)^T z(t)}{\min(x^0)}\right]^{(1-\frac{\gamma_2}{\gamma_1})}.$$

Since $0 < 1 - \frac{\gamma_2}{\gamma_1} < 1$, there should exist a constant $M > 0$ such that $x(t)^T z(t) < M$, and M depends only on x^0, z^0, γ_1, and γ_2.

For $\gamma_1 < \gamma_2$, we can prove this similarly. For $\gamma_1 = \gamma_2$ and any $1 \leq i \leq n$,

$$x(t)_i z(t)_i = (x_i^{\gamma_1} z_i^{\gamma_1})^{\frac{1}{\gamma_1}} \leq (n\mu_0)^{\frac{1}{\gamma_1}} e^{-(1-\sigma)(t-t_0)/\gamma_1},$$

hence $x(t)^T z(t) \leq n(n\mu_0)^{\frac{1}{\gamma_1}} e^{-(1-\sigma)(t-t_0)/\gamma_1} \leq n(n\mu_0)^{\frac{1}{\gamma_1}}$. □

After we get the boundedness of the solution curve, we can extend the existence interval of the solution to infinity.

Theorem 4.4. *Let the solution of the weighted primal-dual path-following ODE system (4.3) on the maximal existence interval* $[t_0, \beta)$ *be* $(x(t), y(t), z(t))$*. Then* $\beta = +\infty$*.*

Proof. Assume $\beta \neq +\infty$. From Theorem 4.1, we can define

$$V_1(t) = \sum_{i=1}^{n} (\gamma_1 \ln x_i + \gamma_2 \ln z_i),$$

then from the third equation in (4.2), we have

$$\gamma_1 X^{-1} \frac{dx}{dt} + \gamma_2 Z^{-1} \frac{dz}{dt} = -X^{-\gamma_1} Z^{-\gamma_2} (X^{\gamma_1} Z^{\gamma_2} e - \sigma \mu w),$$

hence

$$\begin{aligned} \frac{dV_1(t)}{dt} &= e^T \gamma_1 X^{-1} \frac{dx}{dt} + e^T \gamma_2 Z^{-1} \frac{dz}{dt} = -e^T X^{-\gamma_1} Z^{-\gamma_2} (X^{\gamma_1} Z^{\gamma_2} e - \sigma \mu w) \\ &= -n + \sigma \mu e^T X^{-\gamma_1} Z^{-\gamma_2} w \geq -n, \end{aligned}$$

therefore

$$V_1(t) \geq V_1(t_0) - n\beta. \tag{4.5}$$

But according to the Extension Theorem in **S**2.5, [3], we know that the solution $(x(t), y(t), z(t))$ will go to the boundary of the open set $(0, +\infty) \times R^n_{++} \times R^m \times R^n_{++}$. Because of the hypothesis, $\beta \neq +\infty$ and the solution $(x(t), y(t), z(t))$ is bounded (Theorem 4.3), there must exist at least one i so that $x_i(t) \to 0$ or $z_i(t) \to 0$ as $t \to \beta$. In this situation $V_1(t)$ will go to $-\infty$ as $t \to \beta$ since $x(t)$ and $z(t)$ are all bounded (Theorem 4.3), which contradicts (4.5). Hence the hypothesis is not true and $\beta = +\infty$. □

From Theorem 4.3 and Theorem 4.4, we can define the limit set for the solution of the weighted primal-dual path-following ODE system (4.3). The limit set of the solution of ODE system (4.3) $\{x(t), y(t), z(t)\}$ can be defined as follows

$$\begin{aligned} \Omega^1(x^0, y^0, z^0) &= \Big\{ (x, y, z) \mid \exists \, \{t_k\}_{k=0}^{+\infty} \text{ with } \lim_{k \to +\infty} t_k = +\infty \text{ such that} \\ &\qquad \lim_{k \to +\infty} x(t_k) = x, \lim_{k \to +\infty} y(t_k) = y \text{ and } \lim_{k \to +\infty} z(t_k) = z \Big\}. \end{aligned}$$

Theorem 4.5. *The limit set $\Omega^1(x^0, y^0, z^0)$ is nonempty, compact, and connected. Furthermore $\Omega^1(x^0, y^0, z^0)$ is contained in $\mathcal{F}_2$.*

Proof. From Theorems 4.1, 4.2, and 4.4, we know that the limit set $\Omega^1(x^0, y^0, z^0)$ is contained in $\mathcal{F}_2$. From Theorem 4.3, we know that the solution $(x(t), y(t), z(t))$ is contained in a bounded closed set. So similar to the proof of Theorem 1.1 on page 390 in [10] (the proof in [10] is for $n = 2$, but it can be easily extended to the general case), it can be verified that $\Omega^1(x^0, y^0, z^0)$ is nonempty, compact, and connected. □

4.1.2 Optimality of The Cluster Point(s)

In this subsection, we show that every accumulation point of the solution of the weighted primal-dual path-following ODE system (4.3) is an optimal solution for problems (P_2) and (D_2).

Theorem 4.6. *Let the unique solution of the weighted primal-dual path-following ODE system (4.3) be $(x(t), y(t), z(t))$. Then*

(i) $x(t)^T z(t) \le L_2 e^{-(1-\sigma)(t-t_0)/\max(\gamma_1,\gamma_2)}$, *where $L_2 > 0$ is a constant which depends only on x^0, z^0, γ_1, and γ_2, hence for any point $(x^1, y^1, z^1) \in \Omega^1(x^0, y^0, z^0)$, x^1 is an optimal solution for problem (P_2), (y^1, z^1) is an optimal solution of problem (D_2).*

(ii) For any $1 \le i \le n$,

$$(x(t)_i^{\gamma_1} z(t)_i^{\gamma_2} - w_i\mu) = e^{-(t-t_0)}(x(t_0)_i^{\gamma_1} z(t_0)_i^{\gamma_2} - w_i\mu^0),$$

where $\mu^0 = \frac{e^T X(t_0)^{\gamma_1} Z(t_0)^{\gamma_2} e}{n}$.

Proof. Proof of (i). From Theorem 4.3, we know there exists a bound $M > 0$ which depends only on x^0, z^0, γ_1, and γ_2 such that for any $1 \le i \le n$,

$$x(t)_i z(t)_i \le M(n\mu)^{\frac{1}{\max(\gamma_1,\gamma_2)}},$$

along with (4.4), we have

$$x(t)^T z(t) \leq L_2 e^{-(1-\sigma)(t-t_0)/\max(\gamma_1,\gamma_2)},$$

where $L_2 = Mn(n\mu_0)^{\frac{1}{\max(\gamma_1,\gamma_2)}}$. Hence for any point $(x^1, y^1, z^1) \in \Omega^1(x^0, y^0, z^0)$, we have

$$0 \leq (x^1)^T z^1 \leq \limsup_{t\to+\infty} x(t)^T z(t) \leq 0,$$

which indicates that x^1 is an optimal solution of problem (P_2), and (y^1, z^1) is an optimal solution of problem (D_2).

Proof of (ii). For any $1 \leq i \leq n$, from the third equation in (4.2), we can get

$$\frac{dx(t)_i^{\gamma_1} z(t)_i^{\gamma_2}}{dt} = -(x(t)_i^{\gamma_1} z(t)_i^{\gamma_2} - \sigma\mu w_i),$$

hence

$$\frac{d(x(t)_i^{\gamma_1} z(t)_i^{\gamma_2} - w_i\mu)}{dt} = -(x(t)_i^{\gamma_1} z(t)_i^{\gamma_2} - w_i\mu),$$

therefore $(x(t)_i^{\gamma_1} z(t)_i^{\gamma_2} - w_i\mu) = e^{-(t-t_0)}(x(t_0)_i^{\gamma_1} z(t_0)_i^{\gamma_2} - w_i\mu^0)$, where $\mu^0 = \frac{e^T X(t_0)^{\gamma_1} Z(t_0)^{\gamma_2} e}{n}$. Thus the theorem is proved. □

4.1.3 Convergence of The Weighted Primal-Dual Path-Following Continuous Trajectory

Now, it comes to the key result of this section. Under some mild conditions, the solution of the weighted primal-dual path-following ODE system (4.3) will converge as $t \to +\infty$. First we give the definition of the analytic center. We define the analytic center of a closed convex set $\Omega \subseteq R^n$ corresponding to convex function $g(x)$ as the unique minimizer of the following problem:

$$\begin{aligned} \min \quad & g(x) \\ \text{s.t.} \quad & x \in \Omega \cap dom\ g(x), \end{aligned}$$

where $dom\ g(x) = \{x \mid g(x) < +\infty\}$. Generally, the minimizer of the above problem may not be unique, but in our context the existence and uniqueness of the minimizer can be guaranteed and no confuse would occur. We denote $\mathcal{S}_P$ and $\mathcal{S}_D$ as the optimal solution sets of problems (P_2) and (D_2) respectively.

Theorem 4.7. *Let x^* and (y^*, z^*) be optimal solutions for problems (P_2) and (D_2), respectively, such that x^* and z^* have the maximal numbers of positive components among all optimal solutions. Let the unique solution of the weighted primal-dual path-following ODE system (4.3) be $(x(t), y(t), z(t))$. Then*

(i) for any $1 \leq i \leq n$,

$$(x(t)_i^{\gamma_1} z(t)_i^{\gamma_2} - w_i\mu) = \frac{x(t_0)_i^{\gamma_1} z(t_0)_i^{\gamma_2} - w_i\mu^0}{w_i\mu_0} e^{-\sigma(t-t_0)} w_i\mu.$$

(ii) a) if $\gamma_1 < \gamma_2$, then $x(t)$ will converge to the analytic center of $\mathcal{S}_P$ corresponding to $-\sum_{x_i^*>0} (w_i)^{1/\gamma_2}(x_i)^{1-\frac{\gamma_1}{\gamma_2}}$;

b) if $\gamma_1 = \gamma_2$, and either there exists a pair of primal and dual optimal solutions satisfying the strict complementarity or $f(x)$ is analytic, then $x(t)$ will converge to the analytic center of $\mathcal{S}_P$ corresponding to $-\prod_{x_i^*>0} (x_i)^{w_i^{1/\gamma_1}}$;

c) if $\gamma_1 > \gamma_2$ and $f(x)$ is analytic, then $x(t)$ will converge to the analytic center of $\mathcal{S}_P$ corresponding to $\sum_{x_i^*>0} (w_i)^{1/\gamma_2}(x_i)^{1-\frac{\gamma_1}{\gamma_2}}$.

(iii) a) if $\gamma_1 < \gamma_2$, then $z(t)$ will converge to the analytic center of $\mathcal{S}_D$ corresponding to $\sum_{z_i^*>0} (w_i)^{1/\gamma_1}(z_i)^{1-\frac{\gamma_2}{\gamma_1}}$;

b) if $\gamma_1 = \gamma_2$, then $z(t)$ will converge to the analytic center of $\mathcal{S}_D$ corresponding to $-\prod_{z_i^*>0} (z_i)^{w_i^{1/\gamma_1}}$;

c) if $\gamma_1 > \gamma_2$, then $z(t)$ will converge to the analytic center of $\mathcal{S}_D$ corresponding to $-\sum_{z_i^*>0} (w_i)^{1/\gamma_1}(z_i)^{1-\frac{\gamma_2}{\gamma_1}}$.

Proof. Proof of (i). From Theorem 4.6 and (4.4), it is straightforward that for any $1 \leq i \leq n$,

$$(x(t)_i^{\gamma_1} z(t)_i^{\gamma_2} - w_i\mu) = c_i^0 e^{-\sigma(t-t_0)} w_i\mu = d_i^0(t) w_i\mu, \tag{4.6}$$

where $c_i^0 = \frac{x(t_0)_i^{\gamma_1} z(t_0)_i^{\gamma_2} - w_i\mu^0}{w_i\mu_0}$ and $d_i^0(t) = c_i^0 e^{-\sigma(t-t_0)}$.

Proof of (ii). If $x^* = 0$, the result is evident, so we assume $x^* \neq 0$ below.

a) If $\gamma_1 < \gamma_2$, from (4.6), for any $1 \leq i \leq n$,

$$z(t)_i = \frac{(w_i\mu(1 + c_i^0 e^{-\sigma(t-t_0)}))^{1/\gamma_2}}{x_i^{\gamma_1/\gamma_2}},$$

then from Theorem 4.2 and Lemma 1.2, we have

$$\begin{aligned} 0 &\geq f(x^*) - f(x(t)) \geq (x^* - x(t))^T \nabla f(x(t)) = (x^* - x)^T (A^T y + z) \\ &= (x^* - x)^T z = \sum_{i=1}^n \frac{(w_i\mu(1 + c_i^0 e^{-\sigma(t-t_0)}))^{1/\gamma_2}(x_i^* - x_i)}{x_i^{\gamma_1/\gamma_2}}. \end{aligned}$$

This indicates

$$\sum_{i=1}^n \frac{(w_i(1 + c_i^0 e^{-\sigma(t-t_0)}))^{1/\gamma_2}(x_i^* - x_i)}{x_i^{\gamma_1/\gamma_2}} \leq 0, \tag{4.7}$$

then if $\bar{x}$ is an cluster point of $x(t)$, since x^* has the maximal number of positive components among all optimal solutions of problem (P_2) and $\bar{x}$ is also an optimal solution, so for any i with $x_i^* = 0$, $\bar{x}_i = 0$ which implies that in (4.7),

$$\sum_{x_i^*=0} \frac{(w_i(1 + c_i^0 e^{-\sigma(t-t_0)}))^{1/\gamma_2}(x_i^* - x_i)}{x_i^{\gamma_1/\gamma_2}} \to -(w_i)^{1/\gamma_2} \bar{x}_i^{1-\frac{\gamma_1}{\gamma_2}} = 0,$$

as $x(t) \to \bar{x}$. Also for any i with $x_i^* > 0$, $\bar{x}_i$ must be positive, since if $\bar{x}_i = 0$, then

$$\frac{(w_i(1 + c_i^0 e^{-\sigma(t-t_0)}))^{1/\gamma_2}(x_i^* - x_i)}{x_i^{\gamma_1/\gamma_2}} \to +\infty,$$

as $x_i \to \bar{x}_i$ which contradicts with (4.7). Hence by (4.7), $\bar{x}$ must have the maximal number of positive components among all optimal solutions of problem (P_2) and satisfy

$$\sum_{x_i^*>0} \frac{(w_i)^{1/\gamma_2}(x_i^* - \bar{x}_i)}{\bar{x}_i^{\gamma_1/\gamma_2}} \leq 0. \tag{4.8}$$

Since function $-a^{1-\frac{\gamma_1}{\gamma_2}}$ is strictly convex for $a > 0$, from Lemma 1.2, we have for each i with $x_i^* > 0$,

$$(x_i^* - \bar{x}_i)(-(1 - \frac{\gamma_1}{\gamma_2})(\bar{x}_i)^{-\frac{\gamma_1}{\gamma_2}}) \leq -(x_i^*)^{1-\frac{\gamma_1}{\gamma_2}} + (\bar{x}_i)^{1-\frac{\gamma_1}{\gamma_2}},$$

hence

$$\begin{aligned}
& \sum_{x_i^*>0} (w_i)^{1/\gamma_2} (x_i^*)^{1-\frac{\gamma_1}{\gamma_2}} - \sum_{x_i^*>0} (w_i)^{1/\gamma_2} (\bar{x}_i)^{1-\frac{\gamma_1}{\gamma_2}} \\
\leq \quad & (1 - \frac{\gamma_1}{\gamma_2}) \sum_{x_i^*>0} \frac{(w_i)^{1/\gamma_2} (x_i^* - \bar{x}_i)}{\bar{x}_i^{\gamma_1/\gamma_2}} \leq 0.
\end{aligned}$$

Therefore $\bar{x}$ is the analytic center of $\mathcal{S}_P$ corresponding to function $-\sum_{x_i^*>0} (w_i)^{1/\gamma_2} (x_i)^{1-\frac{\gamma_1}{\gamma_2}}$.

b) If $\gamma_1 = \gamma_2$, and there exist a pair of primal and dual optimal solutions satisfying the strict complementarity, we know that x^* and z^* satisfy the strict complementarity. Since (y^*, z^*) is an optimal solution for problem (D_2), we have

$$\begin{aligned}
f(x^*) = L(y^*, z^*) \quad &= \quad \inf_{x \in R^n} f(x) + (b - Ax)^T y^* - x^T z^* \\
&\leq \quad f(x^*) + (b - Ax^*)^T y^* - (x^*)^T z^* = f(x^*) - (x^*)^T z^* \leq f(x^*),
\end{aligned}$$

hence

$$(x^*)^T z^* = 0, \tag{4.9}$$

and

$$\inf_{x \in R^n} f(x) + (b - Ax)^T y^* - x^T z^* = f(x^*) + (b - Ax^*)^T y^* - (x^*)^T z^*,$$

which indicates

$$A^T y^* + z^* = \nabla f(x^*). \tag{4.10}$$

Therefore from Theorem 4.2, we have

$$\begin{aligned}
(x(t) - x^*)^T (z(t) - z^*) \quad &= \quad (x - x^*)^T (A^T (y^* - y) + \nabla f(x) - \nabla f(x^*)) \\
&= \quad (x - x^*)^T (\nabla f(x) - \nabla f(x^*)) \geq 0,
\end{aligned}$$

or

$$\sum_{x_i^*>0} (x_i^* z(t)_i) + \sum_{z_i^*>0} z_i^* x(t)_i \leq x(t)^T z(t).$$

By using (4.6), we can get

$$\sum_{x_i^*>0} \frac{x_i^*}{x(t)_i} (w_i \mu (1 + d_i^0(t)))^{\frac{1}{\gamma_1}} + \sum_{z_i^*>0} \frac{z_i^*}{z(t)_i} (w_i \mu (1 + d_i^0(t)))^{\frac{1}{\gamma_1}} \leq \sum_{i=1}^{n} (w_i \mu (1 + d_i^0(t)))^{\frac{1}{\gamma_1}},$$

or

$$\sum_{x_i^*>0} \frac{x_i^*}{x(t)_i}(w_i(1+d_i^0(t)))^{\frac{1}{\gamma_1}} + \sum_{z_i^*>0} \frac{z_i^*}{z(t)_i}(w_i(1+d_i^0(t)))^{\frac{1}{\gamma_1}} \le \sum_{i=1}^{n}(w_i(1+d_i^0(t)))^{\frac{1}{\gamma_1}},$$

hence for any point $(\bar{x}, \bar{y}, \bar{z}) \in \Omega^1(x^0, y^0, z^0)$, $\bar{x}$ and $\bar{z}$ also have the maximal numbers of positive components among all optimal solutions and satisfy the strict complementarity, so we must have

$$\sum_{x_i^*>0} \frac{x_i^*}{\bar{x}_i} \frac{w_i^{1/\gamma_1}}{\sum_{i=1}^{n} w_i^{1/\gamma_1}} + \sum_{z_i^*>0} \frac{z_i^*}{\bar{z}_i} \frac{w_i^{1/\gamma_1}}{\sum_{i=1}^{n} w_i^{1/\gamma_1}} \le 1,$$

and

$$\sum_{x_i^*>0} \frac{w_i^{1/\gamma_1}}{\sum_{i=1}^{n} w_i^{1/\gamma_1}} + \sum_{z_i^*>0} \frac{w_i^{1/\gamma_1}}{\sum_{i=1}^{n} w_i^{1/\gamma_1}} = 1.$$

Therefore

$$e^{\sum_{x_i^*>0} \frac{w_i^{1/\gamma_1}}{\sum_{i=1}^{n} w_i^{1/\gamma_1}} \ln \frac{x_i^*}{\bar{x}_i} + \sum_{z_i^*>0} \frac{w_i^{1/\gamma_1}}{\sum_{i=1}^{n} w_i^{1/\gamma_1}} \ln \frac{z_i^*}{\bar{z}_i}} \le e^{\ln \sum_{x_i^*>0} \frac{x_i^*}{\bar{x}_i} \frac{w_i^{1/\gamma_1}}{\sum_{i=1}^{n} w_i^{1/\gamma_1}} + \sum_{z_i^*>0} \frac{z_i^*}{\bar{z}_i} \frac{w_i^{1/\gamma_1}}{\sum_{i=1}^{n} w_i^{1/\gamma_1}}} \le 1,$$

which implies

$$(\prod_{x_i^*>0} (\frac{x_i^*}{\bar{x}_i})^{w_i^{1/\gamma_1}})(\prod_{z_i^*>0} (\frac{z_i^*}{\bar{z}_i})^{w_i^{1/\gamma_1}}) \le 1,$$

or

$$(\prod_{x_i^*>0} (x_i^*)^{w_i^{1/\gamma_1}})(\prod_{z_i^*>0} (z_i^*)^{w_i^{1/\gamma_1}}) \le (\prod_{x_i^*>0} (\bar{x}_i)^{w_i^{1/\gamma_1}})(\prod_{z_i^*>0} (\bar{z}_i)^{w_i^{1/\gamma_1}}).$$

Therefore $\bar{x}$ is the analytic center of $\mathcal{S}_P$ corresponding to function $-\prod_{x_i^*>0} (x_i)^{w_i^{1/\gamma_1}}$.

