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DETERMINING SCATTERING SUPPORT OF ANISOTROPIC ACOUSTIC MEDIUMS AND OBSTACLES

HONGYU LIU, HONGKAI ZHAO, AND CHANGJIAN ZOU

Abstract. We consider the inverse acoustic scattering problem for a complex scatterer, which might be composed of both inhomogeneous anisotropic mediums and impenetrable obstacles. It is shown that the fixed-frequency scattering amplitude uniquely determines the scattering support of a complex scatterer, disregarding its contents.

1. Introduction

This paper is concerned with the inverse problem of determining the scattering support of a complex scatterer, possibly consisting of inhomogenous mediums and impenetrable obstacles, by the acoustic far-field measurements. More precisely, let Ω and D be open bounded domains in $\mathbb{R}^n$ of C^2 class, $n \geq 2$, such that both $\Omega_e := \mathbb{R}^n \backslash \overline{\Omega}$ and $D_e := \mathbb{R}^n \backslash \overline{D}$ are connected. In the physical situation, D denotes an impenetrable obstacle, and Ω denotes the inhomogeneity support of an acoustic medium. The medium in $\mathbb{R}^n$ is described by a pair of material parameters $(\gamma(x), \eta(x))$, $x \in \mathbb{R}^n$. $\gamma = (\gamma^{ij})$ on $\mathbb{R}^n$ is a symmetric positive definite matrix-valued function, and $\eta(x)$ is a bounded complex-valued scalar function such that $\Im\eta \geq 0$. γ^{-1} denotes the acoustic density tensor, and η is the acoustic modulus. It is noted that the medium could be absorbing if $\Im\eta \neq 0$. It is always assumed that the inhomogeneity of the medium is compactly supported in the sense that $\gamma^{ij} = \delta_{ij}$ and $\eta = 1$ in Ω_e, where δ_{ij} is the Kronecker delta function. The wave propagation in the presence the obstacle and the inhomogeneous medium introduced above in the whole space $\mathbb{R}^n$ is governed by the following wave equation,

$$\eta U_{tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\gamma^{ij} \frac{\partial U}{\partial x^j} \right) = 0 \qquad \text{for } (x,t) \in \mathbb{R}^n \backslash \overline{D} \times \mathbb{R}_+ \tag{1.1}$$

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where $U(x,t)$ represents the wave field. We shall consider our study in the time-harmonic regime, namely, $U(x,t) = u(x)e^{-ikt}$, where $k \in \mathbb{R}_+$ represents the wave number. By factorizing out the time-dependent part, the wave pressure $u(x)$ satisfies the reduced wave equation,

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\gamma^{ij}\frac{\partial u}{\partial x^j}\right) + k^2 \eta u = 0 \quad \text{in } \mathbb{R}^n \backslash \overline{D}. \tag{1.2}$$

We shall seek scattering solution $u \in H^1_{loc}(\mathbb{R}^n)$ to (1.2) which admits the following asymptotic expansion as $|x| \to +\infty$,

$$u(x) = e^{ik\theta'\cdot x} + a_k\left(\frac{x}{|x|}, \theta'\right)\frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} + \mathcal{O}\left(\frac{1}{|x|^{\frac{n+1}{2}}}\right). \tag{1.3}$$

In (1.3), $a_k(\theta,\theta')$, $\theta = \frac{x}{|x|} \in \mathbb{S}^{n-1}$ and $\theta' \in \mathbb{S}^{n-1}$, is known as the *scattering amplitude*. The inverse problem we want to address is to recover the inhomogeneity of (γ,η) and D from $a_k(\theta,\theta')$ for a fixed $k \in \mathbb{R}_+$ and all $(\theta,\theta') \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. It is noted that $a_k(\theta,\theta')$ is (real) analytic in the variables θ and θ', and hence if it is known on any open subset of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, then it is known on the whole spheres by analytic continuation.

This physical problem could be geometrically reformulated as follows. Let $g = (g_{ij})$ be a Riemannian metric on $\mathbb{R}^n$ such that $g = g_e$ in Ω_e, where $g_e = (\delta_{ij})$ denotes the Euclidean metric. If $g = \alpha g_e$ with α a scalar function, then it is called isotropic, otherwise it is called anisotropic. Let Δ_g denote the Laplace-Beltrami operator associated to the metric g,

$$\Delta_g u = \sum_{i,j=1}^{n} (\det\, g_{lm})^{-1/2}\frac{\partial}{\partial x^i}\left((\det\, g_{lm})^{1/2}\, g^{ij}\frac{\partial u}{\partial x^j}\right),$$

where $(g^{ij}) = (g_{ij})^{-1}$. Let q be a bounded complex-valued scalar function such that $\Im q \geq 0$ and $q = 1$ in Ω_e. The time-harmonic wave scattering is governed by the Helmholtz equation whose weak solution is $u = u(x; k\theta')$,

$$\begin{cases} \Delta_g u + k^2 q u = 0 & \text{in } \mathbb{R}^n \backslash \overline{D}, \\ \mathcal{B}(u(x;k\theta')) = 0 & \text{on } \partial D, \\ u(x;k\theta') - e^{ikx\cdot\theta'} \text{ satisfies the radiation condition.} & \end{cases} \tag{1.4}$$

