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Stationarity Tests for Spatial Point Processes using Discrepancies

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Summary: For testing stationarity of a given spatial point pattern, Guan (2008) proposed a model-free statistic, based on the deviations between observed and expected counts of points in expanding regions within the sampling window. This paper extends his method to a general class of statistics by incorporating also such information when points are projected to the axes and by allowing different ways to construct regions in which the deviations are considered. The limiting distributions of the new statistics can be expressed in terms of integrals of a Brownian sheet and hence asymptotic critical values can be approximated. A simulation study shows that the new tests are always more powerful than that of Guan. When applied to the longleaf pine data where Guan's test gave an inconclusive answer, the new tests indicate a clear rejection of the stationarity hypothesis.

Key words: Discrepancy; Longleaf pine data; Spatial point process; Stationarity test.

1. Introduction

The point pattern of the locations of 584 longleaf pine trees in a 200m $\times$ 200m window of an old-growth forest in Thomas County, Georgia in 1979, is shown in Figure **??**(a). On one hand, the kernel smoothed intensity estimate shown in Figure **??**(b) reveals a trend of increasing intensity from the lower half to the upper half and hence suggests non-stationarity (Cressie, 1993, p. 600). On the other hand, some stationary models for cluster processes offer good fits to these data (Stoyan and Stoyan, 1996; Ghorbani, 2013). Guan (2008) proposed a test of stationarity yielding a p-value of 0.0524 (estimated by our simulation) and he inclined to reject the stationarity hypothesis, but the value 0.0524 is probably not small enough to be conclusive.

[Figure 1 about here.]

Although modern statistical methods for spatial point process data (see e.g. Møller and Waagepetersen, 2007) allow inference for non-stationary models, whether a given point pattern is stationary is still a question that has to be answered before any statistical analysis is done.

If a spatial Poisson process is assumed, then many statistics are available to test the equality among point densities in two or more *a priori* chosen non-overlapping regions (Illian et al., 2008, p. 82, Chiu and Wang, 2009, and Chiu, 2010). However, a model-free stationarity test, surprisingly, had not been available until Guan (2008), in which he generalized the KPSS (Kwiatkowski–Phillips–Schmidt–Shin) test for time series data to spatial point patterns and derived the asymptotic null distribution for the new test statistic. He used models with exponential intensity functions to show by simulation that his proposed test for stationarity is reasonably powerful.

However, as can be seen in later sections, if the intensity function is not monotonically increasing but, for example, has a peak at the middle of the sampling window, Guan's test does not work well. This paper aims to extend Guan's test to a wider class of stationarity tests so that more appropriate (more powerful) test statistics can be chosen from the class, when some prior knowledge of the nature of the possible deviation from stationarity is known or some kind of deviation is of particular interest. Using this strategy, we apply our new test to the longleaf pine data and obtain a p-value of 0.0120, indicating a clear rejection of the stationarity hypothesis.

2. Guan's Test

Denote by Φ a random set of points, or a spatial point process, in $\mathbb{R}^2$, and each member of Φ is called an event. Let $\lambda(x_1, x_2)$ denote the intensity function of Φ at location (x_1, x_2), meaning that $\mathbf{E}\{\#(\Phi \cap A)\} = \int\int_A \lambda(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$ for any Borel set $A \subset \mathbb{R}^2$. A process is stationary if its distribution is invariant under translation. Consequently, the intensity function of a stationary point process is a constant.

Suppose a realization of Φ is observed over the rectangular sampling window $[0, n_1] \times [0, n_2]$, where $n_i = c_i n$ for some positive c_i and n, and $i = 1, 2$. Let $\Phi_n = \Phi \cap [0, n_1] \times [0, n_2]$ denote the set of sampled events. Consider the hypotheses

$$H_0 : \lambda(x_1, x_2) = \lambda,$$

$$H_\mathrm{A} : \lambda(x_1, x_2) \neq \lambda$$

for some positive constant λ.

Let $N(u_1, u_2) = \#(\Phi_n \cap [0, u_1] \times [0, u_2])$, where $0 < u_i \leqslant n_i$ and $i = 1, 2$. Under the null hypothesis of constant intensity, an unbiased estimator for λ is $\hat{\lambda}_n = N(n_1, n_2)/(n_1 n_2)$. Imitating the KPSS test for the stationarity of time series data, Guan (2008) proposed the following test statistic for stationarity of Φ:

$$T_n = \frac{1}{n_1^2 n_2^2 \sigma^2}\, K^2, \tag{1}$$

where

$$K^2 = \int_0^{n_1} \int_0^{n_2} \left\{ N(u_1, u_2) - u_1 u_2 \hat{\lambda}_n \right\}^2 \mathrm{d}u_1 \mathrm{d}u_2, \tag{2}$$

and $\sigma^2 = \mathrm{var}\{N(n_1, n_2)\}/(n_1 n_2)$. For large n, the asymptotic value of σ^2 can be expressed (see Illian et al., 2008, pp. 225–226) as:

$$\sigma^2 \simeq \lambda^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{(g(l_1, l_2) - 1\} \, \mathrm{d}l_1 \, \mathrm{d}l_2 + \lambda,$$