If $\gamma_1 = \gamma_2$ and $f(x)$ is analytic, the proof is contained in the proof of c).

c) If $f(x)$ is analytic, then for any accumulation point $\bar{x}$ of $x(t)$ with $\bar{x} \neq x^*$, from Theorem 4.6 and Lemma 1.6, we have

$$(x^* - \bar{x})^T \nabla f(x) = 0,$$

for any $x \in R^n$. Hence from Theorem 4.2 and (4.6), we have

$$\begin{aligned} 0 &= (x^* - \bar{x})^T \nabla f(x(t)) = (x^* - \bar{x})^T (A^T y(t) + z(t)) \\ &= (x^* - \bar{x})^T z(t) = \sum_{i=1}^{n} \frac{(w_i \mu (1 + c_i^0 e^{-\sigma(t-t_0)}))^{1/\gamma_2} (x_i^* - \bar{x}_i)}{x(t)_i^{\gamma_1/\gamma_2}}. \end{aligned}$$

Similarly, we know that $\bar{x}$ must have the maximal number of positive components among all optimal solutions of problem (P_2) and satisfy

$$\sum_{x_i^*>0} \frac{(w_i)^{1/\gamma_2}(x_i^* - \bar{x}_i)}{\bar{x}_i^{\gamma_1/\gamma_2}} = 0. \tag{4.11}$$

If $\gamma_1 = \gamma_2$, from (4.11), we have

$$\sum_{x_i^*>0} \frac{(w_i)^{1/\gamma_2}}{\sum_{x_i^*>0}(w_i)^{1/\gamma_2}} \frac{x_i^*}{\bar{x}_i} = 1,$$

which implies

$$e^{\sum_{x_i^*>0} \frac{w_i^{1/\gamma_1}}{\sum_{x_i^*>0} w_i^{1/\gamma_1}} \ln \frac{x_i^*}{\bar{x}_i}} \leq e^{\ln \sum_{x_i^*>0} \frac{x_i^*}{\bar{x}_i} \frac{w_i^{1/\gamma_1}}{\sum_{x_i^*>0} w_i^{1/\gamma_1}}} = 1,$$

thus

$$(\prod_{x_i^*>0} (\frac{x_i^*}{\bar{x}_i})^{w_i^{1/\gamma_1}}) \leq 1 \quad \text{or} \quad (\prod_{x_i^*>0} (x_i^*)^{w_i^{1/\gamma_1}}) \leq (\prod_{x_i^*>0} (\bar{x}_i)^{w_i^{1/\gamma_1}}).$$

Therefore $\bar{x}$ is the analytic center of $\mathcal{S}_P$ corresponding to function $-\prod_{x_i^*>0}(x_i)^{w_i^{1/\gamma_1}}$.

If $\gamma_1 > \gamma_2$, since the function $a^{1-\frac{\gamma_1}{\gamma_2}}$ is strictly convex for $a > 0$, from Lemma 1.2, we have for every i such that $x_i^* > 0$,

$$(x_i^* - \bar{x}_i)\left[(1 - \frac{\gamma_1}{\gamma_2})(\bar{x}_i)^{-\frac{\gamma_1}{\gamma_2}}\right] \leq (x_i^*)^{1-\frac{\gamma_1}{\gamma_2}} - (\bar{x}_i)^{1-\frac{\gamma_1}{\gamma_2}},$$

so from (4.11),

$$\begin{aligned} & \sum_{x_i^*>0}(w_i)^{1/\gamma_2}(x_i^*)^{1-\frac{\gamma_1}{\gamma_2}} - \sum_{x_i^*>0}(w_i)^{1/\gamma_2}(\bar{x}_i)^{1-\frac{\gamma_1}{\gamma_2}} \\ \geq \; & (1 - \frac{\gamma_1}{\gamma_2}) \sum_{x_i^*>0} \frac{(w_i)^{1/\gamma_2}(x_i^* - \bar{x}_i)}{\bar{x}_i^{\gamma_1/\gamma_2}} = 0. \end{aligned}$$

Therefore $\bar{x}$ is the analytic center of $\mathcal{S}_P$ corresponding to function $\sum_{x_i^*>0}(w_i)^{1/\gamma_2}(x_i)^{1-\frac{\gamma_1}{\gamma_2}}$.

Proof of (iii). If $z^* = 0$, the result is evident, so we assume $z^* \neq 0$ below. For any point $(\bar{x}, \bar{y}, \bar{z}) \in \Omega^1(x^0, y^0, z^0)$, if $\bar{z} \neq z^*$, from Theorem 4.6, same as the analysis for (4.9) and (4.10), we can get

$$A^T\bar{y} + \bar{z} = \nabla f(\bar{x}) = \nabla f(x^*), \tag{4.12}$$

and

$$L(y^*, z^*) = f(x^*) - \nabla f(x^*)^T x^* + b^T y^* = L(\bar{y}, \bar{z}) = f(x^*) - \nabla f(x^*)^T x^* + b^T \bar{y},$$

which implies

$$b^T \bar{y} = b^T y^*. \tag{4.13}$$

Since z^* has the maximal number of positive components among all optimal solutions of problem (D_2), so from Theorem 4.6, for any i with $z_i^* = 0$, $\bar{z}_i = 0$. Hence from (4.6), (4.12), and (4.13),

$$\begin{aligned}\sum_{z_i^*>0} \frac{(w_i \mu (1 + d_i^0(t)))^{1/\gamma_1} (z_i^* - \bar{z}_i)}{z(t)_i^{\gamma_2/\gamma_1}} &= x(t)^T (z^* - \bar{z}) = x(t)^T (A^T \bar{y} - A^T y^*) \\ &= b^T \bar{y} - b^T y^* = 0.\end{aligned}$$

Thus

$$\sum_{z_i^*>0} \frac{(w_i (1 + d_i^0(t)))^{1/\gamma_1} (z_i^* - \bar{z}_i)}{z(t)_i^{\gamma_2/\gamma_1}} = 0,$$

which indicates that $\bar{z}$ must have the maximal number of positive components among all optimal solutions of problem (D_2) and satisfy

$$\sum_{z_i^*>0} \frac{(w_i)^{1/\gamma_1} (z_i^* - \bar{z}_i)}{\bar{z}_i^{\gamma_2/\gamma_1}} = 0.$$

Therefore, similar to the claim in the proof of (ii), we know if $\gamma_1 < \gamma_2$, $\bar{z}$ is the analytic center of $\mathcal{S}_D$ corresponding to $\sum_{z_i^*>0} (w_i)^{1/\gamma_1} (z_i)^{1-\frac{\gamma_2}{\gamma_1}}$. If $\gamma_1 = \gamma_2$, $\bar{z}$ is the analytic center of $\mathcal{S}_D$ corresponding to $-\prod_{z_i^*>0} (z_i)^{w_i^{1/\gamma_1}}$. If $\gamma_1 > \gamma_2$, $\bar{z}$ is the analytic center of $\mathcal{S}_D$ corresponding to $-\sum_{z_i^*>0} (w_i)^{1/\gamma_1} (z_i)^{1-\frac{\gamma_2}{\gamma_1}}$. Thus the theorem is proved. □

4.2 The Extended Primal-Dual Affine Scaling Continuous Trajectory

In this section, we study the extended primal-dual affine scaling continuous trajectory which is defined by the solution curve of the following ODE system

$$\begin{cases} -\nabla^2 f(x)\frac{dx}{dt} + A^T\frac{dy}{dt} + \frac{dz}{dt} = 0, \\ A\frac{dx}{dt} = 0, \\ \gamma_1 X^{\gamma_1 - 1}Z^{\gamma_2}\frac{dx}{dt} + \gamma_2 X^{\gamma_1}Z^{\gamma_2 - 1}\frac{dz}{dt} = -X^{\gamma_1}Z^{\gamma_2}e, \\ (x(t_0), y(t_0), z(t_0)) = (x^0, y^0, z^0) \in \mathcal{F}_2^0 \end{cases} \tag{4.14}$$

where

$$t_0 > 0,\ \mu = \frac{e^T X^{\gamma_1}Z^{\gamma_2}e}{n},\ \gamma_1 > 0,\ \gamma_2 > 0,\ x \in R^n_{++},\ X = \text{diag }(x) \in R^{n\times n},$$
$$z \in R^n_{++},\ Z = \text{diag }(z) \in R^{n\times n}.$$

Note: ODE system (4.14) is just ODE system (4.2) with $\sigma = 0$. We call this ODE system the extended primal-dual affine scaling ODE system. The unique solution of ODE system (4.14) defines the extended primal-dual affine scaling continuous trajectory for problem (P_2). Similar to the weighted primal-dual path-following ODE system (4.2), this ODE system also has two explicit forms which are same as that of ODE system (4.2) but with $\sigma = 0$. The following assumption is made throughout Section 3.

Assumption 4.2. *The set $\mathcal{F}_2^0$ is nonempty.*

4.2.1 Fundamental Properties of The Extended Primal-Dual Affine Scaling Continuous Trajectory

Theorems 4.1, 4.2, 4.3, and 4.4 in the former section also hold for this extended primal-dual affine scaling ODE system (4.14), and the proofs are almost the same, hence we omit them. Now we can define the limit set for the unique solution $(x(t), y(t), z(t))$ of the extended primal-dual affine scaling ODE system (4.14) as well, which is denoted by $\Omega^2(x^0, y^0, z^0)$, and have the following similar result.

Theorem 4.8. *The limit set $\Omega^2(x^0, y^0, z^0)$ is nonempty, compact, and connected. Furthermore $\Omega^2(x^0, y^0, z^0)$ is contained in $\mathcal{F}_2$.*

4.2.2 Optimality of The Cluster Point(s)

In this subsection, we just show that every accumulation point of the solution of the extended primal-dual affine scaling ODE system (4.14) is an optimal solution for problems (P_2) and (D_2).

Theorem 4.9. *Let the unique solution of the extended primal-dual affine scaling ODE system (4.14) be $(x(t), y(t), z(t))$. Then*

(i) $x(t)^T z(t) \leq L_3 e^{-(t-t_0)/\max(\gamma_1,\gamma_2)}$, *where $L_3 > 0$ is a constant which depends only on x^0, z^0, γ_1, and γ_2, hence for any point $(x^1, y^1, z^1) \in \Omega^2(x^0, y^0, z^0)$, x^1 is an optimal solution for problem (P_2), (y^1, z^1) is an optimal solution for problem (D_2).*

(ii) *For any $1 \leq i \leq n$,*

$$(x(t)_i^{\gamma_1} z(t)_i^{\gamma_2} - \mu) = e^{-(t-t_0)}(x(t_0)_i^{\gamma_1} z(t_0)_i^{\gamma_2} - \mu^0),$$

where $\mu^0 = \frac{e^T X(t_0)^{\gamma_1} Z(t_0)^{\gamma_2} e}{n}$.

Proof. The proof is similar to the proof of Theorem 4.6. □

4.2.3 Convergence of The Extended Primal-Dual Affine Scaling Continuous Trajectory

Now, it comes to the key result of this section. Theorem 4.10 below shows that the solution of the extended primal-dual affine scaling ODE system (4.14) converges to the analytic centers of the optimal solution sets corresponding to some convex functions which depend on the initial point as $t \to +\infty$ and is actually a weighted central path. Similar to Theorem 4.7, we let x^* and (y^*, z^*) be optimal solutions for problems (P_2) and (D_2), respectively, such that x^* and z^* have the maximal numbers of positive components among all optimal solutions.

Theorem 4.10. *Let the unique solution of the extended primal-dual affine scaling ODE system (4.14) be* $(x(t), y(t), z(t))$, $c_i^0 = \frac{(x_i^0)^{\gamma_1}(z_i^0)^{\gamma_2}}{\mu^0}$ *and* $\mu^0 = \frac{e^T X(t_0)^{\gamma_1} Z(t_0)^{\gamma_2} e}{n}$. *Then*

(i) for any $1 \le i \le n$,

$$x(t)_i^{\gamma_1} z(t)_i^{\gamma_2} = \frac{x(t_0)_i^{\gamma_1} z(t_0)_i^{\gamma_2}}{\mu_0} \mu.$$

(ii) same as (ii) of Theorem 4.7 but with $w_i = c_i^0$.

(iii) same as (iii) of Theorem 4.7 but with $w_i = c_i^0$.

Proof. The proof is similar to the proof of Theorem 4.7. □

Chapter 5

Four Primal Interior Point Continuous Trajectories for Convex Semidefinite Programming

In this chapter, we study four primal interior continuous trajectories for Problem (P_3). We restate the following four ordinary differential equation (ODE) systems

$$
\begin{aligned}
\mathrm{svec}(\dot{X}) &= -\big(I-(X\otimes_s X)P_{\mathcal{A}X}\big)(X\otimes_s X)\mathrm{svec}(\frac{\partial f}{\partial X}), & (5.1)\\
\mathrm{svec}(\dot{X}) &= -\Big(I+t\big(I-(X\otimes_s X)P_{\mathcal{A}X}\big)\big((X\otimes_s X)\nabla^2 f(X)\big)\Big)^{-1} & \\
&\quad \big(I-(X\otimes_s X)P_{\mathcal{A}X}\big)(X\otimes_s X)\mathrm{svec}(\frac{\partial f}{\partial X}), & (5.2)\\
\mathrm{svec}(\dot{X}) &= -\big(I-(X\otimes_s X^{\frac{1}{2}})P_{\mathcal{A}X^{\frac{1}{2}}}\big)(X\otimes_s X^{\frac{1}{2}})\mathrm{svec}(\frac{\partial f}{\partial X}), & (5.3)\\
\mathrm{svec}(\dot{X}) &= -\Big(I+t\big(I-(X\otimes_s X^{\frac{1}{2}})P_{\mathcal{A}X^{\frac{1}{2}}}\big)\big((X\otimes_s X^{\frac{1}{2}})\nabla^2 f(X)\big)\Big)^{-1} & \\
&\quad \big(I-(X\otimes_s X^{\frac{1}{2}})P_{\mathcal{A}X^{\frac{1}{2}}}\big)(X\otimes_s X^{\frac{1}{2}})\mathrm{svec}(\frac{\partial f}{\partial X}), & (5.4)
\end{aligned}
$$

which have the same initial condition: $X(t_0) = X^0 \in \mathcal{P}_3^{++}$ and $t_0 > 0$, where

$X \in \mathcal{S}^n_{++}$, $\quad X^{\frac{1}{2}} \in \mathcal{S}^n_{++}$ is the unique square root matrix of X,

$P_{\mathcal{A}X^\gamma} = \mathcal{A}^T(\mathcal{A}(X\otimes_s X^\gamma)\mathcal{A}^T)^{-1}\mathcal{A},\ \gamma \in \{\frac{1}{2},\ 1\}$,

I stands for the $\dfrac{n(n+1)}{2} \times \dfrac{n(n+1)}{2}$ identity matrix.

For ODE systems (5.2) and (5.4), we sometimes use the following equivalent implicit forms

$$\mathrm{svec}(\dot{X}) = -\big(I - (X \otimes_s X)P_{\mathcal{A}X}\big)(X \otimes_s X)\big(t\nabla^2 f(X)\mathrm{svec}(\dot{X}) + \mathrm{svec}(\frac{\partial f}{\partial X})\big) \quad (5.5)$$

$$\mathrm{svec}(\dot{X}) = -\big(I - (X \otimes_s X^{\frac{1}{2}})P_{\mathcal{A}X^{\frac{1}{2}}}\big)(X \otimes_s X^{\frac{1}{2}})\big(t\nabla^2 f(X)\mathrm{svec}(\dot{X}) + \mathrm{svec}(\frac{\partial f}{\partial X})\big) \quad (5.6)$$

For ODE systems (5.1) and (5.3), we need $f(X) \in C^2$ on $\mathcal{S}^n_+$, and for ODE systems (5.2) and (5.4), we need $f(X) \in C^3$ on $\mathcal{S}^n_+$. By assuming the boundedness of the level set, we establish the optimality and convergence of the first and the second trajectories for linear SDP. For the convex case, we show that, starting from any interior feasible point, the third trajectory converges to an optimal solution that has the maximal rank among all optimal solutions, by assuming that there exists an optimal solution and that the maximal rank among all optimal solutions is one. Finally, we obtain the strongest result under the weakest assumption for the fourth trajectory, namely, by only assuming the existence of an optimal solution, we show that the trajectory converges to an optimal solution that has the maximal rank among all optimal solutions.

5.1 Fundamental Properties of The Continuous Trajectories

The following assumptions are made throughout this chapter.