In (1.4), $\mathcal{B}u = u$ if D is a *sound-soft* obstacle; $\mathcal{B}u = \partial u/\partial \nu_g$ if D is a *sound-hard* obstacle; and $\mathcal{B}u = \partial u/\partial \nu_g + i\lambda u$ with λ a positive constant, if D is an *impedance obstacle*. Here and in the following,

$$\frac{\partial u}{\partial \nu_g}(x) := \sum_{l,m} g^{lm} \frac{\partial u}{\partial x^l} \nu_m(x) \left(\sum_{p,r} g^{pr}(x) \nu_p(x) \nu_r(x) \right)^{-\frac{1}{2}},$$

where $\nu = (\nu_l)_{l=1}^n$ is the outward unit Euclidean normal vector to ∂D. The last statement in (1.4) means that if one lets $u^s(x; k\theta') = u(x; k\theta') - e^{ikx\cdot\theta'}$, then

$$\lim_{|x|\to+\infty} |x|^{\frac{n-1}{2}} \left(\frac{\partial u^s}{\partial |x|}(x; k\theta') - iku^s(x; k\theta') \right) = 0, \tag{1.5}$$

which holds uniformly for every direction $\theta = x/|x| \in \mathbb{S}^{n-1}$. (1.5) is known as the *Sommerfeld radiation condition*, which guarantees the same asymptotic expansion (1.3) for u to (1.4). We refer to [23, 28, 30] for the unique existence of an $H^1_{loc}(D_e)$-solution to the PDE system (1.4). (1.4) describes the acoustic scattering from an inhomogeneous medium $(\Omega \cap D_e; \gamma^{ij}, \eta) := (\Omega \cap D_e; (\det\, g_{lm})^{1/2} g^{ij}, (\det\, g_{lm})^{1/2} q)$ together with an impenetrable obstacle D. In the following, we shall denote by $M := \Omega \cap D_e$ the support of the inhomogeneity of the medium and write $(M; g, q) \oplus D$ to denote a *complex scatterer* as described above. In the current study, we shall consider the inverse scattering problem of recovering a complex scatterer by its scattering amplitude at a fixed frequency k. It is noted that a complex scatterer could be very general, and it may happen that $\Omega \cap D = \emptyset$ meaning the medium component and the obstacle component are separated; or it may happen that $D \Subset \Omega$ meaning the obstacle component is embedded into the medium component; or it may happen that $M \cap D = \emptyset$ and $\overline{M} \cap \overline{D} \neq \emptyset$ meaning the medium component and obstacle component are attached to each other; or it may even happen that $D = \emptyset$ or $M = \emptyset$ meaning there is medium component only or obstacle component only for the underlying scattering object. The generality of a complex scatterer is of practical interest if little *a priori* information is available for the underlying scattering object. Before we proceed to discuss the uniqueness results obtained for the present study, some general remarks about related studies in the literature are in order.

Due to the transformation invariance of the wave equation, there are obstructions in determining a complex scatterer (cf. [9, 41]). We first consider the case with $D = \emptyset$. Let $\psi : \overline{M} \to \overline{M}$ be a diffeomorphism which is the identity on ∂M. Then one has that $a_k(M; g, q) = a_k(M; \psi^* g, \psi^* q)$. As usual $\psi^* g$ denotes the pull back of the metric g by the diffeomorphism ψ. Hence, the best one can expect is to recover $(M; g, q)$ in a gauge equivalence class up to a diffeomrophism.