where $g(l_1, l_2)$ is the pair correlation function at lag (l_1, l_2). A probabilistic interpretation of g is that for any two infinitesimally small discs of area $\mathrm{d}x_1$ and $\mathrm{d}x_2$ separated by a lag (l_1, l_2), the probability that both discs contain events is $\lambda^2 g(l_1, l_2) \mathrm{d}x_1 \mathrm{d}x_2$. To estimate σ^2 for a particular sample of Φ_n, Guan (2008) used

$$\hat{\sigma}_n^2 = \sum_{\substack{(x_1 - x_1')^2 + (x_2 - x_2')^2 \leqslant m_n^2 \\ (x_1, x_2) \neq (x_1', x_2')}} \frac{1}{(n_1 - |x_1 - x_1'|)(n_2 - |x_2 - x_2'|)} - \hat{\lambda}_n^2 \pi m_n^2 + \hat{\lambda}_n \tag{3}$$

where the summation is over all distinct pairs of events in Φ_n and m_n is the bandwidth parameter, and suggested choosing m_n to be the smallest lag distance starting from which the empirical pair correlation function becomes flat and close to one.

Guan (2008) showed that for any strongly mixing stationary process satisfying some regularity conditions, under the null hypothesis,

$$T_n \xrightarrow{\mathrm{d}} \zeta = \int_0^1 \int_0^1 \{W(t_1, t_2) - t_1 t_2 W(1, 1)\}^2 \, \mathrm{d}t_1 \mathrm{d}t_2, \qquad \text{as } n \to \infty, \tag{4}$$

where $W(t_1, t_2)$ is the two-dimensional Brownian sheet. The convergence is independent of the type of point process and hence a model-free test is obtained.

Note that any one of the four corners of the rectangular sampling window could be the origin $(0, 0)$ in the integral in (2) and the numerical value of the integral is not rotationally invariant. To achieve rotation-invariance, Guan (2008) proposed to use the sum of the four integrals, each of which takes a different corner as the origin, i.e.

$$T_n^* = \frac{1}{n_1^2 n_2^2 \sigma^2} \sum_{i=1}^{4} \int_0^{n_1} \int_0^{n_2} \left\{ N_i(u_1, u_2) - u_1 u_2 \hat{\lambda}_n \right\}^2 \mathrm{d}u_1 \mathrm{d}u_2, \tag{5}$$

where $N_i(u_1, u_2)$ is the number of events in $[0, u_1] \times [0, u_2]$ when the window is rotated by

$(i-1)\pi/2$, so that ith corner is taken as the origin $(0,0)$, and then derived that

$$\begin{aligned}
T_n^* \xrightarrow{\mathrm{d}} \zeta^* = & \int_0^1 \int_0^1 \{W(t_1,t_2) - t_1 t_2 W(1,1)\}^2 \,\mathrm{d}t_1 \mathrm{d}t_2 \\
& + \int_0^1 \int_0^1 \{W(t_1,1) - W(t_1, 1-t_2) - t_1 t_2 W(1,1)\}^2 \,\mathrm{d}t_1 \mathrm{d}t_2 \\
& + \int_0^1 \int_0^1 \{W(1,t_2) - W(1-t_1, t_2) - t_1 t_2 W(1,1)\}^2 \,\mathrm{d}t_1 \mathrm{d}t_2 \\
& + \int_0^1 \int_0^1 \{W(1,1) - W(1-t_1,1) - W(1,1-t_2) \\
& \qquad + W(1-t_1,1-t_2) - t_1 t_2 W(1,1)\}^2 \mathrm{d}t_1 \mathrm{d}t_2, \qquad \text{as } n \to \infty. \qquad (6)
\end{aligned}$$

His simulation results showed the "four corner" statistic T_n^* always had higher power than the "one corner" version T_n. Hence, allowing different corners to be the origin is not a cosmetic change but is indeed an important consideration. For the rest of the paper, a location playing the role of the origin is called an anchor and taking a particular position as an anchor is called anchoring at that position.

Consider again the statistic T_n in (1), which is anchored at the lower left corner of the sampling window. If the intensity is monotonic, such as the exponential function (20) considered in the simulation of Guan (2008), anchoring at the lower left corner will lead to a high value for T_n, because the integrand is expected to increase monotonically when (u_1, u_2) is moving away from the anchor. However, it is not the case when the intensity function has a peak at the middle of the sampling window, such as a Gaussian type function, as considered below in Section 6.2.

As an illustration, consider Figure **??**, which shows three simulated realizations of the Matérn first type hard core process (more details will be given in Section 6) in a 10×10 square (a) with a constant intensity, (b) thinned to yield an exponential and (c) thinned to yield a Gaussian type intensity. Figure **??**(d) plots the value of T_n, calculated based on the sub-square $[0,x]^2$ with $m_n = 0.4$, against the distance $\|(x,x)\| = \sqrt{2}x$, and the horizontal line represents the asymptotic critical value at the 0.05 nominal level. Ignoring the instability caused by the estimation of σ for small x, we can see that

(i) for the constant case, the values of T_n are fluctuating well below the critical value;

(ii) for the exponential case, T_n increases as x increases;

(iii) for the Gaussian case with the peak at the middle part $x_1 = 5$ of the square, because the intensity function drops when $x_1 > 5$, unlike in the exponential case, the increase in the discrepancy is decelerating after $x = 5$ but the normalizing term is ever decreasing at the constant rate $O(x^{-2})$. Thus, when x increases (in this sample when $x > 7$, i.e. $\|(x, x)\| > 10$), the increase in discrepancy cannot catch up with the decrease in the normalizing term, leading to a decrease in T_n, whose final value is less than the critical value; moreover, because of symmetry, anchoring at all four corners does not help.