Assumption 5.1. *There exists a point $X^* \in \mathcal{P}_3^+$ such that $f(X^*)$ is the optimal value of problem (P_3).*

Assumption 5.2. *The matrix $\mathcal{A}$ has full row rank m.*

Assumption 5.3. *For ODE systems (5.1) and (5.3), we assume $f(X) \in C^2$ on $\mathcal{S}^n_+$, and for ODE systems (5.2) and (5.4), we assume $f(X) \in C^3$ on $\mathcal{S}^n_+$.*

Theorem 5.1. *$P_{\mathcal{A}X^\gamma} \in C^1$ on $\mathcal{S}^n_{++}$, $\gamma \in \{\frac{1}{2},\ 1\}$.*

Proof. According to Assumption 5.2, $\mathcal{A}$ has full row rank. If $X \in \mathcal{S}^n_{++}$, $X \otimes_s X$ and $X \otimes_s X^{\frac{1}{2}}$ are both symmetric and positive definite. So $\mathcal{A}(X \otimes_s X)\mathcal{A}^T$ and $\mathcal{A}(X \otimes_s X^{\frac{1}{2}})\mathcal{A}^T$ are also symmetric and positive definite, thus invertible.

Notice that the inverse of a matrix and $X \otimes_s X$ are both continuous differentiable, we get $P_{\mathcal{A}X} \in C^1$ on $\mathcal{S}^n_{++}$. Furthermore, according to Chapter 6 in [27] we have

$$X^{\frac{1}{2}} = \frac{2}{\pi} X \int_0^\infty (t^2 I + X)^{-1} dt,$$

which indicates that the square root of a symmetric positive definite matrix is continuous differentiable. So $P_{\mathcal{A}X^{\frac{1}{2}}} \in C^1$ on $\mathcal{S}^n_{++}$. Thus the proof is complete. $\square$

Theorems 5.2 and 5.3 below guarantee the existence, uniqueness, and feasibility for the solutions of the four ODE systems (5.1), (5.2), (5.3), and (5.4).

Theorem 5.2. *For each of the four ODE systems (5.1), (5.2), (5.3), and (5.4), there exists a solution $X(t)$ with is unique on a maximal existence interval $[t_0, \alpha_1)$, $[t_0, \alpha_2)$, $[t_0, \beta_1)$, and $[t_0, \beta_2)$, respectively, in addition, $X(t) \succ 0$ on the existence intervals for all four ODE systems.*

Proof. For ODE system (5.2), notice that

$$\big(I - (X \otimes_s X) P_{\mathcal{A}X}\big)(X \otimes_s X) = (X^{\frac{1}{2}} \otimes_s X^{\frac{1}{2}})(I - X^{\frac{1}{2}} \otimes_s X^{\frac{1}{2}} P_{\mathcal{A}X} X^{\frac{1}{2}} \otimes_s X^{\frac{1}{2}})(X^{\frac{1}{2}} \otimes_s X^{\frac{1}{2}}).$$

Since $I - X^{\frac{1}{2}} \otimes_s X^{\frac{1}{2}} P_{\mathcal{A}X} X^{\frac{1}{2}} \otimes_s X^{\frac{1}{2}}$ is symmetric and idempotent, we know it is positive semidefinite. So $\big(I - (X \otimes_s X) P_{\mathcal{A}X}\big)(X \otimes_s X)$ is symmetric and positive semidefinite. Since $f(X) \in C^3$ on $\mathcal{S}^n_{+}$ and is convex, we have $\nabla^2 f(X)$ is symmetric and positive semidefinite. From Lemma 1.1, we know that for any $t > 0$, $I + t\big(I - (X \otimes_s X) P_{\mathcal{A}X}\big)\big((X \otimes_s X)\nabla^2 f(X)\big)$ is always invertible.

For ODE system (5.4), since $X \otimes_s X^{\frac{1}{2}}$ is also symmetric and positive definite, similarly, we can get that for any $t > 0$, $I + t\big(I - (X \otimes_s X^{\frac{1}{2}}) P_{\mathcal{A}X^{\frac{1}{2}}}\big)\big((X \otimes_s X^{\frac{1}{2}})\nabla^2 f(X)\big)$ is also invertible.

Now from Assumption 5.3 and Theorem 5.1, along with the fact that the inverse of a matrix is continuous differentiable, we know the right-hand sides of the four ODE systems are all continuous differentiable and thus locally Lipschitz continuous on $(0,+\infty)\times\mathcal{S}^n_{++}$. Since $(0,+\infty)\times\mathcal{S}^n_{++}$ is an open set, from Theorem IV.1.2 in [8], for each of the four ODE systems (5.1), (5.2), (5.3), and (5.4), a unique solution $X(t)$ is existed on a maximal existence interval $[t_0,\alpha_1)$, $[t_0,\alpha_2)$, $[t_0,\beta_1)$, and $[t_0,\beta_2)$, respectively.

Because the right-hand sides of the four ODE systems are all defined on the open set $(0,+\infty)\times\mathcal{S}^n_{++}$ and the initial points are also symmetric and positive definite, the solutions of the four ODE systems are of course in the open set $\mathcal{S}^n_{++}$, so they are all symmetric and positive definite on the existence intervals. □

Later in this section, it will be shown that $\alpha_1=+\infty$, $\alpha_2=+\infty$ (Theorem 5.9) and $\beta_1=+\infty$, $\beta_2=+\infty$ (Theorem 5.10). For simplicity, in the following, let X stand for $X(t)$.

Theorem 5.3. *Each of the unique solutions $X(t)$ of the four ODE systems (5.1), (5.2), (5.3), and (5.4) satisfies $\mathcal{A}\mathrm{svec}(X(t))=b$ on its own maximal existence interval.*

Proof. For the four ODE systems (5.1), (5.2), (5.3), and (5.4), we know that for any t belonging to their own maximal existence interval, the unique solutions $X(t)$ satisfy

$$X(t)=X^0+\int_{t_0}^{t}(\dot{X}|_{t=\tau})d\tau.$$

Notice

$$\mathcal{A}\big(I-(X\otimes_s X)P_{\mathcal{A}X}\big)(X\otimes_s X)=\mathcal{A}(X\otimes_s X)-\mathcal{A}(X\otimes_s X)=0$$

and

$$\mathcal{A}\big(I-(X\otimes_s X^{\frac{1}{2}})P_{\mathcal{A}X^{\frac{1}{2}}}\big)(X\otimes_s X^{\frac{1}{2}})=\mathcal{A}(X\otimes_s X^{\frac{1}{2}})-\mathcal{A}(X\otimes_s X^{\frac{1}{2}})=0,$$

we can get (for ODE systems (5.2) and (5.4), we use the implicit forms (5.5) and (5.6) instead)

$$\mathcal{A}\text{svec}(\dot{X}) = 0,$$

so

$$\mathcal{A}\text{svec}(X(t)) = \mathcal{A}X^0 + \int_{t_0}^{t} \mathcal{A}\text{svec}(\dot{X}|_{t=\tau})d\tau = \mathcal{A}X^0 = b.$$

Thus the theorem is proved. □

Theorem 5.4. *Let $X(t)$ be any unique solution of the four systems (5.1), (5.2), (5.3), and (5.4). Then $f(X(t))$ is a nonincreasing function of t on its own maximal existence interval.*

Proof. For ODE systems (5.1) and (5.2), we use $\mathcal{X}$ to denote $X^{\frac{1}{2}} \otimes_s X^{\frac{1}{2}}$, and $\mathcal{P}$ to denote $I - \mathcal{X}P_{\mathcal{A}X}\mathcal{X}$. For ODE systems (5.3) and (5.4), we use $\mathcal{X}$ to denote $(X \otimes_s X^{\frac{1}{2}})^{\frac{1}{2}}$, and $\mathcal{P}$ to denote $I - \mathcal{X}P_{\mathcal{A}X^{\frac{1}{2}}}\mathcal{X}$. From Theorem 5.2 it is clear that $\mathcal{X}$ and $\mathcal{P}$ are all symmetric and positive semidefinite and $\mathcal{P}^2 = \mathcal{P}$.

Now we can write ODE systems (5.1) and (5.3) in the same form as

$$\text{svec}(\dot{X}) = -\mathcal{X}\mathcal{P}\mathcal{X}\text{svec}(\frac{\partial f}{\partial X}), \tag{5.7}$$

and can write ODE systems (5.2) and (5.4) in the same form as

$$\text{svec}(\dot{X}) = -(I + t\mathcal{X}\mathcal{P}\mathcal{X}\nabla^2 f(X))^{-1}\mathcal{X}\mathcal{P}\mathcal{X}\text{svec}(\frac{\partial f}{\partial X}). \tag{5.8}$$

So for ODE systems (5.1) and (5.3), we have

$$\begin{aligned} \frac{df(X(t))}{dt} &= \text{tr}(\frac{\partial f}{\partial X}\dot{X}) = \text{svec}(\frac{\partial f}{\partial X})^T\text{svec}(\dot{X}) = -\text{svec}(\frac{\partial f}{\partial X})^T\mathcal{X}\mathcal{P}\mathcal{X}\text{svec}(\frac{\partial f}{\partial X}) \\ &= -\|\mathcal{P}\mathcal{X}\text{svec}(\frac{\partial f}{\partial X})\|_2^2 \le 0. \end{aligned}$$

Thus $f(X(t))$ is a nonincreasing function of t on its own maximal existence interval for ODE systems (5.1) and (5.3).

Similarly, we can prove the same conclusion for ODE systems (5.2) and (5.4) if we can show that $\left(I + t\mathcal{X}\mathcal{P}\mathcal{X}\nabla^2 f(X)\right)^{-1}\mathcal{X}\mathcal{P}\mathcal{X}$ is a symmetric and positive semidefinite matrix. This is actually true because we have

$$\big(I + t\mathcal{X}\mathcal{P}\mathcal{X}\nabla^2 f(X)\big)^{-1}\mathcal{X}\mathcal{P}\mathcal{X} = \mathcal{X}\mathcal{P}\big(I + t\mathcal{P}\mathcal{X}\nabla^2 f(X)\mathcal{X}\mathcal{P}\big)^{-1}\mathcal{P}\mathcal{X}. \tag{5.9}$$

Thus the theorem is proved. □

Now we will introduce four potential functions for the four ODE systems, respectively. The potential function $I_1(X,Y)$ for ODE system (5.1) can be defined as

$$I_1(X,Y) = \ln\det X + \mathrm{tr}(X^{-1}Y), \tag{5.10}$$

where $X \in \mathcal{S}^n_{++}$ is the variable and $Y \in \mathcal{S}^n_{+}$ is a parameter.

The potential function $I_2(t,X,Y)$ for ODE system (5.2) can be defined as

$$I_2(t,X,Y) = I_1(X,Y) + t\left[f(Y) - f(X) + tr\big((X-Y)\frac{\partial f}{\partial X}\big)\right]. \tag{5.11}$$

where $X \in \mathcal{S}^n_{++}$ and $t > 0$ are variables, and $Y \in \mathcal{S}^n_{+}$ is a parameter.

The potential function $I_3(X,Y)$ for ODE system (5.3) can be defined as

$$I_3(X,Y) = 2\mathrm{tr}(X^{-\frac{1}{2}}Y) + 2\mathrm{tr}(X^{\frac{1}{2}}), \tag{5.12}$$

where $X \in \mathcal{S}^n_{++}$ is the variable and $Y \in \mathcal{S}^n_{+}$ is a parameter.

The potential function $I_4(t,X,Y)$ for ODE system (5.4) can be defined as

$$I_4(t,X,Y) = I_3(X,Y) + t\left[f(Y) - f(X) + tr\big((X-Y)\frac{\partial f}{\partial X}\big)\right]. \tag{5.13}$$

where $X \in \mathcal{S}^n_{++}$ and $t > 0$ are variables, and $Y \in \mathcal{S}^n_{+}$ is a parameter.

A direct application of function $I_3(X,Y)$ and $I_4(t,X,Y)$ is the boundedness of the solutions of ODE systems (5.3) and (5.4).

Theorem 5.5. *The unique solution $X(t)$ of ODE system (5.3) is contained in a bounded set in $\mathcal{S}^n_{+}$, and the bound only depends on X^0 and X^*, where X^* is a finite optimal solution for problem (P_3).*

Proof. According Theorem 5.2 and Assumption 5.1, we can define

$$V_1(t) = I_3(X, X^*) = 2\mathrm{tr}(X^{-\frac{1}{2}}X^*) + 2\mathrm{tr}(X^{\frac{1}{2}}) \quad \forall t \in [t_0, \beta_1). \tag{5.14}$$

Then from Theorem 5.3 and ODE system (5.3), we can obtain

$$\begin{aligned}
\frac{dV_1(t)}{dt} &= -\mathrm{svec}(X^*)^T(X^{-\frac{1}{2}} \otimes_s I)^{-1}(X \otimes_s X)^{-1}\mathrm{svec}(\dot{X}) \\
&\quad +\mathrm{svec}(I)^T(X^{\frac{1}{2}} \otimes_s I)^{-1}\mathrm{svec}(\dot{X}) \\
&= -\mathrm{svec}(X^*)^T(X^{-\frac{1}{2}} \otimes_s I)^{-1}(X \otimes_s X)^{-1}\mathrm{svec}(\dot{X}) \\
&\quad +\mathrm{svec}(X)^T(X^{-\frac{1}{2}} \otimes_s I)^{-1}(X \otimes_s X)^{-1}\mathrm{svec}(\dot{X}) \\
&= \mathrm{svec}(X - X^*)^T(X^{-\frac{1}{2}} \otimes_s I)^{-1}(X \otimes_s X)^{-1}\mathrm{svec}(\dot{X}) \\
&= \mathrm{svec}(X^* - X)^T\big(I - P_{\mathcal{A}X^{\frac{1}{2}}}(X \otimes_s X^{\frac{1}{2}})\big)\mathrm{svec}(\frac{\partial f}{\partial X}) \\
&= \mathrm{svec}(X^* - X)^T\mathrm{svec}(\frac{\partial f}{\partial X}).
\end{aligned}$$

From Lemma 1.2, we know

$$\mathrm{svec}(X^* - X)^T\mathrm{svec}(\frac{\partial f}{\partial X}) \leq f(X^*) - f(X) \leq 0,$$

so we get

$$\frac{dV_1(t)}{dt} \leq f(X^*) - f(X) \leq 0. \tag{5.15}$$

Then for any $T \in [t_0, \beta_1)$, we have

$$2\mathrm{tr}(X(T)^{-\frac{1}{2}}X^*) + 2\mathrm{tr}(X(T)^{\frac{1}{2}}) = V_1(T) \leq V_1(t_0) = 2\mathrm{tr}(X(t_0)^{-\frac{1}{2}}X^*) + 2\mathrm{tr}(X(t_0)^{\frac{1}{2}}).$$

Since $\|X(T)\|_2 \leq \mathrm{tr}(X(T))$ and $\mathrm{tr}(X(T)^{-\frac{1}{2}}X^*) \geq 0$ (Lemma 1.1), we get

$$\|X(T)\|_2 \leq \|X(T)^{\frac{1}{2}}\|_2^2 \leq \big(\mathrm{tr}(X(T)^{\frac{1}{2}})\big)^2 \leq \frac{V_1(t_0)^2}{4},$$

where $\frac{V_1(t_0)^2}{4}$ only depends on X^0 and X^*. □

Theorem 5.6. *The unique solution $X(t)$ of ODE system (5.4) is contained in a bounded set in $\mathcal{S}_+^n$, and the bound only depends on X^0 and X^*, where X^* is a finite optimal solution for problem (P_3).*

Proof. According Theorem 5.2 and Assumption 5.1, we can define

$$V_2(t) = I_4(t, X, X^*) = I_3(X, X^*) + t\left[f(X^*) - f(X) + tr\big((X - X^*)\frac{\partial f}{\partial X})\right], \quad (5.16)$$

where $t \in [t_0, \beta_2)$.

Then from Theorem 5.3 and the implicit form (5.6), we have

$$\begin{aligned}\frac{dV_2(t)}{dt} &= \text{svec}(X^* - X)^T \text{svec}\big(t\nabla^2 f(X)\text{svec}(\dot{X}) \\ &\quad + \frac{\partial f}{\partial X}) + t\text{svec}(\dot{X})^T \nabla^2 f(X)\text{svec}(X - X^*) \\ &\quad + \left[f(X^*) - f(X) + tr\big((X - X^*)\frac{\partial f}{\partial X})\right] \\ &= f(X^*) - f(X) \le 0,\end{aligned}$$

so for any $T \in [t_0, \beta_2)$, we have

$$V_2(T) \le V_2(t_0) = I_3(X^0, X^*) + t_0\left[f(X^*) - f(X^0) + tr\big((X^0 - X^*)\frac{\partial f}{\partial X}|_{X=X^0}\big)\right].$$

From Lemma 1.2, we know

$$f(X^*) - f(X(T)) + tr\big((X(T) - X^*)\frac{\partial f}{\partial X}|_{X=X(T)}\big) \ge 0,$$

this along with $\|X(T)\|_2 \le \text{tr}(X(T))$ and $\text{tr}(X(T)^{-\frac{1}{2}}X^*) \ge 0$ imply

$$\|X(T)\|_2 \le \|X(T)^{\frac{1}{2}}\|_2^2 \le \big(\text{tr}(X(T)^{\frac{1}{2}})\big)^2 \le \frac{V_2(t_0)^2}{4},$$

where $\frac{V_2(t_0)^2}{4}$ only depends on X^0 and X^*. □

For ODE systems (5.1) and (5.2), we need additional conditions to guarantee the boundness of the solutions for the general convex function $f(X)$.

Lemma 5.1. *(Theorem 4.3.26, [28]) Let A be Hermitian. The vector of diagonal entries of A majorizes the vector of eigenvalues of A.*

According to [28], a vector β is said to majorize a vector α if

$$\min\{\sum_{j=1}^{k} \beta_{i_j} : 1 \le i_1 < \cdots < i_k \le n\} \ge \min\{\sum_{j=1}^{k} \alpha_{i_j} : 1 \le i_1 < \cdots < i_k \le n\},$$

for any $k = 1, 2, \ldots, n$ with equality for $k = n$.

Theorem 5.7. *For ODE systems (5.1) and (5.2), if the level set* $\{X \in \mathcal{P}_3^+ | f(X) \leq f(X^0)\}$ *is bounded, then the unique solutions* $X(t)$ *of ODE systems (5.1) and (5.2) are contained in a bounded set in* $\mathcal{S}_+^n$.

Proof. From Theorem 5.4, for the unique solutions $X(t)$ of ODE systems (5.1) and (5.2) we have $f(X(t)) \leq f(X^0)$, then $X(t)$ will be contained in the set $\{X \in \mathcal{P}_3^+ | f(X) \leq f(X^0)\}$ which is bounded according to the assumption. □

However for ODE systems (5.1) and (5.2), if $f(X)$ is linear, then we do not need the boundedness of the level set to guarantee the boundedness of the solutions.

Theorem 5.8. *If* $f(X) = \mathrm{tr}(CX)$, *where* $C \in R^{n \times n}$ *is a symmetric matrix, then the unique solutions* $X(t)$ *of ODE systems (5.1) and (5.2) are contained in a bounded set in* $\mathcal{S}_+^n$.

Proof. For ODE system (5.1), from Theorem 5.2, for any $T \in [t_0, \alpha_1)$, $X(T) \succ 0$, so we define

$$V_C(t) = tr\big(X^{-1}(X(T) - X^*)\big),$$

where $t \in [t_0, \alpha_1)$. From Theorem 5.3,

$$\begin{aligned} \frac{dV_C(t)}{dt} &= -tr\big(X^{-1}(X(T) - X^*)X^{-1}\dot{X}\big) \\ &= -\mathrm{svec}(X(T) - X^*)^T(X^{-1} \otimes_s X^{-1})\mathrm{svec}(\dot{X}) \\ &= \mathrm{svec}(X(T) - X^*)^T\mathrm{svec}(C) = f(X(T)) - f(X^*) \geq 0, \end{aligned}$$

then

$$V_C(t_0) \leq V_C(T) = n - tr\big(X(T)^{-1}X^*\big) \leq n. \tag{5.17}$$

From the eigenvalue decomposition, we have $X(T) = Q(T)\Lambda(T)Q(T)^T$, then

$$\begin{aligned} V_C(t_0) &= \mathrm{tr}((X^0)^{-1}X(T)) - \mathrm{tr}((X^0)^{-1}X^*) \\ &= tr\big(Q(T)^T(X^0)^{-1}Q(T)\Lambda(T)\big) - \mathrm{tr}((X^0)^{-1}X^*). \end{aligned}$$

From Lemma 5.1, the diagonal entries of $Q(T)^T(X^0)^{-1}Q(T)$ are all greater than

$$\lambda_{\min}(Q(T)^T(X^0)^{-1}Q(T)) = \lambda_{\min}(X^0)^{-1},$$

therefore from (5.17) we have

$$\begin{aligned}\|X(T)\|_2 &\leq \operatorname{tr}(X(T)) \leq \frac{1}{\lambda_{\min}(X^0)^{-1}}\operatorname{tr}(Q(T)^T(X^0)^{-1}Q(T)\Lambda(T)) \\ &\leq \frac{1}{\lambda_{\min}(X^0)^{-1}}\big(n + \operatorname{tr}((X^0)^{-1}X^*)\big).\end{aligned}$$

So $X(T)$ is bounded, and the bound depends only on X^0 and X^*. Notice that if $f(X)$ is linear, ODE systems (5.1) and (5.2) are the same. □

Theorem 5.9. *If the unique solutions $X(t)$ of ODE systems (5.1) and (5.2) are contained in a bounded set in $\mathcal{S}^n_+$, then the maximal existence interval for $X(t)$ can be extended to infinity, that is, $\alpha_1 = \alpha_2 = +\infty$.*

Proof. First, we prove this for ODE system (5.1) by contradiction. According to the Extension Theorem in **S**2.5, [3], we know that the solution $X(t)$ will go to the boundary of the open set $(0, +\infty) \times \mathcal{S}^n_{++}$. If $\alpha_1 \neq +\infty$, $X(t)$ will go to the boundary of $\mathcal{S}^n_{++}$, but from the condition, $X(t)$ is contained in a bounded set in $\mathcal{S}^n_+$. Then $\lambda_{\min}(X(t)) \to 0$ as $t \to \alpha_1$. So $\ln\det(X(t)) \to -\infty$ as $t \to \alpha_1$.

Let us define

$$V_4(t) = \ln\det(X),$$

where $t \in [t_0, \alpha_1)$ and X (or $X(t)$) is the unique solution of ODE system (5.1). Then from (5.7) and using the same notations $\mathcal{X}$ and $\mathcal{P}$ as in the proof of Theorem 5.4, we have

$$\frac{dV_4(t)}{dt} = \operatorname{tr}(X^{-1}\dot{X}) = \operatorname{svec}(I)^T(X^{-\frac{1}{2}} \otimes_s X^{-\frac{1}{2}})\operatorname{svec}(\dot{X}) = -\operatorname{svec}(I)^T\mathcal{P}\mathcal{X}\operatorname{svec}(\frac{\partial f}{\partial X}).$$

Since $\|\mathcal{P}\|_2 \leq 1$, along with Assumption 5.3 and the assumption that $X(t)$ is contained in a bounded set in $\mathcal{S}^n_+$, there exists a bound $M_1 > 0$ such that

$$|\frac{dV_4(t)}{dt}| \leq M_1.$$

Then for any $t \in [t_0, \alpha_1)$, we have

$$V_4(t) \geq V_4(t_0) - M_1(\alpha_1 - t_0) > -\infty,$$

but this is contrary with $\ln\det(X) \to -\infty$ as $t \to \alpha_1$. So the hypothesis is not true, thus $\alpha_1 = +\infty$.

For ODE system (5.2), from (5.8) and (5.9),

$$\text{svec}(\dot{X}) = -\mathcal{X}\mathcal{P}(I + t\mathcal{P}\mathcal{X}\nabla^2 f(X)\mathcal{X}\mathcal{P})^{-1}\mathcal{P}\mathcal{X}\text{svec}(\frac{\partial f}{\partial X}),$$

since

$$\|(I + t\mathcal{P}\mathcal{X}\nabla^2 f(X)\mathcal{X}\mathcal{P})^{-1}\|_2 = \lambda_{\max}((I + t\mathcal{P}\mathcal{X}\nabla^2 f(X)\mathcal{X}\mathcal{P})^{-1}) \leq 1,$$

we can prove $\alpha_2 = +\infty$ in the similar way. □

For ODE systems (5.3) and (5.4), if we can prove that the matrix

$$(\mathcal{A}(X \otimes_s X^{\frac{1}{2}})\mathcal{A}^T)^{-1}\mathcal{A}(X \otimes_s X^{\frac{1}{2}}),$$

is bounded for any bounded subset of $\mathcal{S}^n_{++}$, then their solutions can be extended to infinity by the same way as in the proof of Theorem 5.9. This is true if X is a diagonal matrix [66], but not correct in the general case. We show this by the following example.

Example: $m = 1$, $n = 2$, $\mathcal{A} = (1, 0, 0)$, $h = (0, 0, 1)^T$. For $\epsilon \in (0, 1)$, $X_\epsilon = Q_\epsilon \begin{pmatrix} \epsilon & \\ & 1-\epsilon \end{pmatrix} Q_\epsilon^T$, where $Q_\epsilon = \begin{pmatrix} \sqrt{1-\epsilon} & \sqrt{\epsilon} \\ -\sqrt{\epsilon} & \sqrt{1-\epsilon} \end{pmatrix}$ is an orthogonal matrix. Then

$$\begin{aligned} & \lim_{\epsilon \to 0^+} (\mathcal{A}(X_\epsilon \otimes_s X_\epsilon^{\frac{1}{2}})\mathcal{A}^T)^{-1}\mathcal{A}(X_\epsilon \otimes_s X_\epsilon^{\frac{1}{2}})h \\ = & \lim_{\epsilon \to 0^+} \frac{(1-\epsilon)\epsilon^4 - \frac{1}{2}\epsilon^2(1-\epsilon)^2(\sqrt{\epsilon} + \sqrt{1-\epsilon})^2 + \epsilon(1-\epsilon)^4}{(1-\epsilon)^2\epsilon^3 + \frac{1}{2}\epsilon^2(1-\epsilon)^2(\sqrt{\epsilon} + \sqrt{1-\epsilon})^2 + \epsilon^2(1-\epsilon)^3} = +\infty. \end{aligned}$$

By this example, we also show that $(\mathcal{A}(X \otimes_s X)\mathcal{A}^T)^{-1}\mathcal{A}(X \otimes_s X)$ can be unbounded on certain bounded subset of $\mathcal{S}^n_{++}$. However, by using some potential functions we can still extend the solutions of ODE systems (5.3) and (5.4) to infinity.

Theorem 5.10. *The maximal existence interval for the unique solutions $X(t)$ of systems (5.3) and (5.4) can be extended to infinity, that is, $\beta_1 = \beta_2 = +\infty$.*

Proof. We first prove this for ODE system (5.3) by contradiction. According to the Extension Theorem in **S**2.5, [3], we know that the solution $X(t)$ will go to the boundary of the open set $(0,+\infty)\times\mathcal{S}^n_{++}$. If $\beta_1 \neq +\infty$, $X(t)$ will go to the boundary of $\mathcal{S}^n_{++}$, but from Theorem 5.5, $X(t)$ is contained in a bounded set in $\mathcal{S}^n_{+}$. So $\lambda_{\min}(X(t)) \to 0$ as $t \to \beta_1$.

Let us define

$$V_3(t) = I_3(X, X^0) = 2\mathrm{tr}(X^{-\frac{1}{2}}X^0) + 2\mathrm{tr}(X^{\frac{1}{2}}),$$

where $t \in [t_0, \beta_1)$ and X (or $X(t)$) is the unique solution of ODE system (5.3).

Then we have

$$\frac{dV_3(t)}{dt} = \mathrm{svec}(X^0 - X)^T \mathrm{svec}(\frac{\partial f}{\partial X}),$$

from Theorem 5.5 and Assumption 5.3, we know there exists a bound $M > 0$ which depends only on X^0, X^* and $f(X)$ such that for every $t \in [t_0, \beta_1)$,

$$|\frac{dV_3(t)}{dt}| = |\mathrm{svec}(X^0 - X)^T \mathrm{svec}(\frac{\partial f}{\partial X})| \leq M.$$

Hence for any $t \in [t_0, \beta_1)$,

$$V_3(t) \leq V_3(t_0) + M(\beta_1 - t_0) < +\infty. \tag{5.18}$$

But $\lambda_{\min}(X(t)) \to 0$ as $t \to \beta_1$. Let $X(t) = Q(t)\Lambda(t)Q(t)^T$ be an eigenvalue decomposition of $X(t)$, then $\mathrm{tr}(X(t)^{-\frac{1}{2}}X^0) = \mathrm{tr}(\Lambda(t)^{-\frac{1}{2}}Q(t)^T X^0 Q(t))$. According to Lemma 5.1, the diagonal entries of $Q(t)^T X^0 Q(t)$ are all greater than

$$\lambda_{\min}(Q(t)^T X^0 Q(t)) = \lambda_{\min}(X^0) > 0.$$

So $\mathrm{tr}(X(t)^{-\frac{1}{2}}X^0) \to +\infty$ as $t \to \beta_1$ which is contrary with (5.18). So the hypothesis is not true, thus $\beta_1 = +\infty$.

For ODE system (5.4), from Theorem 5.6, Lemmas 1.2 and 5.1, we can prove $\beta_2 = +\infty$ in the similar way. $\square$

From Theorems 5.9 and 5.10, we can define the limit set for the unique solutions $X(t)$ of the four ODE systems (5.1), (5.2), (5.3), and (5.4). For $i = 1, 2, 3, 4$, the limit set $\Omega^i(X^0)$ of $\{X(t)\}$ of the ODE system $(i+1)$ can be defined as follows

$$\Omega^i(X^0) = \left\{ X \in \mathcal{S}^n \mid \exists \{t_k\}_{k=0}^{+\infty} \text{ with } \lim_{k\to+\infty} t_k = +\infty \text{ such that } \lim_{k\to+\infty} X(t_k) = X \right\}.$$

Theorem 5.11. *If the unique solutions $X(t)$ of ODE systems (5.1) and (5.2) are contained in a bounded set in $\mathcal{S}^n_+$, then for each $i = 1, 2, 3, 4$, the limit set $\Omega^i(X^0)$ is nonempty, compact, and connected. Furthermore $\Omega^i(X^0)$ is contained in $\mathcal{P}^+_3$, $i = 1, 2, 3, 4$.*

Proof. From Theorems 5.2, 5.3, 5.10, and 5.9, we know that the limit set $\Omega^i(X^0)$ is contained in $\mathcal{P}^+_3$, $i = 1, 2, 3, 4$. From the proof of Theorem 5.10 and the assumption, it is easy to see that the unique solutions $X(t)$ of the four ODE systems are contained in a bounded closed set. So similar to the proof of Theorem 1.1 on page 390 in [10] (the proof in [10] is for $n = 2$, but it can be easily extended to the general case), it can be verified that for each $i = 1, 2, 3, 4$, $\Omega^i(X^0)$ is nonempty, compact, and connected. □

At the end of this section, we prove the weak convergence of ODE system (5.1), i.e., $\text{svec}(\dot{X}) \to 0$ as $t \to +\infty$.

Theorem 5.12. *If the unique solution $X(t)$ of ODE system (5.1) is contained in a bounded set in $\mathcal{S}^n_+$, then*

$$\lim_{t\to+\infty} \big(I - (X \otimes_s X)P_{\mathcal{A}X})(X \otimes_s X)\text{svec}(\frac{\partial f}{\partial X}) = 0.$$

Proof. From (5.7) and using the same notations $\mathcal{X}$ and $\mathcal{P}$ as in the proof of Theorem 5.4, we know

$$\begin{aligned} \frac{df(X(t))}{dt} &= -\text{svec}(\frac{\partial f}{\partial X})^T \mathcal{X}\mathcal{P}\mathcal{X}\text{svec}(\frac{\partial f}{\partial X}) \\ &= -\text{svec}(\frac{\partial f}{\partial X})^T((X \otimes_s X) - (X \otimes_s X)P_{\mathcal{A}X}(X \otimes_s X))\text{svec}(\frac{\partial f}{\partial X}). \end{aligned}$$

From Assumption 5.3, $f(X) \in C^2$ on $\mathcal{S}^n_+$. Furthermore, $\text{svec}(\dot{X})$ is bounded because $X(t)$ is contained in a bounded set in $\mathcal{S}^n_+$. So if we want to show $\frac{d^2 f(X(t))}{dt^2}$ is bounded, we only need to show that $\frac{d(X \otimes_s X)}{dt}$ and $\frac{d((X \otimes_s X) P_{\mathcal{A}X} (X \otimes_s X))}{dt}$ are both bounded. Notice

$$\frac{d(X \otimes_s X)}{dt} = 2X \otimes_s \dot{X},$$

since $X(t)$ and $\dot{X}$ are both bounded, thus $\frac{d(X \otimes_s X)}{dt}$ is bounded. Notice

$$\begin{aligned} & \frac{d((X \otimes_s X) P_{\mathcal{A}X} (X \otimes_s X))}{dt} \\ = \; & 2(X \otimes_s \dot{X}) P_{\mathcal{A}X} (X \otimes_s X) + 2(X \otimes_s X) P_{\mathcal{A}X} (X \otimes_s \dot{X}) \\ & -2(X \otimes_s X) P_{\mathcal{A}X} (X \otimes_s \dot{X}) P_{\mathcal{A}X} (X \otimes_s X) \\ = \; & 2(X \otimes_s \dot{X}) \mathcal{X}^{-1} \mathcal{X} P_{\mathcal{A}X} \mathcal{X}^2 + 2\mathcal{X}^2 P_{\mathcal{A}X} \mathcal{X} \mathcal{X}^{-1} (X \otimes_s \dot{X}) \\ & -2\mathcal{X}^2 P_{\mathcal{A}X} \mathcal{X} \mathcal{X}^{-1} (X \otimes_s \dot{X}) \mathcal{X}^{-1} \mathcal{X} P_{\mathcal{A}X} \mathcal{X}^2. \end{aligned}$$

Since $\mathcal{X} P_{\mathcal{A}X} \mathcal{X}$ is symmetric and idempotent, it's always bounded. So if we can show $\mathcal{X}^{-1}(X \otimes_s \dot{X})\mathcal{X}^{-1}$ is bounded, then $\frac{d((X \otimes_s X) P_{\mathcal{A}X} (X \otimes_s X))}{dt}$ will be bounded. Let smat be the inverse map of svec. From (5.7), if we denote $B(t) = -\text{smat}(\mathcal{P}\mathcal{X}\text{svec}(\frac{\partial f}{\partial X}))$, then $B(t)$ is also bounded, and $\dot{X} = X^{\frac{1}{2}} B(t) X^{\frac{1}{2}}$. Therefore we get

$$\mathcal{X}^{-1}(X \otimes_s \dot{X})\mathcal{X}^{-1} = (I \otimes_s B(t)),$$

which is bounded. Thus $\frac{d^2 f(X(t))}{dt^2}$ is bounded, and as a consequence, $\frac{df(X(t))}{dt}$ is uniformly continuous. Furthermore from Theorem 5.4 and Assumption 5.1, $f(X(t))$ has a finite limit as $t \to +\infty$. So from Barbalat's Lemma, we have

$$\lim_{t \to +\infty} \frac{df(X(t))}{dt} = \lim_{t \to +\infty} -\|\mathcal{P}\mathcal{X}\text{svec}(\frac{\partial f}{\partial X})\|_2^2 = 0.$$

□

5.2 Optimality of The Cluster Point(s)

In this part, we will show that every accumulation point of the solutions of the four ODE systems (5.1), (5.2), (5.3), and (5.4) is an optimal solution for problem (P_3).

Theorem 5.13. *If the unique solutions $X(t)$ of ODE systems (5.1) and (5.2) are contained in a bounded set in $\mathcal{S}_+^n$, then for any $X^{(1)} \in \Omega^1(X^0)$ and $X^{(2)} \in \Omega^2(X^0)$, both $X^{(1)}$ and $X^{(2)}$ are optimal solutions for problem (P_3).*

Proof. We prove this by contradiction. From Theorems 5.4 and 5.9, we know $\lim_{t\to+\infty} f(X(t))$ exists since $f(X)$ is bounded below in $\mathcal{P}_3^+$. Then if $X^{(1)} \in \Omega^1(X^0)$ is not an optimal solution for problem (P_3), we have

$$f(X^{(1)}) = \lim_{k\to+\infty} f(X(t_k)) > f(X^*).$$

Let us define

$$Y^{(1)} = \frac{f(X^{(1)}) - f(X^*)}{2(f(X^0) - f(X^*))}X^0 + (1 - \frac{f(X^{(1)}) - f(X^*)}{2(f(X^0) - f(X^*))})X^*,$$

then $Y^{(1)} \in \mathcal{P}_3^{++}$. Since $f(X)$ is convex, we have

$$\begin{aligned} f(Y^{(1)}) &\leq \frac{f(X^{(1)}) - f(X^*)}{2(f(X^0) - f(X^*))}f(X^0) + (1 - \frac{f(X^{(1)}) - f(X^*)}{2(f(X^0) - f(X^*))})f(X^*) \\ &= \frac{f(X^{(1)}) + f(X^*)}{2}. \end{aligned}$$

From (5.10), we can define

$$V_5(t) = I_1(X, Y^{(1)}) = \ln\det X + \mathrm{tr}(X^{-1}Y^{(1)}),$$

where $t \in [t_0, +\infty)$ and X (or $X(t)$) is the unique solution of ODE system (5.1). Then from Theorem 5.3 and Lemma 1.2, we have

$$\begin{aligned} \frac{dV_5(t)}{dt} &= \mathrm{tr}(X^{-1}\dot{X}) - \mathrm{tr}(X^{-1}Y^{(1)}X^{-1}\dot{X}) \\ &= \mathrm{svec}(X)^T(X^{-1} \otimes_s X^{-1})\mathrm{svec}(\dot{X}) - \mathrm{svec}(Y^{(1)})^T(X^{-1} \otimes_s X^{-1})\mathrm{svec}(\dot{X}) \\ &= -\mathrm{svec}(X - Y^{(1)})^T(I - P_{\mathcal{A}X})\mathrm{svec}(\frac{\partial f}{\partial X}) = \mathrm{svec}(Y^{(1)} - X)^T\mathrm{svec}(\frac{\partial f}{\partial X}) \\ &\leq f(Y^{(1)}) - f(X) \leq f(Y^{(1)}) - f(X^{(1)}) \leq \frac{f(X^{(1)}) + f(X^*)}{2} - f(X^{(1)}) \\ &= \frac{f(X^*) - f(X^{(1)})}{2} < 0, \end{aligned}$$

so $V_5(t) \to -\infty$ as $t \to +\infty$. We next show $V_5(t)$ is bounded below.