[Figure 2 about here.]

In view of the empirical intensity shown in Figure **??**(b), observation (iii) above provides a plausible explanation why Guan's test failed to reject the null hypothesis of constant intensity for the longleaf pine data.

3. Discrepancy

The integral K^2 in (2) is very similar to (the square of) the discrepancy, denoted by D^2, of a point set in a unit square, defined in Warnock (1972) as

$$D^2 = \int_0^1 \int_0^1 \left\{ \frac{N(u_1, u_2)}{N} - u_1 u_2 \right\}^2 \mathrm{d}u_1 \mathrm{d}u_2, \tag{7}$$

where N is the total number of events in the unit square. This serves as a measure of the uniformity of the points, which is a problem of great importance in the literature of quasi-Monte Carlo methods (see e.g. Niederreiter, 1992).

However, there are two differences between K^2 and D^2:

(i) D^2 is defined for a point set in a unit square but not a rectangle;

(ii) the integrand of D^2, which is called the local discrepancy, measures the difference between the proportion of the events in the sub-rectangle $[0, u_1] \times [0, u_2]$ and the corresponding area of it.

Nevertheless, one can transform the point set $\Phi_n = \{(x_{11}, x_{12}),\ \ldots,\ (x_{N1}, x_{N2})\}$ in the rectangle $[0, n_1] \times [0, n_2]$ to the point set

$$\Phi'_n = \{(x_{11}/n_1, x_{12}/n_2),\ \ldots,\ (x_{N1}/n_1, x_{N2}/n_2)\} \tag{8}$$

in the unit square, and then denote by $K^2(\Phi_n)$ the K^2-value of Φ_n and by $D^2(\Phi'_n)$ the D^2-value of Φ'_n, so that they are simply related by

$$K^2(\Phi_n) = n_1\, n_2\, N^2\, D^2(\Phi'_n),$$

where in this case $N = N(n_1, n_2)$. Hence, the test statistic in (1) can be expressed in terms of the discrepancy of the transformed point set Φ'_n

$$T_n = \frac{N^2}{n_1 n_2 \sigma^2} \int_0^1 \int_0^1 \left\{ \frac{N(n_1 u_1, n_2 u_2)}{N} - u_1 u_2 \right\}^2 \mathrm{d}u_1 \mathrm{d}u_2 = \frac{N^2}{n_1 n_2 \sigma^2}\ D^2(\Phi'_n). \tag{9}$$

Expressing T_n in terms of the discrepancy is more appealing since the exact computational formulas of (7) and its generalizations can be derived. Some illustrating examples will be given in Section 4.

The weakness of anchoring at a corner or at all four corners has long been explored in the quasi-Monte Carlo methods literature, and Hickernell (1998a,b, 1999a,b) proposed several better anchoring schemes, giving rise to various generalized discrepancies. These discrepancies have been applied in testing complete spatial randomness of spatial point patterns (Zimmermann, 1993, Ho and Chiu, 2007, and Ong et al., 2012) and testing goodness-of-fit for multivariate distributions (Liang et al., 2000, and Chiu and Liu, 2009). In these two applications it was found that statistics based on generalized discrepancies have notably higher power than those based on Warnock's discrepancy.

In addition to different anchoring schemes, another feature of the generalized discrepancies in the quasi-Monte Carlo methods literature is that the uniformity of the points projected to the axes is also measured. In the spatial point process context, we introduce

$$T_n^{(1)} = \frac{N^2}{n_1 n_2 \sigma^2} \int_0^1 \left\{ \frac{N(n_1 u_1, n_2)}{N} - u_1 \right\}^2 \mathrm{d}u_1, \tag{10}$$

$$T_n^{(2)} = \frac{N^2}{n_1 n_2 \sigma^2} \int_0^1 \left\{ \frac{N(n_1, n_2 u_2)}{N} - u_2 \right\}^2 \mathrm{d}u_2, \tag{11}$$

and following Kwiatkowski et al. (1992) and Guan (2008), we can show that under the null hypothesis of constant intensity,

$$\left(T_n, T_n^{(1)}, T_n^{(2)}\right) \xrightarrow{\mathrm{d}} \left(\zeta, \zeta^{(1)}, \zeta^{(2)}\right), \qquad \text{as } n \to \infty, \tag{12}$$

where ζ is given in (4) and

$$\zeta^{(i)} = \int_0^1 \left\{ B_i(t) - tB(1) \right\}^2 \mathrm{d}t, \tag{13}$$

in which

$$B_1(t) = \frac{1}{\sqrt{3}} \int_0^1 W(t_1, t)\,\mathrm{d}t_1 \qquad \text{and} \qquad B_2(t) = \frac{1}{\sqrt{3}} \int_0^1 W(t, t_2)\,\mathrm{d}t_2$$

are the two one-dimensional Brownian motions obtained by projecting the Brownian sheet $W(t_1, t_2)$ in the integrand of ζ to the two axes. The two statistics in (10) and (11) can be re-expressed as

$$T_n^{(i)} = \frac{N^2}{n_1 n_2 \sigma^2} D^2({\Phi'}_n^{(i)}), \qquad i = 1,\ 2, \tag{14}$$

where $D^2({\Phi'}_n^{(i)})$ is the one-dimensional Warnock's discrepancy of the set

$${\Phi'}_n^{(i)} = \{x_{1i}/n_i,\ \ldots,\ x_{Ni}/n_i\} \tag{15}$$

of transformed points projected to the ith axis. The asymptotic distributions remain valid when σ^2 is replaced by $\hat{\sigma}_n^2$ in (3).