For any $t \in [t_0, +\infty)$, let $X(t) = Q(t)\Lambda(t)Q(t)^T$ be an eigenvalue decomposition of $X(t)$, and $\{\lambda_i(t)\}_{i=1}^n$ be the eigenvalues of $X(t)$. Then

$$\begin{aligned} V_5(t) &= \ln\det X(t) + tr\big(Q(t)\Lambda(t)^{-1}Q(t)^T Y^{(1)}\big) \\ &= \sum_{i=1}^n \ln\lambda_i(t) + tr\big(\Lambda(t)^{-1}Q(t)^T Y^{(1)} Q(t)\big), \end{aligned}$$

since $Y^{(1)} \in \mathcal{P}^{++}$, we have

$$\lambda_{\min}(Q(t)^T Y^{(1)} Q(t)) = \lambda_{\min}(Y^{(1)}) > 0.$$

Hence from Lemma 5.1, we have

$$\begin{aligned} V_5(t) &= \sum_{i=1}^n \ln\lambda_i(t) + \mathrm{tr}(\Lambda(t)^{-1}Q(t)^T Y^{(1)} Q(t)) \\ &\geq \sum_{i=1}^n \ln\lambda_i(t) + \sum_{i=1}^n \lambda_i(t)^{-1}\lambda_{\min}(Y^{(1)}) = \sum_{i=1}^n (\ln\lambda_i(t) + \lambda_i(t)^{-1}\lambda_{\min}(Y^{(1)})) \\ &\geq \sum_{i=1}^n (\ln\lambda_{\min}(Y^{(1)}) + 1) = n(\ln\lambda_{\min}(Y^{(1)}) + 1) > -\infty, \end{aligned}$$

which is contrary with $V_5(t) \to -\infty$ as $t \to +\infty$. So the hypothesis is not true, thus $X^{(1)}$ is an optimal solution.

As for $X^{(2)} \in \Omega^2(X^0)$, let the unique solution of ODE system (5.2) be X (or $X(t)$). Notice if $Y \in \mathcal{P}_3^+$, then

$$\begin{aligned} \frac{dI_2(t,X,Y)}{dt} &= \mathrm{svec}(Y-X)^T \mathrm{svec}(t\nabla^2 f(X)\mathrm{svec}(\dot{X}) + \frac{\partial f}{\partial X}) \\ &\quad + t\mathrm{svec}(\dot{X})^T \nabla^2 f(X)\mathrm{svec}(X-Y) + \left[f(Y) - f(X) + \mathrm{tr}(X-Y)\frac{\partial f}{\partial X}\right] \\ &= f(Y) - f(X), \end{aligned}$$

and we can prove that $X^{(2)}$ is also an optimal solution for problem (P_3) in the similar way. □

Now we are ready to prove the weak convergence of ODE system (5.2).

Theorem 5.14. *For ODE system (5.2), if the level set* $\{X \in \mathcal{P}^+ | f(X) \leq f(X^0)\}$ *is bounded, then the unique solution* $X(t)$ *of ODE system (5.2) satisfies*

$$\lim_{t\to+\infty} \dot{X} = 0.$$

Proof. From (5.9) and using the same notations $\mathcal{X}$ and $\mathcal{P}$ as in the proof of Theorem 5.4, we know that if we can prove

$$\lim_{t\to+\infty} -\|\mathcal{P}\mathcal{X}\mathrm{svec}(\frac{\partial f}{\partial X})\|_2 = 0, \tag{5.19}$$

then the theorem holds.

We prove this by contradiction. If (5.19) is not true, there must exist a constant $c_0 > 0$ such that for any $T > t_0$, there always exists a $t > T$ such that $\|\mathcal{P}\mathcal{X}\mathrm{svec}(\frac{\partial f}{\partial X})\|_2 > c_0$.

Let us consider the following cluster of trajectories: each trajectory is defined by the solution of ODE system (5.1) with initial point $X(t)$ at initial time t_0, where $X(t)$ denotes the solution of ODE system (5.2) at time t. We use $\tilde{X}(\tau, t)$ to denote this trajectory. From Theorem 5.4, each trajectory $\tilde{X}(\tau, t)$ is contained in the bounded level set. Then with the same analysis as in the proof of Theorem 5.12, we know there exists an $L_0 > 0$ so that

$$|\frac{d^2 f(\tilde{X}(\tau,t))}{d\tau^2}| \leq L_0. \tag{5.20}$$

From the hypothesis and Theorem 5.13, there exists a $T_1 > t_0$, such that

$$f(\tilde{X}(t_0, T_1)) = f(X(T_1)) < f(X^*) + \frac{c_0^4}{4L_0}, \tag{5.21}$$

and

$$\|\mathcal{P}\mathcal{X}\mathrm{svec}(\frac{\partial f}{\partial X})\|_2|_{X=\tilde{X}(t_0,T_1)} = \|\mathcal{P}\mathcal{X}\mathrm{svec}(\frac{\partial f}{\partial X})\|_2|_{X=X(T_1)} > c_0. \tag{5.22}$$

From (5.20) and (5.22), we have

$$\|\mathcal{P}\mathcal{X}\mathrm{svec}(\frac{\partial f}{\partial X})\|_2^2|_{X=\tilde{X}(\tau,T_1)} \geq \max(c_0^2 - L_0(\tau - t_0), 0),$$

then

$$\int_{t_0}^{+\infty} -\frac{df(\tilde{X}(\tau,T_1))}{d\tau} d\tau = \int_{t_0}^{+\infty} \|\mathcal{P}\mathcal{X}\mathrm{svec}(\frac{\partial f}{\partial X})\|_2^2|_{X=\tilde{X}(\tau,T_1)} d\tau \geq \frac{c_0^4}{2L_0},$$

however, from Theorem 5.13 and (5.21), we have

$$\int_{t_0}^{+\infty} -\frac{df(\tilde{X}(\tau,T_1))}{d\tau} d\tau = f(X(T_1)) - f(X^*) < \frac{c_0^4}{4L_0},$$

which contradicts with the previous inequality. Thus the theorem is proved. □

Theorem 5.15. *For any $X^{(3)} \in \Omega^3(X^0)$ and $X^{(4)} \in \Omega^4(X^0)$, both $X^{(3)}$ and $X^{(4)}$ are optimal solutions for problem (P_3).*

Proof. We prove this by contradiction. Similar to the proof of Theorem 5.13, if $X^{(3)} \in \Omega^3(X^0)$ is not an optimal solution, then

$$f(X^{(3)}) = \lim_{k \to +\infty} f(X(t_k)) > f(X^*).$$

From (5.15), we can see $V_1(t)$ defined by (5.14) will go to $-\infty$ as $t \to +\infty$. However,

$$V_1(t) = I_3(X, X^*) = 2\text{tr}(X^{-\frac{1}{2}}X^*) + 2\text{tr}(X^{\frac{1}{2}}) \geq 0 \quad \forall t \in [t_0, +\infty),$$

where X (or $X(t)$) is the unique solution for ODE system (5.3). So the hypothesis is not true, thus $X^{(3)}$ must be an optimal solution for problem (P_3).

As for $X^{(4)} \in \Omega^4(X^0)$, by using $V_2(t)$ defined in (5.16), we can prove that $X^{(4)}$ is also an optimal solution for problem (P_3) in the similar way. □

5.3 Convergence of The Continuous Trajectories

Now, it comes to the key results of this chapter. Theorem 5.16 below shows that if the maximal rank among the optimal solution set of problem (P_3) is equal to one, then the solution of ODE system (5.3) converges as $t \to +\infty$. Theorem 5.17 shows that the solution of ODE system (5.4) always converges as $t \to +\infty$. Theorem 5.20 shows that in the linear case of $f(X)$, the solutions of ODE systems (5.1) and (5.2) also converge as $t \to +\infty$.

Theorem 5.16. *Every point in the limit set $\Omega^3(X^0)$ has the maximal rank among the optimal solution set of problem (P_3). Furthermore, if the maximal rank among the optimal solution set of problem (P_3) is equal to one, then the limit set $\Omega^3(X^0)$ only contains a single point.*

Proof. From Theorem 5.11, we know that $\Omega^3(X^0)$ is not empty. So we can choose a point $\bar{X} \in \Omega^3(X^0)$, and evidently $\bar{X} \in \mathcal{P}_3^+$. Without loss of generality, we assume the optimal solution X^* has the maximal rank among the optimal solution set of problem (P_3), and $rank(X^*) = r$. Let $X^* = Q\Lambda Q^T$ be an eigenvalue decomposition of X^* and

$$\Lambda = \begin{pmatrix} \Lambda_1 & \\ & 0 \end{pmatrix},$$

where Λ_1 is a $r \times r$ diagonal matrix and Λ_1 is invertible. Since X^* has the maximal rank among the optimal solution set and $\bar{X}$ is an optimal solution (Theorem 5.13), $rank(\bar{X}) \leq rank(X^*)$. Following the same claim as Lemma 4.1 in [20], there exists an eigenvalue decomposition $\bar{X} = \bar{Q}\bar{\Lambda}\bar{Q}^T$ with

$$\bar{\Lambda} = \begin{pmatrix} \bar{\Lambda}_1 & \\ & 0 \end{pmatrix},$$

where $\bar{\Lambda}_1$ is a $r \times r$ diagonal matrix, and a sequence $\{\bar{t}_k\}_{k=1}^{+\infty}$ with $\lim_{k\to+\infty} \bar{t}_k = +\infty$ such that $X(\bar{t}_k) \to \bar{X}$, $Q(\bar{t}_k) \to \bar{Q}$, and $\Lambda(\bar{t}_k) \to \bar{\Lambda}$, where $Q(\bar{t}_k)\Lambda(\bar{t}_k)Q(\bar{t}_k)^T$ is an eigenvalue decomposition of $X(\bar{t}_k)$ with $\Lambda(\bar{t}_k) = \begin{pmatrix} \Lambda_1(\bar{t}_k) & \\ & \Lambda_2(\bar{t}_k) \end{pmatrix}$, $\Lambda_1(\bar{t}_k) \in R^{r\times r}$. Notice $V_1(t)$ defined by (5.14) is a nonincreasing function in $[t_0, +\infty)$ and bounded below, we know $V_1(t)$ has a finite limit as $t \to +\infty$. Therefore,

$$\begin{aligned} \lim_{t\to+\infty} V_1(t) &= \lim_{k\to+\infty} V_1(\bar{t}_k) = \lim_{k\to+\infty} \left[2\mathrm{tr}(X(\bar{t}_k)^{-\frac{1}{2}}X^*) + 2\mathrm{tr}(X(\bar{t}_k)^{\frac{1}{2}})\right] \\ &= \lim_{k\to+\infty} \left[2\mathrm{tr}(Q(\bar{t}_k)\Lambda(\bar{t}_k)^{-\frac{1}{2}}Q(\bar{t}_k)^TQ\Lambda Q^T) + 2\mathrm{tr}(Q(\bar{t}_k)\Lambda(\bar{t}_k)^{\frac{1}{2}}Q(\bar{t}_k)^T)\right] \\ &= \lim_{k\to+\infty} \left[2\mathrm{tr}(\Lambda(\bar{t}_k)^{-\frac{1}{2}}Q(\bar{t}_k)^TQ\Lambda Q^TQ(\bar{t}_k)) + 2\mathrm{tr}(\Lambda(\bar{t}_k)^{\frac{1}{2}})\right]. \end{aligned}$$

Let $Q(\bar{t}_k)^TQ = \begin{pmatrix} (Q(\bar{t}_k)^TQ)_{11} & (Q(\bar{t}_k)^TQ)_{12} \\ (Q(\bar{t}_k)^TQ)_{21} & (Q(\bar{t}_k)^TQ)_{22} \end{pmatrix}$, where $(Q(\bar{t}_k)^TQ)_{11} \in R^{r\times r}$, then

$$\begin{aligned} \lim_{t\to+\infty} V_1(t) &= \lim_{k\to+\infty} \Big[2\mathrm{tr}(\Lambda_1(\bar{t}_k)^{-\frac{1}{2}}(Q(\bar{t}_k)^TQ)_{11}\Lambda_1(Q(\bar{t}_k)^TQ)_{11}^T) \\ &\quad + 2\mathrm{tr}(\Lambda_2(\bar{t}_k)^{-\frac{1}{2}}(Q(\bar{t}_k)^TQ)_{21}\Lambda_1(Q(\bar{t}_k)^TQ)_{21}^T) + 2\mathrm{tr}(\Lambda(\bar{t}_k)^{\frac{1}{2}})\Big]. \end{aligned}$$

Since $\Lambda_1(\bar{t}_k) \to \bar{\Lambda}_1$, $\Lambda_2(\bar{t}_k) \to 0$, and $Q(\bar{t}_k) \to \bar{Q}$, we know the diagonal entries of $(\bar{Q}^TQ)_{21}\Lambda_1(\bar{Q}^TQ)_{21}^T$ are all zero which leads to $(\bar{Q}^TQ)_{21} = 0$. Since $\bar{Q}^TQ$ is an orthogonal matrix, $(\bar{Q}^TQ)_{21}(\bar{Q}^TQ)_{21}^T + (\bar{Q}^TQ)_{22}(\bar{Q}^TQ)_{22}^T = I$. But $(\bar{Q}^TQ)_{21} = 0$, we have $(\bar{Q}^TQ)_{22}(\bar{Q}^TQ)_{22}^T = I$, so $(\bar{Q}^TQ)_{22}$ is an orthogonal matrix. Then from $(\bar{Q}^TQ)_{11}(\bar{Q}^TQ)_{21}^T + (\bar{Q}^TQ)_{12}(\bar{Q}^TQ)_{22}^T = 0$ and $(\bar{Q}^TQ)_{21} = 0$, we have $(\bar{Q}^TQ)_{12} = 0$. From $(\bar{Q}^TQ)_{11}(\bar{Q}^TQ)_{11}^T + (\bar{Q}^TQ)_{12}(\bar{Q}^TQ)_{12}^T = I$, we know $(\bar{Q}^TQ)_{11}(\bar{Q}^TQ)_{11}^T = I$ and so $(\bar{Q}^TQ)_{11}$ is also an orthogonal matrix. From Lemma 5.1, we know the diagonal entries of $(\bar{Q}^TQ)_{11}\Lambda_1(\bar{Q}^TQ)_{11}^T$ are all positive. So $\bar{\Lambda}_1$ must be invertible which indicates that $\bar{X}$ has the maximal rank among the optimal solution set of problem (P_3), and

$$\begin{aligned}\lim_{t\to+\infty} V_1(t) &= 2\mathrm{tr}(\bar{\Lambda}_1^{-\frac{1}{2}}(\bar{Q}^TQ)_{11}\Lambda_1(\bar{Q}^TQ)_{11}^T) + 2\mathrm{tr}(\bar{\Lambda}_1^{\frac{1}{2}}) \\ &\quad + \lim_{k\to+\infty} 2\mathrm{tr}((Q(\bar{t}_k)^TQ)_{21}^T\Lambda_2(\bar{t}_k)^{-\frac{1}{2}}(Q(\bar{t}_k)^TQ)_{21}\Lambda_1).\end{aligned}$$

If the maximal rank among the optimal solution set of problem (P_3) is equal to one, then the optimal solutions have the form $\lambda q_1 q_1^T$, where $\lambda \geq 0$ and $\|q\| = 1$. Hence in this case, the orthogonal matrix in the eigenvalue decomposition of every optimal solution can have the same Q as X^*. If $\bar{X}$ is not the only point of $\Omega^3(X^0)$, there must exist another point $\tilde{X} \in \Omega^3(X^0)$. Let us define

$$V_6(t) = I_3(X, \bar{X}) = 2\mathrm{tr}(X^{-\frac{1}{2}}\bar{X}) + 2\mathrm{tr}(X^{\frac{1}{2}}),$$

where $t \in [t_0, +\infty)$ and X (or $X(t)$) is the unique solution of ODE system (5.3). Since $\tilde{X} \in \Omega^3(X^0)$, for the same reason, $\tilde{X}$ has the maximal rank among the optimal solution set of problem (P_3) and there exists an eigenvalue decomposition $\tilde{X} = Q\tilde{\Lambda}Q^T = Q\begin{pmatrix}\tilde{\Lambda}_1 & \\ & 0\end{pmatrix}Q^T$, where $\tilde{\Lambda}_1 \in R$ is positive, and a sequence $\{\tilde{t}_k\}_{k=1}^{+\infty}$ with $\lim_{k\to+\infty}\tilde{t}_k = +\infty$ such that $X(\tilde{t}_k) \to \tilde{X}$, $Q(\tilde{t}_k) \to Q$, and $\Lambda(\tilde{t}_k) \to \tilde{\Lambda}$, where $Q(\tilde{t}_k)\Lambda(\tilde{t}_k)Q(\tilde{t}_k)^T$ is an eigenvalue decomposition of $X(\tilde{t}_k)$ with $\Lambda(\tilde{t}_k) = \begin{pmatrix}\Lambda_1(\tilde{t}_k) & \\ & \Lambda_2(\tilde{t}_k)\end{pmatrix}$, $\Lambda_1(\tilde{t}_k) \in R$, such that

$$\lim_{t\to+\infty} V_6(t) = 2\tilde{\Lambda}_1^{-\frac{1}{2}}\bar{\Lambda}_1 + 2\tilde{\Lambda}_1^{\frac{1}{2}} + \lim_{k\to+\infty} 2(Q(\tilde{t}_k)^TQ)_{21}^T\Lambda_2(\tilde{t}_k)^{-\frac{1}{2}}(Q(\tilde{t}_k)^TQ)_{21}\bar{\Lambda}_1. \quad (5.23)$$

Hence, $\lim_{k\to+\infty} 2(Q(\tilde{t}_k)^TQ)_{21}^T\Lambda_2(\tilde{t}_k)^{-\frac{1}{2}}(Q(\tilde{t}_k)^TQ)_{21}$ exists and we denote it by $\tilde{\epsilon}$. However, $\bar{X}$ is also an accumulation point, therefore

$$\lim_{t\to+\infty} V_6(t) = 4\bar{\Lambda}_1^{\frac{1}{2}} + \bar{\epsilon}\bar{\Lambda}_1, \tag{5.24}$$

where $\bar{\epsilon} = \lim_{k\to+\infty} 2\text{tr}((Q(\bar{t}_k)^T Q)_{21}^T \Lambda_2(\bar{t}_k)^{-\frac{1}{2}}(Q(\bar{t}_k)^T Q)_{21}$.

Combining (5.23) and (5.24), we get

$$(\bar{\epsilon} - \tilde{\epsilon})\bar{\Lambda}_1 = 2\tilde{\Lambda}_1^{-\frac{1}{2}}\bar{\Lambda}_1 + 2\tilde{\Lambda}_1^{\frac{1}{2}} - 4\bar{\Lambda}_1^{\frac{1}{2}} = 2\bar{\Lambda}_1^{\frac{1}{2}}(\tilde{\Lambda}_1^{-\frac{1}{2}}\bar{\Lambda}_1^{\frac{1}{2}} + \tilde{\Lambda}_1^{\frac{1}{2}}\bar{\Lambda}_1^{-\frac{1}{2}} - 2) \geq 0,$$

which implies $\bar{\epsilon} \geq \tilde{\epsilon}$. If we replace $\bar{X}$ in $V_6(t)$ by $\tilde{X}$, from the similar claim, we can get

$$(\tilde{\epsilon} - \bar{\epsilon})\tilde{\Lambda}_1 \geq 0,$$

which indicates $\tilde{\epsilon} \geq \bar{\epsilon}$. Therefore $\tilde{\epsilon} = \bar{\epsilon}$, and then $\tilde{\Lambda}_1 = \bar{\Lambda}_1$, hence $\bar{X} = \tilde{X}$ and the limit set $\Omega^3(X^0)$ is a singleton. □

Theorem 5.17. *The limit set $\Omega^4(X^0)$ only contains a single point, and the limit point has the maximal rank among the optimal solution set of problem (P_3).*

Proof. From Theorem 5.11, we know that $\Omega^4(X^0)$ is not empty. So we can choose a point $\bar{X} \in \Omega^4(X^0)$, and evidently $\bar{X} \in \mathcal{P}_3^+$. Similar to the proof of Theorem 5.16, by using $V_2(t)$ defined by (5.16), we can show every accumulation point in $\Omega^4(X^0)$ has the maximal rank among the optimal solution set of problem (P_3). From (5.13), we can define $V_7(t)$ as follows

$$V_7(t) = I_4(t, X, \bar{X}) - 4\text{tr}(\bar{X}^{\frac{1}{2}}),$$

where $t \in [t_0, +\infty)$ and X (or $X(t)$) is the unique solution of ODE system (5.4). Since $\frac{dV_7(t)}{dt} = f(\bar{X}) - f(X(t)) \leq 0$, $V_7(t)$ is a nonincreasing function, furthermore $V_7(t)$ is bounded below, so $\lim\limits_{t\to+\infty} V_7(t)$ exists. Similar to the claim in the proof of Theorem 5.16, if $\tilde{X} = \tilde{Q}\tilde{\Lambda}\tilde{Q}^T \in \Omega^4(X^0)$ is another point, we can have

$$\begin{aligned}\lim_{t\to+\infty} V_7(t) \;\; = \;\; & 2\text{tr}(\tilde{\Lambda}_1^{-\frac{1}{2}}(\tilde{Q}^T\bar{Q})_{11}\bar{\Lambda}_1(\tilde{Q}^T\bar{Q})_{11}^T) + 2\text{tr}(\tilde{\Lambda}_1^{\frac{1}{2}}) - 4\text{tr}(\bar{\Lambda}_1^{\frac{1}{2}}) \\ & + \lim_{k\to+\infty}\Big\{2\text{tr}((Q(\tilde{t}_k)^T\bar{Q})_{21}^T\Lambda_2(\tilde{t}_k)^{-\frac{1}{2}}(Q(\tilde{t}_k)^T\bar{Q})_{21}\bar{\Lambda}_1) \\ & + \tilde{t}_k\left[f(\bar{X}) - f(X(\tilde{t}_k)) + tr\big((X(\tilde{t}_k) - \bar{X})\frac{\partial f}{\partial X}|_{X=X(\tilde{t}_k)}\big)\right]\Big\},\end{aligned}$$

where $\tilde{\Lambda}_1$, $\tilde{t}_k$, $(\tilde{Q}^T\bar{Q})_{11}$, $\bar{\Lambda}_1$, $Q(\tilde{t}_k)$, $(Q(\tilde{t}_k)^T\bar{Q})_{21}$, and $\Lambda_2(\tilde{t}_k)$ have the same meanings as that in the proof of Theorem 5.16. From Lemma 1.2, we know for any k,

$$\begin{aligned}&2\text{tr}((Q(\tilde{t}_k)^T\bar{Q})_{21}^T\Lambda_2(\tilde{t}_k)^{-\frac{1}{2}}(Q(\tilde{t}_k)^T\bar{Q})_{21}\bar{\Lambda}_1)\\ &+\tilde{t}_k\left[f(\bar{X})-f(X(\tilde{t}_k))+tr\big((X(\tilde{t}_k)-\bar{X})\frac{\partial f}{\partial X}|_{X=X(\tilde{t}_k)}\big)\right]\geq 0,\end{aligned}$$

hence its limit must be nonnegative. First we assume $\lim_{t\to+\infty}V_7(t)=0$, then we can get

$$\begin{aligned}&0\geq \text{tr}(\tilde{\Lambda}_1^{-\frac{1}{2}}(\tilde{Q}^T\bar{Q})_{11}\bar{\Lambda}_1(\tilde{Q}^T\bar{Q})_{11}^T)+\text{tr}(\tilde{\Lambda}_1^{\frac{1}{2}})-2\text{tr}(\bar{\Lambda}_1^{\frac{1}{2}})\\ &=\text{tr}(\bar{\Lambda}_1^{\frac{1}{2}}(\tilde{Q}^T\bar{Q})_{11}^T\tilde{\Lambda}_1^{-\frac{1}{2}}(\tilde{Q}^T\bar{Q})_{11}\bar{\Lambda}_1^{\frac{1}{2}})+\text{tr}((\tilde{Q}^T\bar{Q})_{11}^T\tilde{\Lambda}_1^{\frac{1}{2}}(\tilde{Q}^T\bar{Q})_{11})-2\text{tr}(\bar{\Lambda}_1^{\frac{1}{2}})\\ &=tr\Big(\bar{\Lambda}_1^{\frac{1}{4}}\left[\bar{\Lambda}_1^{\frac{1}{4}}(\tilde{Q}^T\bar{Q})_{11}^T\tilde{\Lambda}_1^{-\frac{1}{2}}(\tilde{Q}^T\bar{Q})_{11}\bar{\Lambda}_1^{\frac{1}{4}}+\bar{\Lambda}_1^{-\frac{1}{4}}(\tilde{Q}^T\bar{Q})_{11}^T\tilde{\Lambda}_1^{\frac{1}{2}}(\tilde{Q}^T\bar{Q})_{11}\bar{\Lambda}_1^{-\frac{1}{4}}-2I\right]\bar{\Lambda}_1^{\frac{1}{4}}\Big),\end{aligned}$$

but for any symmetric positive definite matrix A, $A+A^{-1}-2I\succeq 0$ and $A+A^{-1}=2I$ if and only if $A=I$. Therefore $\bar{\Lambda}_1^{\frac{1}{4}}(\tilde{Q}^T\bar{Q})_{11}^T\tilde{\Lambda}_1^{-\frac{1}{2}}(\tilde{Q}^T\bar{Q})_{11}\bar{\Lambda}_1^{\frac{1}{4}}=I$ which leads to $(\tilde{Q}^T\bar{Q})_{11}^T\tilde{\Lambda}_1(\tilde{Q}^T\bar{Q})_{11}=\bar{\Lambda}_1$. Notice $(\tilde{Q}^T\bar{Q})_{12}=0$, $(\tilde{Q}^T\bar{Q})_{21}=0$, $(\tilde{Q}^T\bar{Q})_{11}$ and $(\tilde{Q}^T\bar{Q})_{22}$ are both orthogonal matrices, we have $(\tilde{Q}^T\bar{Q})_{11}^T\tilde{\Lambda}_1(\tilde{Q}^T\bar{Q})_{11}=\bar{\Lambda}_1 \iff \bar{X}=\tilde{X}$.

Now let us prove that $\lim_{t\to+\infty}V_7(t)=0$. For any $T>t_0$ and $X(T)\succ 0$ (guaranteed by Theorems 5.2), we can define

$$V_8(t)=I_4(t,X(t),X(T))-4\text{tr}(X(T)^{\frac{1}{2}}),$$

where $t\in[t_0,+\infty)$. Then we have

$$\frac{V_8(t)}{dt}=f(X(T))-f(X(t)),$$

and

$$\frac{d(V_7(t)-V_8(t))}{dt}=f(\bar{X})-f(X(T))\leq 0.$$

But $V_8(T)=I_4(T,X(T),X(T))-4\text{tr}(X(T)^{\frac{1}{2}})=0$, so we have

$$V_7(T)-V_8(T)=V_7(T)\leq V_7(t_0)-V_8(t_0).$$

Notice $I_4(t_0,X^0,Y)-4\text{tr}(Y^{\frac{1}{2}})$ is continuous with respect to Y at $\bar{X}$, and $\bar{X}\in\Omega^4(X^0)$ is an accumulation point, so for any $\epsilon>0$, we can choose $T>t_0$ such that $V_7(t_0)-$

$V_8(t_0) < \epsilon$. Then we get $V_7(T) < \epsilon$, furthermore $V_7(t)$ is an nonincreasing function in $[t_0, +\infty)$, therefore we have

$$\lim_{t \to +\infty} V_7(t) = 0.$$

Thus the proof is completed. □

For ODE systems (5.1) and (5.2), we cannot prove the convergence for the general convex $f(X)$, however we can prove the convergence in the linear case where $f(X) = \mathrm{tr}(CX)$ and $C \in \mathcal{S}^n$. For linear SDP, ODE systems (5.1) and (5.2) are actually the same, hence we only discuss ODE system (5.1) below. In [20], Goldfarb and Scheinberg used some auxiliary optimization problems and the auxiliary continuous trajectories $y(\mu)$ and $Z(\mu)$ to study the limiting behavior of the infeasible central paths for linear SDP. Here we adopt the same strategy. In order to propose the auxiliary optimization problems, we choose a $y^0 \in \mathbb{R}^m$, and let $Z^0 = C - \sum_{k=1}^m y_k^0 A_k$, $P = t_0 Z^0 - X(t_0)^{-1}$. Then we get the following lemma.

Lemma 5.2. *For any $t \geq t_0$, the following optimization problem*

$$\begin{aligned} \min_{X \in \mathcal{S}^n} \quad & C \bullet X - \tfrac{1}{t}(P \bullet X + \ln\det X) \\ s.t. \quad & A_k \bullet X = b_k, \ k = 1, \ldots, m, \\ & X \succ 0, \end{aligned} \tag{Pt}$$

has a unique optimal solution.

Proof. For $t = t_0$, this is evident since (X^0, y^0, Z^0) satisfies the following KKT system

$$\begin{cases} \sum_{k=1}^m y_k^0 A_k + Z^0 = C, \\ A_k \bullet X^0 = b_k, \ k = 1, \ldots, m, \ X^0 \succ 0, \\ t_0 Z^0 = X(t_0)^{-1} + P. \end{cases} \tag{5.25}$$

Since for $t = t_0$, the objective function is strictly convex, we know the optimal solution set is a single point which must be bounded. Hence from Theorem 24 on page 93 in [18], the level set is bounded as well. For any $t > t_0$, $\alpha > 0$, and $X \in \mathcal{P}_3^{++}$, if

$$C \bullet X - \frac{1}{t}(P \bullet X + \ln\det X) \leq \alpha,$$

then for any optimal solution X^* to problem (P_3),

$$\begin{aligned} & C \bullet X - \frac{1}{t_0}(P \bullet X + \ln\det X) \\ \leq \ & \frac{t}{t_0}\alpha - (\frac{t}{t_0} - 1)C \bullet X \leq \frac{t}{t_0}\alpha - (\frac{t}{t_0} - 1)C \bullet X^* \leq \frac{t}{t_0}(\alpha + |C \bullet X^*|), \end{aligned}$$

which implies X is bounded, hence the level set for problem (Pt) for any given $t > t_0$ is bounded. Since the objective function in problem (Pt) is strictly convex for any $t \geq t_0$, it has a unique optimal solution. □

From the above lemma, we can obtain the auxiliary continuous trajectories $y(t)$ and $Z(t)$, and have the following results.

Theorem 5.18. *There exists two auxiliary continuous trajectories $y(t)$ and $Z(t)$ for $t \geq t_0$ such that $(X(t), y(t), Z(t))$ satisfies the following system*

$$\begin{cases} \sum_{k=1}^{m} y(t)_k A_k + Z(t) = C, \\ A_k \bullet X(t) = b_k, \ \ k = 1, \ldots, m, \ \ X(t) \succ 0, \\ tZ(t) = X(t)^{-1} + P, \end{cases} \tag{5.26}$$

where the unique solution of system (5.1) is $X(t)$.

Proof. From Lemma 5.2, for any $t \geq t_0$, there exists a unique solution for system (5.26). Then we can take derivative with respect to t and get the svec$(\frac{dX}{dt})$ which is actually the right-hand side of ODE system (5.1). Hence the unique solution $X(t)$ of system (5.26) is actually the unique solution of ODE system (5.1). □

From Theorem 5.8, we know $X(t)$ is bounded. Next we show that $y(t)$ and $Z(t)$ are also bounded.

Theorem 5.19. *The auxiliary continuous trajectories $y(t)$ and $Z(t)$ in (5.26) are bounded.*

Proof. From system (5.26), we have

$$\mathrm{tr}(X(t)Z(t)) = \frac{1}{t}\left[n + \mathrm{tr}(X(t)P)\right], \tag{5.27}$$

from Theorem 5.8, $X(t)$ is bounded, we know $\text{tr}(X(t)Z(t)) \to 0$ as $t \to +\infty$. From system (5.26) and Theorem 5.3, we can have

$$\text{tr}(X(t) - X^0)(Z(t) - Z^0) = 0,$$

hence

$$\text{tr}(X^0 Z(t)) = \text{tr}(X(t)Z(t)) + \text{tr}(X^0 Z^0) - \text{tr}(X(t)Z^0),$$

which implies that $\text{tr}(X^0 Z(t))$ is bounded for $t \geq t_0$. Let $X^0 = Q_0 \Lambda_0 Q_0^T$ be the eigenvalue decomposition of X^0, then

$$\text{tr}(X^0 Z(t)) = \text{tr}(\Lambda_0 Q_0^T Z(t) Q_0).$$

From $Z(t) = \frac{1}{t} X(t)^{-1} + \frac{1}{t} P$ in system (5.26) and the Weyl theorem, we have

$$\lambda_{\min}(Q_0^T Z(t) Q_0) = \lambda_{\min}(Z(t)) \geq \frac{1}{t} \lambda_{\min}(P),$$

then from Lemma 5.1, the diagonal entries of $Q_0^T Z(t) Q_0$ must be bounded below by $\frac{1}{t}\lambda_{\min}(P)$. If $\|Z(t)\|_2$ is unbounded, consider $\lambda_{\min}(Z(t)) \geq \frac{1}{t}\lambda_{\min}(P)$, $\lambda_{\max}(Z(t))$ will go to $+\infty$ as $t \to +\infty$, then

$$\text{tr}(Q_0^T Z(t) Q_0) = \text{tr}(Z(t)) \geq \lambda_{\max}(Z(t)) + \frac{n-1}{t} \lambda_{\min}(P) \to +\infty,$$

as $t \to +\infty$, which indicates at least one diagonal entry of $Q_0^T Z(t) Q_0$ will go to $+\infty$ as $t \to +\infty$. But $\Lambda_0 \succ 0$, hence $\text{tr}(X^0 Z(t)) = \text{tr}(\Lambda_0 Q_0^T Z(t) Q_0)$ is unbounded. This is a contradiction, so $Z(t)$ is bounded. From $\sum_{k=1}^m y(t)_k A_k + Z(t) = C$ in system (5.26) and Assumption 5.2, $y(t)$ can be determined by $Z(t)$, so $y(t)$ is also bounded. □

Now we prove the convergence for ODE systems (5.1) and (5.2) in the linear case. In the proof, we use the similar method as Theorem A.3 in [24] where the curve selection lemma (Lemma A.2 in [24])will be used.

Theorem 5.20. *If $f(X) = \text{tr}(CX)$ is linear, where $C \in \mathcal{S}^n$, then each of $\Omega^1(X^0)$ and $\Omega^2(X^0)$ contains a single point, and each of the two limit points is on the optimal face of problem (P_3).*

Proof. Since in the linear SDP, ODE systems (5.1) and (5.2) are the same, we only need to prove ODE system (5.1). From Theorem 5.8 and Theorem 5.19, let $(\tilde{X}, \tilde{y}, \tilde{Z})$ be an accumulation point of the continuous trajectory $(X(t), y(t), Z(t))$. Let $\mu = \frac{1}{t}$ for $t \geq t_0$, and $X(\mu) = X(\frac{1}{t})$, $y(\mu) = y(\frac{1}{t})$, $Z(\mu) = Z(\frac{1}{t})$. Let the real algebraic set V be defined via

$$V = \left\{ (\bar{X}, \bar{Z}, \bar{y}, \mu) \;\middle|\; \begin{array}{l} A_k \bullet \bar{X} = 0 \ (k = 1, \dots, m), \\ \sum_{k=1}^{m} \bar{y}_k A_k + \bar{Z} = 0, \\ (\bar{X} + \tilde{X})(\bar{Z} + \tilde{Z}) - \mu I - \mu(\bar{X} + \tilde{X})P = 0, \end{array} \right\}$$

and let the open set U be defined as the set of all $(\bar{X}, \bar{Z}, \bar{y}, \mu)$ such that all principal minors of $(\bar{X} + \tilde{X})$ are positive and $\mu > 0$.

Now from Lemma 5.2 and Theorem 5.18, we can see that $V \cap U$ corresponds to the continuous trajectory $(X(\mu), y(\mu), Z(\mu))$ excluding its limit points, in the sense that if $(\bar{X}, \bar{Z}, \bar{y}, \mu) \in V \cap U$ then $X(\mu) = \bar{X} + \tilde{X}$ and $Z(\mu) = \bar{Z} + \tilde{Z}$. Moreover, the zero element is in the closure of $V \cap U$, by construction. Then similar to the proof of Theorem A.3 in [24], we can prove that $(\tilde{X}, \tilde{y}, \tilde{Z})$ is the only limit point of the continuous trajectory $(X(t), y(t), Z(t))$.