4. Generalized Test Statistics for Stationarity

Given a two-dimensional finite point pattern Φ_n, we need discrepancies of both the two-dimensional set of original points and the one-dimensional sets of points projected to the axes. Fortunately, the definitions and the exact computational formulas for the generalized discrepancies can be expressed for a general dimension. We consider $[0,1]^s$ (in our context $s = 1$ or 2) and denote the s-dimensional transformed points by

$$\boldsymbol{y}_p = (y_{p1},\ \ldots,\ y_{ps}) = (x_{p1}/n_1,\ \ldots,\ x_{ps}/n_s)$$

for $p = 1, \ldots, N$. Let $N(\boldsymbol{u})$ be the number of events (or projected events if $s = 1$) in $[\boldsymbol{0}, \boldsymbol{u}]$, where $\boldsymbol{u} = (u_1,\ \ldots,\ u_s)$.

The generalized discrepancies to be introduced can be summarized in the following expression:

$$D^2_{\text{generalized}} = \int_0^1 \cdots \int_0^1 \left\{ \frac{N(J(\boldsymbol{a},\boldsymbol{u}))}{N} - |J(\boldsymbol{a},\boldsymbol{u})| \right\}^2 \mathrm{d}u_1 \cdots \mathrm{d}u_s\, \mathrm{d}a_1 \cdots \mathrm{d}a_s, \tag{16}$$

where $\boldsymbol{a} = (a_1,\ \ldots,\ a_s)$. The set $J(\boldsymbol{a},\boldsymbol{u})$ defines the region to be considered in the local discrepancy, $N(J(\boldsymbol{a},\boldsymbol{u}))$ is the number of events inside $J(\boldsymbol{a},\boldsymbol{u})$, and $|\cdot|$ is the volume measure. When $J(\boldsymbol{a},\boldsymbol{u})$ depends solely on $\boldsymbol{u}$, then the discrepancies are anchored at some fixed positions; the introduction of the extra variable $\boldsymbol{a}$ allows the discrepancies to have variable anchors, or, in a certain sense, be unanchored.

Replacing the discrepancy D^2 by $D^2_{\text{generalized}}$ in the test statistics (9) and (14) leads to generalized test statistics, denoted by $T_n^{\text{generalized}}$ and $T_n^{\text{generalized},(i)}$:

$$T_n^{\text{generalized}} = \frac{N^2}{n_1 n_2 \sigma^2}\, D^2_{\text{generalized}}(\Phi'_n), \tag{17}$$

$$T_n^{\text{generalized},(i)} = \frac{N^2}{n_1 n_2 \sigma^2} D^2_{\text{generalized}}(\Phi'^{(i)}_n), \qquad i = 1,\ 2, \tag{18}$$

where Φ'_n and $\Phi'^{(i)}_n$ are given in (8) and (15), respectively.

Denoted the limit by

$$(T_n^{\text{generalized}}, T_n^{\text{generalized},(1)}, T_n^{\text{generalized},(2)}) \xrightarrow{\mathrm{d}} (\zeta^{\text{generalized}}, \zeta^{\text{generalized},(1)}, \zeta^{\text{generalized},(2)}), \tag{19}$$

where the limiting distribution of $(\zeta^{\text{generalized}}, \zeta^{\text{generalized},(1)}, \zeta^{\text{generalized},(2)})$ will be given for each particular discrepancy below, and also the subscript and superscript "generalized" will be replaced by its own name, except Warnock's discrepancy, for which the subscript and superscript will be omitted. (Note that in the quasi-Monte Carlo methods literature, the discrepancy of Φ'_n usually refers to the sum $D^2_{\text{generalized}}(\Phi'_n)+D^2_{\text{generalized}}(\Phi'^{(1)}_n)+D^2_{\text{generalized}}(\Phi'^{(2)}_n)$, but in this paper each term stands alone individually.)

In theory, any measurable region can be used as $J(\boldsymbol{a},\boldsymbol{u})$, but in practice, $J(\boldsymbol{a},\boldsymbol{u})$ should be chosen such that one can derive

(i) a closed-form expression of the corresponding test statistic, and

(ii) a computationally tractable limiting distribution of the corresponding test statistic.

The following particular choices of $J(\boldsymbol{a},\boldsymbol{u})$ fulfil this criterion for strongly mixing stationary spatial point processes satisfying some mild regularity conditions stated in Web Appendix A, and the derivations of their limiting distributions are given in Web Appendix B.

4.1 *Warnock's discrepancy*

Consider

$$J(\boldsymbol{a},\boldsymbol{u}) = \prod_{k=1}^{s} [0, u_k].$$

Then $D^2_{\text{generalized}}$ is just D^2, and a simple computational formula to calculate D^2 exactly has been given in Warnock (1972). In order to illustrate the method for the other discrepancies, the procedure is described briefly as follows.

The local discrepancy, denoted by $q(\boldsymbol{u})$, can be expressed by using the Heaviside function $H(z) = 1$ for $z \geqslant 0$ and $H(z) = 0$ for $z < 0$ as

$$q(\boldsymbol{u}) = \frac{N(\boldsymbol{u})}{N} - \prod_{i=1}^{s} u_i = \frac{1}{N}\sum_{p=1}^{N}\prod_{i=1}^{s} H(u_i - y_{pi}) - \prod_{i=1}^{s} u_i,$$

and hence

$$
\begin{aligned}
D^2 &= \int_0^1 \cdots \int_0^1 q(\boldsymbol{u})^2 \,\mathrm{d}u_1 \cdots \mathrm{d}u_s \\
&= \int_0^1 \cdots \int_0^1 \left\{ \frac{1}{N^2} \sum_{p=1}^{N} \sum_{q=1}^{N} \prod_{i=1}^{s} H(u_i - y_{pi}) H(u_i - y_{qi}) \right. \\
&\qquad \left. - \frac{2}{N} \sum_{p=1}^{N} \prod_{i=1}^{s} u_i H(u_i - y_{pi}) + \prod_{i=1}^{s} u_i^2 \right\} \mathrm{d}u_1 \cdots \mathrm{d}u_s \\
&= \frac{1}{N^2} \sum_{p=1}^{N} \sum_{q=1}^{N} \prod_{i=1}^{s} \{1 - \max(y_{pi}, y_{qi})\} - \frac{1}{2^{s-1}N} \sum_{p=1}^{N} \prod_{i=1}^{s} (1 - y_{pi}^2) + \frac{1}{3^s}.
\end{aligned}
$$

4.2 *Centred discrepancy*

Instead of taking the sum of the discrepancies anchored at each of the four corners, the centred discrepancy D^2_{centred} forces its local discrepancy to anchor at the nearest corner so that the discrepancy value will also be invariant under rotation by a multiple of $\pi/2$ and reflection of the point set. Thus, all four corners serve as anchors, but for each (u_1, u_2) we will anchor at one and only one corner. To achieve such an anchoring scheme, let

$$
J(a_k, u_k) = \begin{cases} [0, u_k] & \text{if } u_k \leqslant 0.5, \\[2ex] [u_k, 1] & \text{if } u_k > 0.5, \end{cases}
$$

$$
J(\boldsymbol{a}, \boldsymbol{u}) = \prod_{k=1}^{s} J(a_k, u_k).
$$

Its local discrepancy, when expressed in terms of the Heaviside function, is

$$
\begin{aligned}
q_{\text{centred}}(\boldsymbol{u}) =& \frac{1}{N} \sum_{p=1}^{N} \prod_{i=1}^{s} \{H(u_i - 0.5)\, H(y_{pi} - u_i) + H(0.5 - u_i)\, H(u_i - y_{pi})\} \\
& - \prod_{i=1}^{s} \{u_i\, H(0.5 - u_i) + (1 - u_i)\, H(u_i - 0.5)\},
\end{aligned}
$$

which leads to the following exact formula

$$
\begin{aligned}
D^2_{\text{centred}} =& \frac{1}{2^s N^2} \sum_{p=1}^{N} \sum_{q=1}^{N} \prod_{i=1}^{s} (|y_{pi} - 0.5| + |y_{qi} - 0.5| - |y_{pi} - y_{qi}|) \\
& - \frac{1}{2^{s-1}N} \sum_{p=1}^{N} \prod_{i=1}^{s} \{|y_{pi} - 0.5| - (y_{pi} - 0.5)^2\} + \frac{1}{12^s}.
\end{aligned}
$$

The test statistics T_n^{centred} and $T_n^{\text{centred},(i)}$, respectively, converge in distribution to ζ^{centred} and $\zeta^{\text{centred},(i)}$, where

$$\begin{aligned}
\zeta^{\text{centred}} = & \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \{W(t_1,t_2) - t_1 t_2 W(1,1)\}^2 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
& + \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \{W(t_1,1) - W(t_1,1-t_2) - t_1 t_2 W(1,1)\}^2 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
& + \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \{W(1,t_2) - W(1-t_1,t_2) - t_1 t_2 W(1,1)\}^2 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
& + \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \{W(1,1) - W(1-t_1,1) - W(1,1-t_2) \\
& \qquad + W(1-t_1,1-t_2) - t_1 t_2 W(1,1)\}^2 \,\mathrm{d}t_1\,\mathrm{d}t_2,
\end{aligned}$$

and $\zeta^{\text{centred},(i)}$ has the same expression as $\zeta^{(i)}$ given in (13).

4.3 *Symmetric discrepancy*

Instead of counting the number of events in a rectangle, when $s = 2$, the symmetric discrepancy counts the number in two rectangles, by anchoring at $(0,0)$ and $(1,1)$ at the same time. The corresponding $J(\boldsymbol{a},\boldsymbol{u})$ of symmetric discrepancy is

$$J(\boldsymbol{a},\boldsymbol{u}) = ([0,u_1]\times[0,u_2]) \cup ([u_1,1]\times[u_2,1]).$$

When $s = 1$, the symmetric discrepancy anchors just at 0 and hence is the same as Warnock's.