Without loss of generality, we assume the optimal solution X^* is on the optimal face of problem (P_3), from system (5.26) and Theorem 5.3, we can get

$$\begin{aligned} \mathrm{tr}(X(t)^{-1} X^*) &= n + tr\left[(X(t) - X^*)P\right] - t \cdot tr\left[C(X(t) - X^*)\right] \\ &\leq n + tr\left[(X(t) - X^*)P\right], \end{aligned}$$

which is bounded above, hence similar to the claim in the proof of Theorem 5.16, we can show that the limit point $\tilde{X}$ is on the optimal face of problem (P_3). □

Chapter 6

Primal Affine Scaling Algorithm for Convex Semidefinite Programming

The affine scaling algorithm is one of the earliest interior point methods developed for linear programming. This algorithm is simple and elegant in terms of its geometric interpretation, but it is notoriously difficult to prove its convergence. It often requires additional restrictive conditions such as nondegeneracy, specific initial solutions, and/or small step length to guarantee its global convergence. This situation is made worse when it comes to apply the affine scaling idea to the solution of semidefinite optimization problems or more general convex optimization problems. In [53], Muramatsu presented an example of linear semidefinite programming that the affine scaling algorithm with either short or long step converges to a non-optimal point.

This chapter aims at developing a strategy that guarantees the global convergence for the affine scaling algorithm in the context of linearly constrained convex semidefinite optimization in a least restrictive manner. We propose a new rule of stepsize, which is similar to the Armijo rule, and prove that such an affine scaling algorithm is globally convergent in the sense that each accumulation point of the sequence generated by the algorithm is an optimal solution as long as the optimal solution set is nonempty and bounded. The algorithm is least restrictive in the sense that it allows the problem to be degenerate and it may start from any interior feasible point.

The affine scaling direction $D(X)$ (see [17] or [53]) for problem (P_3) is

$$\text{svec}(D(X)) = -\left(I - (X \circledast X)P_{\mathcal{A}X}\right)(X \circledast X)\text{svec}\left(\frac{\partial f}{\partial X}\right), \tag{6.1}$$

where

$$X \in \mathcal{S}^n_{++}, \quad P_{\mathcal{A}X} = \mathcal{A}^T(\mathcal{A}(X \circledast X)\mathcal{A}^T)^{-1}\mathcal{A},$$
$$I \text{ stands for the } \frac{n(n+1)}{2} \times \frac{n(n+1)}{2} \text{ identity matrix.}$$

6.1 Properties of The Affine Scaling Direction

The following assumptions are made throughout this chapter.

Assumption 6.1. *$\mathcal{P}_3^{++}$ is nonempty.*

Assumption 6.2. *The matrix $\mathcal{A}$ has full row rank m.*

Assumption 6.3. *The optimal solution set for problem (P_3) is nonempty and bounded.*

For linear SDP, the properties of the affine scaling direction have been discussed in [53]. For convex SDP, these properties can be obtained similarly. First we introduce two notations $u(X)$ and $S(X)$ for any $X \in \mathcal{S}^n_{++}$ which are defined as

$$u(X) = (\mathcal{A}(X \circledast X)\mathcal{A}^T)^{-1}\mathcal{A}(X \circledast X)\text{svec}\left(\frac{\partial f}{\partial X}\right) \in R^m, \tag{6.2}$$

and

$$S(X) = \frac{\partial f}{\partial X} - \sum_{i=1}^{m} u_i(X)A_i. \tag{6.3}$$

Note $\mathcal{A}^T u(X) = P_{\mathcal{A}X}(X \circledast X)\text{svec}\left(\frac{\partial f}{\partial X}\right)$, so we have

$$\text{svec}(S(X)) = (I - P_{\mathcal{A}X}(X \circledast X))\text{svec}\left(\frac{\partial f}{\partial X}\right), \tag{6.4}$$

and

$$D(X) = -XS(X)X. \tag{6.5}$$

$(u(X), S(X))$ is called the dual estimate in [53], which is actually the unique solution of the following optimization problem

$$\begin{array}{cl} \min\limits_{S \in \mathcal{S}^n, u \in R^m} & \|X^{\frac{1}{2}} S X^{\frac{1}{2}}\|_F^2 \\ \text{s.t.} & S + \sum\limits_{i=1}^m u_i A_i = \frac{\partial f}{\partial X}, \end{array}$$

where $X^{\frac{1}{2}} \in \mathcal{S}^n_{++}$ is the unique square root matrix of X. Then we have the following theorem.

Theorem 6.1. *We have*

$$A_i \bullet D(X) = 0, \tag{6.6}$$

for all $i = 1, ..., m$, *and*

$$\frac{\partial f}{\partial X} \bullet D(X) = -\|X^{\frac{1}{2}} S(X) X^{\frac{1}{2}}\|_F^2. \tag{6.7}$$

Proof. Notice

$$\begin{aligned} \mathcal{A}\text{svec}(D(X)) &= -\mathcal{A}\left(I - (X \circledast X)P_{\mathcal{A}X}\right)(X \circledast X)\text{svec}\left(\frac{\partial f}{\partial X}\right) \\ &= -\left[\mathcal{A}(X \circledast X) - \mathcal{A}(X \circledast X)P_{\mathcal{A}X}(X \circledast X)\right]\text{svec}\left(\frac{\partial f}{\partial X}\right) \\ &= -\left[\mathcal{A}(X \circledast X) - \mathcal{A}(X \circledast X)\right]\text{svec}\left(\frac{\partial f}{\partial X}\right) \\ &= 0, \end{aligned}$$

hence $A_i \bullet D(X) = 0$ for all $i = 1, ..., m$. Since $\mathcal{P} = I - (X^{\frac{1}{2}} \circledast X^{\frac{1}{2}})P_{\mathcal{A}X}(X^{\frac{1}{2}} \circledast X^{\frac{1}{2}})$ is an idempotent matrix, we have

$$\begin{aligned} \frac{\partial f}{\partial X} \bullet D(X) &= -\text{svec}\left(\frac{\partial f}{\partial X}\right)^T (X^{\frac{1}{2}} \circledast X^{\frac{1}{2}})\mathcal{P}(X^{\frac{1}{2}} \circledast X^{\frac{1}{2}})\text{svec}\left(\frac{\partial f}{\partial X}\right) \\ &= -\|\mathcal{P}(X^{\frac{1}{2}} \circledast X^{\frac{1}{2}})\text{svec}\left(\frac{\partial f}{\partial X}\right)\|^2 \\ &= -\|X^{\frac{1}{2}}\frac{\partial f}{\partial X}X^{\frac{1}{2}} - X^{\frac{1}{2}}\sum_{i=1}^m u_i(X)A_i X^{\frac{1}{2}}\|_F^2 \\ &= -\|X^{\frac{1}{2}} S(X) X^{\frac{1}{2}}\|_F^2. \end{aligned}$$

Thus the proof is completed. ∎ □

6.2 A New Step Size Rule

It has been shown in [53] that the affine scaling algorithm for linear SDP can fail no matter with a long step strategy or a short step strategy by a counterexample. Hence here we try to use a new step size strategy which is similar to the Armijo-type rule [6]. First we state the algorithm.

Algorithm 6.2.1

Step 0: Initialize $X_0 \in \mathcal{P}_3^{++}$, and $k = 0$.

Step 1: Calculate $D(X_k)$ from (6.1) or (6.5).

Step 2: If the norm of $D(X_k)$ is very small, stop; otherwise, calculate the step size α^k from (6.9) and (6.10). Then take

$$X_{k+1} = X_k + \alpha^k D(X_k). \tag{6.8}$$

and $k = k + 1$; go to Step 1.

In order to state our step size strategy, we first introduce some notations. Let

$$\rho(X) = \sup\{\rho > 0 | X + \rho D(X) \succ 0\},$$

for any $X \in \mathcal{S}^n_{++}$ and select a positive sequence $\{a_i\}_{i=0}^{+\infty}$ such that $\lim_{i \to +\infty} a_i = 0$ and $\lim_{s \to +\infty} \sum_{i=0}^{s} a_i = +\infty$. For instance, the sequence can be $\{\frac{1}{(i+1)^\alpha}\}_{i=0}^{+\infty}$ with $0 < \alpha \le 1$ or $\{\frac{1}{\ln(i+2)}\}_{i=0}^{+\infty}$. Then α^k in (6.8) can be defined from the following two steps:

Step 1:

$$\alpha_0^k = \min\left\{\frac{a_k}{\|S_k X_k\| c(\|S_k\|)}, \tau\rho(X_k)\right\} > 0, \tag{6.9}$$

where $S_k = S(X_k)$, $0 < \tau < 1$ is a constant, and $c(x)$ is a scalar function which satisfies $c_1 x \le c(x) \le \max(c_2 x, c_3)$, where $0 < c_1 \le c_2$, $c_3 > 0$ are constants.

Step 2: α^k is the largest $\alpha \in \{\alpha_0^k \beta^l\}_{l=0,1,\ldots}$ satisfying

$$f(X_k + \alpha D(X_k)) \leq f(X_k) + \sigma\alpha G_k \bullet D(X_k), \tag{6.10}$$

where $G_k = \frac{\partial f}{\partial X}|_{X=X_k}$, $0 < \beta,\ \sigma < 1$ are constants.

It should be noticed that in (6.9) S_k should be a nonzero matrix. In fact, if $S_k = 0$, then it is easy to verify that X_k is actually an optimal solution from the KKT condition, and the iteration should stop, hence we will not consider this trivial case for simplicity.

6.3 Optimality of The Affine Scaling Algorithm

In this section, we will show that the affine scaling algorithm with our step size strategy (6.10) will succeed without nondegeneracy assumptions. We begin our discussions with the following lemma.

Lemma 6.1. *The level set $\mathcal{F} = \{X \in \mathcal{P}_3^+ | f(X) \leq f(X_0)\}$ is bounded.*

Proof. Let $\delta_{\mathcal{P}_3^+}(X)$ be the indicator function of $\mathcal{P}_3^+$ which is defined by

$$\delta_{\mathcal{P}_3^+}(X) = \begin{cases} 0 & \text{if } X \in \mathcal{P}_3^+, \\ +\infty & \text{Otherwise.} \end{cases}$$

Then $f(X) + \delta_{\mathcal{P}_3^+}(X)$ is a closed proper convex function, and the optimal solution set of problem (P_3) can be expressed as

$$\{X \in \mathcal{S}^n | f(X) + \delta_{\mathcal{P}_3^+}(X) \leq f^*\},$$

where f^* is the optimal objective value for problem (P_3). According to Assumption 6.3, the optimal solution set of problem (P_3) is nonempty and bounded, hence the Corollary 8.7.1 in [58] implies that

$$\{X \in \mathcal{S}^n | f(X) + \delta_{\mathcal{P}_3^+}(X) \leq f(X_0)\} = \mathcal{F} = \{X \in \mathcal{P}_3^+ | f(X) \leq f(X_0)\}, \tag{6.11}$$

is also bounded. Thus the proof is completed. ▮ □

Theorem 6.2. *Let $\{X_k\}$ be generated by the affine scaling algorithm (6.8) with the step size $\{\alpha^k\}$ chosen by (6.10). Then*

(i) $X_k \in \mathcal{P}_3^{++}$, $\{f(X_k)\}$ is nonincreasing, $\{X_k\}$ and $\{D(X_k)\}$ are bounded;

(ii) every accumulation point of $\{X_k\}$ is an optimal solution for problem (P_3).

Proof. Proof of (i). Since $X_0 \in \mathcal{P}_3^{++}$ and $\alpha^k \leq \tau\rho(X_k)$ at each step, by using an induction argument on k, we have that $X_k \in \mathcal{S}_{++}^n$ for all k. Moreover, from (6.6) in Theorem 6.1 and $X_0 \in \mathcal{P}_3^{++}$, we have $X_k \in \mathcal{P}_3^{++}$ for all k. Also, since $f(X)$ is continuous differentiable and $\alpha_0^k > 0$, $0 < \sigma < 1$ in (6.10), we know $\alpha^k > 0$ for all k.

Combining (6.7) and (6.10), we have for all k

$$f(X_{k+1}) - f(X_k) \leq \sigma\alpha^k G_k \bullet D(X_k) = -\sigma\alpha^k \|X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}\|_F^2 \leq 0, \tag{6.12}$$

thus $\{f(X_k)\}$ is nonincreasing. Then $X_k \in \mathcal{F}$ for all k. Since the level set $\mathcal{F}$ in (6.11) is bounded from Lemma 6.1, we know $\{X_k\}$ is bounded as well. For $D(X_k)$, since

$$\text{svec}(D(X_k)) = (X_k^{\frac{1}{2}} \circledast X_k^{\frac{1}{2}})\mathcal{P}_k(X_k^{\frac{1}{2}} \circledast X_k^{\frac{1}{2}})G_k,$$

where $\mathcal{P}_k = I - (X_k^{\frac{1}{2}} \circledast X_k^{\frac{1}{2}})P_{\mathcal{A}X_k}(X_k^{\frac{1}{2}} \circledast X_k^{\frac{1}{2}})$ is an idempotent matrix which implies $\|\mathcal{P}_k\| \leq 1$ for all k. Then along with the facts that $\{X_k\}$ is bounded and $f(X)$ is continuous differentiable, we have $\{D(X_k)\}$ is also bounded.

Proof of (ii). From (i), we have $\{X_k\}$ is bounded, hence $\{X_k\}$ must have at least one accumulation point. Let $\bar{X}$ be any accumulation point of $\{X_k\}$, we will show it is actually an optimal solution for problem (P_3) by contradiction. Assume $\bar{X}$ is not an optimal solution of problem (P_3).

First, since $X_k \in \mathcal{P}_3^{++}$, we have $\bar{X} \in \mathcal{P}_3^{+}$. From Assumption 6.3, we can choose a point $X^* \in \mathcal{P}_3^{+}$ such that X^* is an optimal solution for problem (P_3). According to the hypothesis and the fact that $\{f(X_k)\}$ is nonincreasing from (i), we have

$$f(X_0) \geq f(\bar{X}) = \lim_{k\to+\infty} f(X_k) > f(X^*).$$

Let us define

$$\bar{Y} = \frac{f(\bar{X}) - f(X^*)}{2(f(X_0) - f(X^*))} X_0 + \left[1 - \frac{f(\bar{X}) - f(X^*)}{2(f(X_0) - f(X^*))}\right] X^*,$$

then $\bar{Y} \in \mathcal{P}_3^{++}$. Since $f(X)$ is convex, we have

$$\begin{aligned} f(\bar{Y}) &\leq \frac{f(\bar{X}) - f(X^*)}{2(f(X_0) - f(X^*))} f(X_0) + \left[1 - \frac{f(\bar{X}) - f(X^*)}{2(f(X_0) - f(X^*))}\right] f(X^*) \\ &= \frac{f(\bar{X}) + f(X^*)}{2}. \end{aligned}$$

Let us define

$$V(X)^1 = \ln\det X + \mathrm{tr}(X^{-1}\bar{Y}),$$

where $X \in \mathcal{S}_{++}^n$. Then at X_k, we can define a scalar function $V_k(\alpha)$ as

$$V_k(\alpha) = V(X_k + \alpha D(X_k)),$$

where $0 \leq \alpha \leq \alpha_0^k$. Obviously, $V_k(0) = V(X_k)$ and $V_k(\alpha^k) = V(X_{k+1})$. Moreover,

$$\begin{aligned} \frac{dV_k(\alpha)}{d\alpha} &= \mathrm{tr}(X(\alpha)^{-1}D(X_k)) - \mathrm{tr}(X(\alpha)^{-1}\bar{Y}X(\alpha)^{-1}D(X_k)) \\ &= \mathrm{tr}\left[(X(\alpha) - \bar{Y})X(\alpha)^{-1}D(X_k)X(\alpha)^{-1}\right] \\ &= \mathrm{tr}\left[(\bar{Y} - X(\alpha))X(\alpha)^{-1}X_kS_kX_kX(\alpha)^{-1}\right], \end{aligned}$$

where $X(\alpha) = X_k + \alpha D(X_k) = X_k - \alpha X_kS_kX_k$. Notice $X_k^{-1}X(\alpha) = I - \alpha S_kX_k$, which implies

$$X(\alpha)^{-1}X_k = I + \alpha S_kX_k(I - \alpha S_kX_k)^{-1}, \tag{6.13}$$

hence

$$\begin{aligned} \frac{dV_k(\alpha)}{d\alpha} &= \mathrm{tr}\left[(\bar{Y} - X(\alpha))X(\alpha)^{-1}X_kS_kX_kX(\alpha)^{-1}\right] \\ &= \mathrm{tr}\left\{(\bar{Y} - X(\alpha))\left[I + \alpha S_kX_kW\right]S_k\left[I + \alpha W^TX_kS_k\right]\right\} \\ &= \mathrm{tr}\left\{(\bar{Y} - X(\alpha))\left[S_k + \alpha S_kX_kWS_k\right]\right\} + \\ &\quad\ \mathrm{tr}\left\{(\bar{Y} - X(\alpha))\left[\alpha S_kW^TX_kS_k + \alpha^2 S_kX_kWS_kW^TX_kS_k\right]\right\}, \end{aligned}$$

[1]The definition of $V(x)$ is inspired by the potential function in Losert and Akin [36].

where $W = X(\alpha)^{-1}X_k = (I - \alpha S_k X_k)^{-1}$. Next we show that when $0 \le \alpha \le \alpha^k$, $\frac{dV_k(\alpha)}{d\alpha}$ is always negative if k is large enough. Since the level set $\mathcal{F}$ in (6.11) is bounded and $f(X)$ is continuous differentiable, there exist $M_1,\ M_2 > 0$ such that $\|X\| \le M_1$, $\|\frac{\partial f}{\partial X}\| \le M_2$ if $X \in \mathcal{F}$. From Theorem 6.1 and Lemma 1.2, we have

$$\begin{aligned} \mathrm{tr}((\bar{Y} - X_k)S_k) &= \mathrm{tr}((\bar{Y} - X_k)G_k) \le f(\bar{Y}) - f(X_k) \\ &\le f(\bar{Y}) - f(\bar{X}) \le \frac{f(X^*) - f(\bar{X})}{2} < 0, \end{aligned}$$

which implies

$$\frac{f(\bar{X}) - f(X^*)}{2} \le \mathrm{tr}((X_k - \bar{Y})S_k) \le \|X_k - \bar{Y}\| \cdot \|S_k\| \le 2M_1\|S_k\|,$$

thus

$$\|S_k\| \ge \frac{f(\bar{X}) - f(X^*)}{4M_1}, \tag{6.14}$$

for all k. Then when $0 \le \alpha \le \alpha^k$ we have

$$\begin{aligned} \mathrm{tr}((\bar{Y} - X(\alpha))S_k) &= \mathrm{tr}\left[(\bar{Y} - X_k + \alpha X_k S_k X_k)G_k\right] \\ &\le \frac{f(X^*) - f(\bar{X})}{2} + \alpha M_1 M_2 \|S_k X_k\| \\ &\le \frac{f(X^*) - f(\bar{X})}{2} + M_1 M_2 \frac{a_k}{c_1\|S_k\|} \\ &\le \frac{f(X^*) - f(\bar{X})}{2} + \frac{4M_1^2 M_2 a_k}{c_1(f(\bar{X}) - f(X^*))}. \end{aligned}$$