Its local discrepancy, in terms of the Heaviside function, is

$$q_{\text{symmetric}}(\boldsymbol{u}) = \frac{1}{2}\left[1 + \frac{1}{N}\sum_{p=1}^{N}\prod_{i=1}^{s}\{H(u_i - y_{pi})\,H(y_{pi} - u_i)\}\right] - \frac{1}{2}\left\{1 + \prod_{i=1}^{s}(2u_i - 1)\right\},$$

which leads to

$$D^2_{\text{symmetric}} = \frac{1}{4}\left\{\frac{1}{N^2}\sum_{p=1}^{N}\sum_{q=1}^{N}\prod_{i=1}^{s}(1 - 2|y_{pi} - y_{qi}|) - \frac{2^{s+1}}{N}\sum_{p=1}^{N}\prod_{i=1}^{s} y_{pi}(1-y_{pi}) + \frac{1}{3^s}\right\},$$

and $T_n^{\text{symmetric}}$ converges in distribution to

$$\begin{aligned}
\zeta^{\text{symmetric}} = & \int_0^1 \int_0^1 [2W(t_1,t_2) - W(t_1,1) - W(1,t_2) \\
& \qquad + \{1 - t_1 t_2 - (1-t_1)(1-t_2)\}\, W(1,1)]^2 \,\mathrm{d}t_1\,\mathrm{d}t_2.
\end{aligned}$$

4.4 *Unanchored discrepancy*

The above three anchoring schemes have fixed anchors. To unanchor a discrepancy, we first anchor at $\boldsymbol{a}$ and then let it move around. More precisely, consider

$$J(a_k, u_k) = \begin{cases} [a_k, u_k] & \text{if } a_k \leqslant u_k, \\ \\ [u_k, u_k] & \text{if } a_k > u_k, \end{cases}$$

$$J(\boldsymbol{a}, \boldsymbol{u}) = \prod_{k=1}^{s} J(a_k, u_k).$$

Since the anchor point becomes a variable, the definition of the local discrepancy for the unanchored discrepancy has two arguments, i.e.

$$q_{\text{unanchored}}(\boldsymbol{a}, \boldsymbol{u}) = \frac{1}{N} \sum_{p=1}^{N} \prod_{i=1}^{s} H(u_i - y_{pi})\, H(y_{pi} - a_i) - \prod_{i=1}^{s} (u_i - a_i)\, H(u_i - a_i),$$

which leads to

$$\begin{aligned} D^2_{\text{unanchored}} =& \frac{1}{N^2} \sum_{p=1}^{N} \sum_{q=1}^{N} \prod_{i=1}^{s} \{1 - \max(y_{pi}, y_{qi})\}\{\min(y_{pi}, y_{qi})\} \\ & - \frac{1}{2^{s-1} N} \sum_{p=1}^{N} \prod_{i=1}^{s} y_{pi}(1 - y_{pi}) + \frac{1}{12^s}, \end{aligned}$$

and $T_n^{\text{unanchored}}$ and $T_n^{\text{unanchored},(i)}$, respectively, converge in distribution to

$$\begin{aligned} \zeta^{\text{unanchored}} =& \int_0^1 \int_0^1 \int_{a_1}^1 \int_{a_2}^1 \{W(t_1, t_2) - W(a_1, t_2) - W(t_1, a_2) + W(a_1, a_2) \\ & -(t_1 - a_1)(t_2 - a_2) W(1,1)\}^2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \, \mathrm{d}a_1 \, \mathrm{d}a_2, \end{aligned}$$

and

$$\zeta^{\text{unanchored},(i)} = \int_0^1 \int_a^1 \{B(t) - B(a) - (t - a) B(1)\}^2 \, \mathrm{d}t \, \mathrm{d}a.$$

4.5 *Wraparound discrepancy*

Wraparound discrepancy is similar to unanchored discrepancy, but the rectangular interval is defined in the wraparound sense, meaning that the unit square is considered as a torus.

Thus,

$$J(a_k, u_k) = \begin{cases} [a_k, u_k] & \text{if} \quad a_k \leqslant u_k, \\ \\ [0, u_k] \cup [a_k, 1) & \text{if} \quad a_k > u_k, \end{cases}$$

$$J(\boldsymbol{a}, \boldsymbol{u}) = \prod_{k=1}^{s} J(a_k, u_k).$$

Its local discrepancy is

$$\begin{aligned} q_{\text{wraparound}}(\boldsymbol{a}, \boldsymbol{u}) =& \frac{1}{N} \sum_{p=1}^{N} \prod_{i=1}^{s} \left[H(u_i - y_{pi})\, H(y_{pi} - a_i) + H(a_i - u_i) \left\{ H(y_i - a_i) + H(u_i - y_i) \right\} \right] \\ & - \prod_{i=1}^{s} \left\{ (u_i - a_i)\, H(u_i - a_i) + H(a_i - u_i)(1 - a_i + u_i) \right\}, \end{aligned}$$

leading to

$$D^2_{\text{wraparound}} = \frac{1}{2^s N^2} \sum_{p=1}^{N} \sum_{q=1}^{N} \prod_{i=1}^{s} \left\{ 1 + 2(y_{pi} - y_{qi})^2 - 2|y_{pi} - y_{qi}| \right\} - \frac{1}{3^s}.$$

Finally, $T_n^{\text{wraparound}}$ and $T_n^{\text{wraparound},(i)}$, respectively, converge in distribution to

$$\begin{aligned} \zeta^{\text{wraparound}} = & \int_0^1 \int_0^1 \Bigg(\int_{a_1}^1 \int_{a_2}^1 \{ W(t_1, t_2) - W(a_1, t_2) - W(t_1, a_2) + W(a_1, a_2) \\ & \qquad -(t_1 - a_1)(t_2 - a_2) W(1,1) \}^2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \; + \\ & \int_0^{a_1} \int_{a_2}^1 \{ W(1, t_2) - W(1, a_2) - W(a_1, t_2) + W(a_1, a_2) + W(t_1, t_2) \\ & \qquad -W(t_1, a_2) - (1 - a_1 + t_1)(t_2 - a_2) W(1,1) \}^2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \; + \\ & \int_{a_1}^1 \int_0^{a_2} \{ W(t_1, 1) - W(t_1, a_2) - W(a_1, 1) + W(a_1, a_2) + W(t_1, t_2) \\ & \qquad -W(a_1, t_2) - (t_1 - a_1)(1 - a_2 + t_2) W(1,1) \}^2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \; + \\ & \int_0^{a_1} \int_0^{a_2} [W(1,1) - W(a_1, 1) - W(1, a_2) + W(a_1, a_2) + W(t_1, t_2) \\ & \qquad -\{ (1 - a_1)(1 - a_2) + t_1 t_2 \} W(1,1)]^2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \Bigg) \, \mathrm{d}a_1 \, \mathrm{d}a_2, \end{aligned}$$

and

$$\zeta^{\text{wraparound},(i)} = \int_0^1 \left[\int_a^1 \{B(t) - B(a) - (t-a)B(1)\}^2 \, \mathrm{d}t + \int_0^a \{B(1) - B(a) + B(t) - (1-a+t)B(1)\}^2 \, \mathrm{d}t \right] \mathrm{d}a.$$

5. Critical values and p-values

When taking the bivariate $(T_n^{\text{generalized}}, T_n^{\text{generalized},(i)})$ or the trivariate $(T_n^{\text{generalized}}, T_n^{\text{generalized},(1)}, T_n^{\text{generalized},(2)})$ as the joint test statistic, for the ease of calculation we restrict the acceptance region to be rectangular in such a way that the marginal type I error rate by using any single statistic are the same. The p-value is determined by the maximal rectangular acceptance region that does not contain in its interior the observed value of the test statistic.

Given a nominal level, when estimating the critical values for the bivariate and trivariate statistics under this rectangular region restriction by simulation, we need a technical twist: Since $T_n^{\text{generalized},(i)}$, $i = 1, 2$, have the same marginal distribution, their corresponding critical values should be the same; however, almost surely in a finite number of simulations, the estimates of these two critical values of them would not be the same, because the empirical distributions of $\zeta^{\text{generalized},(1)}$ and $\zeta^{\text{generalized},(2)}$ would not be identical. This paper proposes to take the average of the two estimates obtained from simulation as the common critical value for them.

For the same reason, when estimating p-values, we would obtain two slightly different estimates when $T_n^{\text{generalized},(i)}$ is compared to the empirical distributions of $\zeta^{\text{generalized},(1)}$ and $\zeta^{\text{generalized},(2)}$ respectively. Again, we take the average of the p-values from these two distributions as the estimated p-value.

Note that the p-value approach and the critical value approach do not always agree. It may happen that a realization with p-value less than the nominal level is lying in the acceptance region.

For ease of implementation, the simulation study in the next section adopts the critical value approach, and the critical values at the 0.01, 0.05 and 0.1 nominal levels, obtained by 10,000 simulations, are tabulated in Table **??**.

[Table 1 about here.]

6. Simulation

Let the sampling window be $[0, n]^2$. Two models were considered in Guan (2008), namely the Neyman–Scott cluster process and the Matérn first type hard core process. To simulate the Neyman–Scott process, centring at each point of a stationary Poisson process of intensity 0.25, we generate an independent Poisson number of daughter points following a symmetric Gaussian distribution with standard deviation σ_0; the daughter points then form a stationary clustered pattern. For the Matérn process, a stationary Poisson process is first generated, and then pairs with inter-point distance less than 0.2 are removed; the surviving points form a stationary regular pattern, see Figure **??**(a). Both models satisfy the conditions for the convergence of the null distributions of the test statistics.

To introduce non-stationarity to the patterns, points are then further thinned according to a density function. Guan (2008) used only the exponential density

$$\lambda(x_1, x_2) = \exp(\beta x_1/n), \tag{20}$$

and the present paper includes also the Gaussian type density

$$\lambda(x_1, x_2) = \exp\left\{-\frac{2\beta^2}{n^2}\left(x_1 - \frac{n}{2}\right)^2\right\}. \tag{21}$$

Figures **??**(b) and (c) show realizations of the Matérn process thinned via the two different intensity functions with $\beta = 1$ in $[0, 10]^2$.

In the simulation below, the mean of the Poisson number of daughter points in the Neyman–Scott process and the intensity of the stationary Poisson process in the Matérn process will be chosen so that the expected number of remaining points after thinning is about n^2.