From (6.13), if $0 < a_k < \frac{c_1(f(\bar{X}) - f(X^*))}{4M_1}$ and $0 \le \alpha \le \alpha^k$, we have

$$\|W\| \le 1 + \alpha\|S_k X_k\| \cdot \|W\| \le 1 + \frac{4M_1 a_k}{c_1(f(\bar{X}) - f(X^*))}\|W\|,$$

which indicates that

$$\|W^T\| = \|W\| \le \frac{1}{1 - \frac{4M_1 a_k}{c_1(f(\bar{X}) - f(X^*))}}. \tag{6.15}$$

Therefore if $0 < a_k < \frac{c_1(f(\bar{X}) - f(X^*))}{4M_1}$ and $0 \le \alpha \le \alpha^k$, we have

$$\begin{aligned} \frac{dV_k(\alpha)}{d\alpha} \le\ & \frac{f(X^*) - f(\bar{X})}{2} + \frac{4M_1^2 M_2 a_k}{c_1(f(\bar{X}) - f(X^*))} + 4\alpha M_1 \|S_k X_k\| \cdot \|W\| \cdot \|S_k\| \\ & + 2\alpha^2 M_1 \|S_k X_k\|^2 \cdot \|W\|^2 \cdot \|S_k\| \\ \le\ & \frac{f(X^*) - f(\bar{X})}{2} + \frac{4M_1^2 M_2 a_k}{c_1(f(\bar{X}) - f(X^*))} + \frac{4M_1 a_k \|W\|}{c_1} + \frac{2M_1 a_k^2 \|W\|^2}{c_1^2 \|S_k\|}, \end{aligned}$$

then from (6.14), (6.15), and the fact that $\lim_{k\to+\infty} a_k = 0$, we know there exists a $K > 0$ such that for all $k \geq K$, if $0 \leq \alpha \leq \alpha^k$, then $\frac{dV_k(\alpha)}{d\alpha} < 0$. Especially, we have

$$V(X_{k+1}) = V_k(\alpha^k) < V_k(0) = V(X_k),$$

for all $k \geq K$. Hence there exists an $M_3 \in R$ such that $V(X_k) \leq M_3$ for all k. When $X \in \mathcal{S}^n_{++}$, let $X = Q\Lambda Q^T$ be an eigenvalue decomposition of X, and $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of X. Then

$$V(X) = \ln\det X + \operatorname{tr}\big(Q\Lambda^{-1}Q^T\bar{Y}\big) = \sum_{i=1}^n \ln\lambda_i + \operatorname{tr}\big(\Lambda^{-1}Q^T\bar{Y}Q\big),$$

since $\bar{Y} \in \mathcal{P}_3^{++}$, we have

$$\lambda_{\min}(Q^T\bar{Y}Q) = \lambda_{\min}(\bar{Y}) > 0.$$

Therefore from Lemma 5.1, we have

$$\begin{aligned} V(X) &= \sum_{i=1}^n \ln\lambda_i + \operatorname{tr}(\Lambda^{-1}Q^T\bar{Y}Q) \\ &\geq \sum_{i=1}^n \ln\lambda_i + \sum_{i=1}^n {\lambda_i}^{-1}\lambda_{\min}(\bar{Y}) \\ &= \sum_{i=1}^n (\ln\lambda_i + {\lambda_i}^{-1}\lambda_{\min}(\bar{Y})). \end{aligned}$$

For each i, $\ln\lambda_i + {\lambda_i}^{-1}\lambda_{\min}(\bar{Y}) \geq \ln\lambda_{\min}(\bar{Y}) + 1$ and $\lim_{\lambda_i\to 0}[\ln\lambda_i + {\lambda_i}^{-1}\lambda_{\min}(\bar{Y})] = +\infty$. Thus, by $V(X_k) \leq M_3$ for all k, we know there exists an $M_4 > 0$ such that for all k, we have

$$\lambda_{\min}(X_k) \geq M_4 > 0.$$

Let us define

$$\mathcal{H} = \{X \in \mathcal{S}^n_{++} |\ \lambda_{min}(X) \geq M_4\}.$$

Then for all k, $X_k \in \mathcal{H} \cap \mathcal{F}$ which implies $\|X_k^{-\frac{1}{2}}\| \leq \frac{1}{\sqrt{M_4}}$. Along with (6.14), we get for all k,

$$\|X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}\|_F \geq \|X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}\| \geq \frac{\|S_k\|}{\|X_k^{-\frac{1}{2}}\|^2} \geq \frac{M_4(f(\bar{X}) - f(X^*))}{4M_1} > 0. \qquad (6.16)$$

From (6.12), (6.16), and $\lim_{k\to+\infty} f(X_{k+1}) - f(X_k) = 0$, we know $\lim_{k\to+\infty} \alpha^k = 0$. Next we show that the index set

$$\mathcal{I} = \{\ k \mid \alpha^k = \alpha_0^k \beta^l,\ l \geq 1 \text{ in (6.10)}\}$$

is finite. If not, then we can choose a subsequence $\{X_k\}_{k\in\mathcal{K}}$ ($\mathcal{K} \subseteq \{0,1,...\}$) such that $\mathcal{K} \subseteq \mathcal{I}$ and $\lim_{k\in\mathcal{K},\ k\to+\infty} X_k = \tilde{X} \in \mathcal{H} \cap \mathcal{F}$. Then for $k \in \mathcal{K}$, the condition (6.10) is violated by $\alpha = \alpha^k/\beta$, i.e.,

$$\frac{f(X_k + \frac{\alpha^k}{\beta} D(X_k)) - f(X_k)}{\frac{\alpha^k}{\beta}} > \sigma G_k \bullet D(X_k). \tag{6.17}$$

Since $\lim_{k\to+\infty} \alpha^k = 0$ and $f(X)$ is continuous differentiable, from (6.17) we have $\tilde{G} \bullet D(\tilde{X}) \geq \sigma \tilde{G} \bullet D(\tilde{X})$ where $\tilde{G} = \frac{\partial f}{\partial X}|_{X=\tilde{X}}$, which implies

$$-\|\tilde{X}^{\frac{1}{2}} S(\tilde{X}) \tilde{X}^{\frac{1}{2}}\|_F^2 = \tilde{G} \bullet D(\tilde{X}) \geq 0,$$

but this contradicts with (6.16). Thus the index set $\mathcal{I}$ is finite and there must exist an $N_1 > 0$ such that for all $k \geq N_1$, $\alpha^k = \alpha_0^k$ in (6.10). From (6.2), we know $\|u(X)\|$ is a continuous function on $\mathcal{H} \cap \mathcal{F}$ which is closed and bounded, thus $\|u(X)\|$ will be bounded on $\mathcal{H} \cap \mathcal{F}$. Along with (6.3) and (6.14), there exists an $M_5 > 0$ such that for all k, $\|S_k\| \leq M_5$ and

$$\frac{c_1(f(\bar{X}) - f(X^*))}{4M_1} \leq c(\|S_k\|) \leq M_5. \tag{6.18}$$

Since $\lim_{k\to+\infty} a_k = 0$, there exists an $N_2 > 0$ such that for all $k \geq N_2$, we have $a_k < \frac{\tau c_1(f(\bar{X}) - f(X^*))}{4M_1}$. Then for $k \geq N_2$, by using Theorem 1.3.20 in [28] and (6.18), for $\alpha = \frac{a_k}{\tau \|S_k X_k\| c(\|S_k\|)}$, we have

$$\begin{aligned} \|\alpha X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}\| &= \alpha r(X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}) = \alpha r(S_k X_k) \\ &\leq \alpha \|S_k X_k\| = \frac{a_k}{\tau c(\|S_k\|)} \leq \frac{4M_1 a_k}{\tau c_1(f(\bar{X}) - f(X^*))} < 1, \end{aligned}$$

where $r(A)$ denotes the spectral radius of matrix A, this implies

$$X_k + \alpha D(X_k) = X_k^{\frac{1}{2}} (I - \alpha X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}) X_k^{\frac{1}{2}} \in \mathcal{S}_{++}^n,$$

therefore $\alpha \leq \rho(X_k)$ and then $\frac{a_k}{\|S_k X_k\| c(\|S_k\|)} \leq \tau\rho(X_k)$. Thus for all $k \geq N_2$, by (6.18) we have

$$\alpha_0^k = \frac{a_k}{\|S_k X_k\| c(\|S_k\|)} \geq \frac{a_k}{\|S_k\| \cdot \|X_k\| c(\|S_k\|)} \geq \frac{a_k}{M_1 M_5^2}. \tag{6.19}$$

Combining (6.12), (6.16), and (6.19), we know for all $k \geq N_3 = \max(N_1, N_2)$,

$$f(X_{k+1}) - f(X_k) \leq -\sigma\alpha_0^k \frac{M_4^2(f(\bar{X}) - f(X^*))^2}{16M_1^2} \leq -\frac{\sigma M_4^2(f(\bar{X}) - f(X^*))^2}{16M_1^3 M_5^2} a_k,$$

this implies

$$f(\bar{X}) - f(X_{N_3}) \leq -\frac{\sigma M_4^2(f(\bar{X}) - f(X^*))^2}{16M_1^3 M_5^2} \sum_{k \geq N_3}^{+\infty} a_k = -\infty,$$

which is a contradiction. Therefore any accumulation point of $\{X_k\}$ is an optimal solution for problem (P_3). ∎ □

6.4 A Special Case of Problem (P_3)

In this section, we consider a special case of problem (P_3) where X and A_i $(i = 1, ..., m)$ are all diagonal. The results of this section are therefore applicable to the linearly constrained convex programming. We will show that in this special case, the step size can be larger in (6.10) in the sense that the positive sequence $\{a_i\}_{i=0}^{+\infty}$ is not required. If $X = diag(x)$, where $x \in R^n_{++}$, then $S(X)$, $D(X)$, and $\frac{\partial f}{\partial X}$ are all diagonal matrices. Let $A \in R^{m \times n}$ such that $A_{ij} = (A_i)_{jj}$ and $\nabla f \in R^n$ such that $(\nabla f)_i = (\frac{\partial f}{\partial X})_{ii}$. Then from (6.2), $u(X)$ can be denoted as

$$u(X) = (AX^2A^T)^{-1}AX^2\nabla f. \tag{6.20}$$

In this special case, the step size α^k is also chosen by (6.10) but α_0^k is defined as

$$0 < \alpha_0^k = \min\left\{\frac{c_5}{\|S_k X_k\|^{c_4}}, \tau\rho(X_k)\right\}, \tag{6.21}$$

where $0 \leq c_4 < 1$, $c_5 > 0$ are constants.

Theorem 6.3. *Let $\{X_k\}$ be generated by the affine scaling algorithm (6.8) with the step size $\{\alpha^k\}$ chosen by (6.10) and (6.21). Then* *(i)* $X_k \in \mathcal{P}_3^{++}$, $\{f(X_k)\}$ *is nonincreasing,* $\{X_k\}$ *and* $\{D(X_k)\}$ *are bounded;*

(ii) $\lim_{k\to+\infty} \|X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}\|_F = 0$*;*

(iii) every accumulation point of $\{X_k\}$ *is an optimal solution for problem* (P_3)*.*

Proof. Proof of (i). Similar to the proof of (i) in Theorem 6.2.

Proof of (ii). We prove this by contradiction. Assume $\lim_{k\to+\infty} \|X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}\|_F = 0$ is not true, then since $\{X_k\}$, $\{D(X_k)\}$ are both bounded, there must exist a subsequence $\{X_k\}_{k\in\mathcal{K}}$ ($\mathcal{K} \subseteq \{0, 1, ...\}$) and an $\bar{M}_1 > 0$ such that $\lim_{k\in\mathcal{K},\ k\to+\infty} \|X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}\|_F = \bar{M}_1$, $\lim_{k\in\mathcal{K},\ k\to+\infty} X_k = \hat{X}$, and $\lim_{k\in\mathcal{K},\ k\to+\infty} D(X_k) = \hat{D}$.

Then from (6.12) and $\lim_{k\to+\infty} f(X_{k+1}) - f(X_k) = 0$, we know

$$\lim_{k\in\mathcal{K},\ k\to+\infty} \alpha^k = 0. \tag{6.22}$$

From Lemma 3 and the Remark in Sun [66], we know that if $x > 0$ every entry of $(AX^2A^T)^{-1}AX^2$ is bounded, and the bound depends only on A and n, thus from (6.20), we know that $u(X_k)$ is bounded which implies $S(X_k)$ is also bounded. Hence there exists an $\bar{M}_2 > 0$ such that $\|S_k X_k\| \le \bar{M}_2$ for all k. If $\alpha = \frac{1}{2\bar{M}_2}$, then

$$\|\alpha X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}\| = \alpha \|S_k X_k\| < 1,$$

which implies

$$X_k + \alpha D(X_k) = X_k^{\frac{1}{2}}(I - \alpha X_k^{\frac{1}{2}} S_k X_k^{\frac{1}{2}}) X_k^{\frac{1}{2}} \in \mathcal{S}_{++}^n,$$

therefore $\rho(X_k) \ge \frac{1}{2\bar{M}_2}$ for all k. Let $\bar{M}_3 = \min(\frac{c_5}{\bar{M}_2^{c_4}}, \tau\frac{1}{2\bar{M}_2})$. Then from (6.21), we have $\alpha_0^k \ge \bar{M}_3 > 0$. Hence from (6.22), we know for all $k \in \mathcal{K}$ sufficiently large, $\alpha^k < \alpha_0^k$, which implies that condition (6.10) is violated by $\alpha = \alpha^k/\beta$, i.e.,

$$\frac{f(X_k + \frac{\alpha^k}{\beta} D(X_k)) - f(X_k)}{\frac{\alpha^k}{\beta}} > \sigma G_k \bullet D(X_k). \tag{6.23}$$

Since $\lim_{k\in\mathcal{K},\ k\to+\infty} \alpha^k = 0$ and $f(X)$ is continuous differentiable, from (6.23) we have $\hat{G} \bullet \hat{D} \ge \sigma \hat{G} \bullet \hat{D}$ where $\hat{G} = \frac{\partial f}{\partial X}|_{X=\hat{X}}$, which implies

$$\hat{G} \bullet \hat{D} = 0,$$

but this contradicts with $\lim\limits_{k\in\mathcal{K},\ k\to+\infty}\|X_k^{\frac{1}{2}}S_kX_k^{\frac{1}{2}}\|_F=\bar{M}_1>0$. Hence the hypothesis is not true, and $\lim\limits_{k\to+\infty}\|X_k^{\frac{1}{2}}S_kX_k^{\frac{1}{2}}\|_F=0$.

Proof of (iii). Similar to Theorem 6.2, we also prove this by contradiction. From (ii), we have

$$\begin{aligned}0\le\liminf_{k\to+\infty}\alpha^k\|S_kX_k\| &\le \limsup_{k\to+\infty}\alpha^k\|S_kX_k\| \\ &\le \limsup_{k\to+\infty}\alpha_0^k\|S_kX_k\|\le\limsup_{k\to+\infty}c_5\|S_kX_k\|^{1-c_4}=0,\end{aligned}$$

which implies $\lim\limits_{k\to+\infty}\alpha^k\|S_kX_k\|=0$. Moreover, $\|S_k\|$ is also bounded, hence similar to the proof of Theorem 6.2, we can also get (6.16) ($M_1,M_4>0$ may be different), but this contradicts with (ii). Hence every accumulation point of $\{X_k\}$ is an optimal solution for problem (P_3). $\blacksquare$ $\square$

Chapter 7

Summary

7.1 Conclusion

In the thesis, we mainly study some interior point continuous trajectories for convex programming and convex semidefinite programming. These interior point continuous trajectories are all characterized as the solution curves of corresponding ODE systems. We study linearly constrained convex programming in two forms, i.e., problem (P_1) and problem (P_2). For problem (P_1), we study two primal interior point continuous trajectories, and the optimality and convergence of both trajectories are obtained under some mild conditions. In addition, for the examples in [19], the central path fails to converge, but our generalized central path which is the solution of ODE system (2.2) do converge. For problem (P_2), we study the first-order affine scaling continuous trajectory and two primal-dual interior point continuous trajectories. For the first-order affine scaling trajectory, the optimality can be obtained under some mild conditions, but to guarantee the convergence, we need the additional condition that the objective function is analytic. The two primal-dual interior point continuous trajectories we studied for problem (P_2) are called the weighted primal-dual path-following continuous trajectory and the extended primal-dual affine scaling continuous trajectory respectively. For the two primal-dual interior point continuous trajectories, the optimality can be obtained under some mild conditions. For the primal continuous trajectory of each primal-dual interior point continuous trajectories, the convergence can be obtained in different cases where some additional conditions, for example the strictly complementarity condition or analyticity of the objective function, may ne needed. However, for the dual continuous trajectory, the convergence can be obtained without any additional conditions.

For the linearly constrained convex semidefinite programming, we study four primal interior point continuous trajectories and a primal affine scaling algorithm. For the four primal interior point continuous trajectories, the optimality can be obtained under some mild conditions. For the first trajectory of the four primal interior point continuous trajectories, if the maximal rank of the optimal solution set is equal to one, we can get the convergence. For the second trajectory of the four primal interior point continuous trajectories, the convergence can be guaranteed without any additional conditions. For the last two trajectories of the four primal interior point continuous trajectories, we can obtain the convergence in the linear case. It was showed in [53] that the affine scaling algorithm for linear SDP may converge to a non-optimal solution with either short or long step. We propose a new step size rule such that every accumulation point of the affine scaling algorithm is actually an optimal solution for convex SDP.

7.2 Future Research

In the thesis, we mainly study the interior point continuous trajectories which are characterized by corresponding ODE systems for convex programming and convex semidefinite programming. For the discrete algorithm, we only study an affine scaling algorithm for convex SDP in Chapter 6. The implementation of the ODE systems could be the future research. The explicit Euler scheme is the simplest implementation of the ODE systems. For ODE system (2.1) in Chapter 2, the explicit Euler scheme have been studied in [73] for problem (P_1) with $s = n$. But for problem (P_1) with $s < n$, the explicit Euler scheme of ODE system (2.1) has not been studied. For ODE system (2.2), we have proposed Algorithm 2.4.1, but the optimality and convergence remains to be studied. For ODE system (3.1), the explicit Euler scheme yields the first-order primal affine scaling algorithm which was studied in [22] and [73], but the convergence of the affine scaling algorithm for the convex case remains open. For ODE systems (4.2) and (4.14), the explicit Euler schemes were only studied for the special case where $\gamma_1 = \gamma_2 = 1$ as we know, and the study of the discrete algorithms from ODE systems (4.2) and (4.14) for the general case could be the future research. For the explicit Euler schemes of the four ODE systems (5.1), (5.2), (5.3), and (5.4), only the explicit Euler scheme of ODE system (5.1) which yields the affine scaling algorithm 6.2.1 was studied in Chapter 6, and the convergence of Algorithm 6.2.1 remains open. The implementations of ODE system (5.2), (5.3), and (5.4) could be the future research.

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