6.1 *Exponential density*

The same parameter values and choices of bandwidth as those in Guan (2008) are used in our simulation study (but we do not report the cases with the largest bandwidth as the conclusions remain the same). The rejection rates in 1,000 realizations are tabulated in Table **??**, from which we can observe the following:

(i) When $\beta = 0$, the intensity is a constant and the rejection rates are close to the nominal level.

(ii) The powers of T_n^* and $T_n^{\text{symmetric}}$ are often very close, which was to be expected since these two anchoring schemes are very similar.

(iii) Including the discrepancies of the projected points on the x_1-axis in the test statistics improves the power, sometimes quite substantially, of all tests.

(iv) Though T_n is less powerful than T_n^*, T_n^{centred} and $T_n^{\text{symmetric}}$, when the discrepancies on the x_1-axis are included, all these four anchoring schemes that have fixed anchors lead to bivariate statistics that are equally powerful (and more powerful than any univariate statistic alone).

(v) The four statistics based on fixed anchors perform better than the two based on variable anchors. This is understandable because fixed anchors can well capture the momentum of the increase in discrepancy caused by the monotonicity of the exponential intensity function, see Figure **??**(d), whilst a variable anchor flattens the increase in discrepancy by averaging. Figure **??**(e) shows how $T_n^{\text{unanchored}}$ changes when the window is expanding; the corresponding curves are equally flat in the constant intensity case and the exponential intensity case.

[Table 2 about here.]

6.2 *Gaussian type intensity*

The same parameter values and choices of bandwidth as those in the exponential case are used and the rejection rates in 1,000 realizations are tabulated in Table **??**, from which we can observe the following:

(i) As in the exponential intensity case, when $\beta = 0$, the rejection rates are close to the nominal level.

(ii) The test statistics based on fixed anchors perform much worse than those based on variable anchors. An explanation for the poor performance of the former has already been provided in observation (iii) of Section 2, and it can also explain why the unanchored and the wraparound statistics succeed here: as the four corners are poor places for anchoring, averaging the discrepancy by moving the anchors around leads to larger values for $T_n^{\text{unanchored}}$ and $T_n^{\text{wraparound}}$.

(iii) Including the discrepancies on the x_1-axis in the test statistics helps.

It may be worth mentioning that for the non-stationary Neyman–Scott cluster processes with small n and large σ_0 considered, some of the tests have very small or even zero power. This suggests that the distributions of the test statistics for some non-stationary models can have upper tails that are even lighter than their null distributions. Thus, we agree with Illian et al. (2008, p. 38) that "[s]tatistical tests can assess only some aspects of stationarity but never all. ... [A]ccepting the stationary hypothesis for a given point pattern based on some test is only a necessary condition. ...[I]t is very helpful to justify stationarity based on non-statistical scientific arguments."

[Table 3 about here.]

6.3 *Some other types of density*

Some other types of density, such as double exponential, linear, a mixture of two Gaussian and Gaussian centred not at the centre of the window, have also been considered and the results are reported in Web Appendix C, from which we can see that

(i) monotonic densities such as double exponential and linear lead to conclusions similar to those of the exponential density in Section 6.1;

(ii) the remaining, non-monotonic densities give similar conclusions to those of the Gaussian type intensity in Section 6.2.

7. Longleaf pine data

For the longleaf pine data introduced in Section 1, with $m_n = 20$m (see Guan, 2008, for an explanation of this choice), Guan's four corner T_n^* gave a p-value of 0.0524, estimated from 10,000 simulated Brownian sheets. Since the intensity seems quite varying on the x_2-axis, we included $T_n^{*,(2)}$ in the test statistic and obtained a p-value of 0.0202. All other bivariate statistics based on fixed anchors also gave p-values around 0.02. As we can see from Figure **??**(b), the empirical intensity is not monotonic. Thus, the bivariate statistics based on the unanchored discrepancy and wraparound discrepancy are expected to be more powerful and hence should be used, and the p-values are 0.0128 and 0.0120, respectively, indicating strong evidence against the null hypothesis of constant intensity.

8. Discussions

A class of statistics, extending the work in Guan (2008), for testing stationarity of a given spatial point pattern has been proposed. This class generalizes Guan's statistic in two aspects: (i) a bivariate or trivariate statistic incorporating a measure of stationarity of the projected points on one axis or both axes is used, and simulation shows that most of the time this extra information can improve the power; (ii) a variety of anchoring schemes are available and more appropriate statistics can be chosen from this class when prior knowledge of the nature of non-stationarity is known. From our simulations we suggest that when a monotonic intensity is suspected, then we choose test statistics based on fixed anchors, otherwise we choose test statistics based on variable anchors. Although we do not have a uniformly best anchoring scheme to construct statistics, the tests are model-free, not only in the sense that the (asymptotic) critical region requires solely the assumption of weak dependence, but also in the sense that the appropriate choice of the anchoring scheme depends only on the nature of non-stationarity and does not depend on the type of the process.

9. Supplementary Materials

The longleaf pine data analyzed in Sections 1 and 7 and the MATLAB codes implementing the new test statistics and simulation study, as well as the Web Appendices, referenced in Sections 4 and 6.3, are available with this paper at the Biometrics website on Wiley Online Library.

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