There are extensive studies in the literature on the unique determination of q by knowing g; see, [39, 38] for the Euclidean metric case, [29] for the case when the metric is conformal to the Euclidean metric, [6] for the general smooth metric case. Very little is known in recovering both g and q in a gauge equivalence class by the fixed frequency scattering amplitude. Nevertheless, we would like to mention that there are extensive studies on the inverse problem of determining g associated with Δ_g only, i.e., without the low order perturbation term q; see [19, 20, 37]. If there is an obstacle presented, i.e. $D \neq \emptyset$, most of the studies are concerned with the practically interesting case that the obstacle is embedded into the medium, namely, $D \Subset \Omega$. If $g = g_e$ and $q = 1$, the unique determination of D is established in [5, 17]; and if $g = g_e$ and q is a known function, the uniqueness is established in [18]; and for general known smooth g and q, the uniqueness is established in [36] in two dimensions, and in [33] in arbitrary dimensions. In all the aforementioned studies with $D \neq \emptyset$, the uniqueness results are established in determining D with the surrounding medium known in advance. In two dimensions, the uniqueness is established in [13] in determining an unknown q in $\Omega\backslash\overline{D}$ ($g = g_e$ is known) provided D is known to be a sound-soft obstacle. If $g = g_e$ and q is assumed to a constant (unknown), the uniqueness is established in [26] in determining an unknown D. Very little is known in determining both unknown $(\Omega\backslash\overline{D}; g, q)$ and D by the fixed frequency scattering amplitude. For this inverse problem, we note that if $\psi : \overline{\Omega}\backslash\widetilde{D} \to \overline{\Omega}\backslash D$ is a diffeomorphism which is the identity on $\partial\Omega$, then one has $a_k(g, q, D) = a_k(\psi^* g, \psi^* q, \widetilde{D})$, and this is known as the virtual reshaping effect in [22]. Due to the gauge equivalence obstruction, the uniqueness established for an anisotropic medium up to a diffeomorphism could not be applied directly to the determination of the unknown in practice. Hence, there are a lot of studies on the qualitative determinations, namely, if g or q possesses singularities across a certain interface, one intends to locate such an interface. Particularly, the case with significant practical interests is that the interface is the one delimiting the inhomogeneity of a scatterer and the homogeneous background space. The corresponding inverse problem is also known as determining the scattering support. Uniqueness in determining the scattering support for an isotropic medium could be found in [14, 16, 18, 35], and for an anisotropic medium could be found in [11, 31, 34] by assuming that the metric g has discontinuity or non-smooth singularity. We refer to the [2, 4, 9, 15, 25, 41] for comprehensive surveys.

In this paper, we shall establish the unique determination of the scattering support of a complex scatterer as described earlier, disregarding its contents, which is obviously of significant practical importance. There are two particular aspects of our study which are new to the literature. First, we consider the case that the metric g is smooth, whereas q has discontinuity. It turns out that the singularity in the lower term q is subtler to recover than the singularity in the metric term g. Indeed, we also consider the relatively simpler case of recovering singularity in the metric term for isotropic mediums. Second, our uniqueness arguments are reconstructive, that is, certain numerical reconstruction procedures could be directly adapted from the proofs. The technique in proving our uniqueness is based on the integral equation method and by making use of the singular point source with 2nd order singularity.

Finally, we would like to briefly mention some closely related research on invisibility cloaking, which is also one of the motivations of the current study. By taking advantage of the uniqueness obstruction due to a gauge transformation, one can devise some anisotropic medium which produces zero scattering amplitude, thus invisible to detections. That is, there exists $(M; g, q)$, such that $a_k(M; g, q) = 0$; see, e.g., [8] for the 3D case and [27] for the 2D case. However, the material parameters are singular, and there are extensive studies on achieving non-singular approximate invisibility cloaking by implementing regularized materials $(M; g_\varepsilon, q_\varepsilon)$, see, e.g., [1, 7, 10, 21, 24]. However, we note that the aforementioned non-singular approximate cloaking materials parameters all possess non-smooth singularities across the outer boundary of the cloaking devices. Hence, our current uniqueness study indicate that it might be of practical importance to smooth out the cloaking material parameters at the outer boundary of a cloaking design.

2. Some results on the direct scattering problem

In this section, we present some results on the direct scattering problem (1.4), especially, the integral representation of the wave field. In order to ease our analysis and exposition, we shall only consider the case that the obstacle D is sound-soft, namely, $\mathcal{B}u = u$ in (1.4). Moreover, throughout the rest of the paper, we shall assume that $n = 3$. But we emphasize that all the obtained results can be shown to hold equally for the case with more general obstacle components, as well as dimensions $n = 2$, by directly modifying our subsequent arguments.

Let

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x, y \in \mathbb{R}^3,\ x \neq y,$$

be the fundamental solution to the differential operator $-\Delta - k^2$ in the free space $\mathbb{R}^3$. Next, we give the Green's function for the anisotropic PDO $\mathcal{L} := -\Delta_g - k^2$, where $k \in \mathbb{R}_+$ and, g is smooth in $\mathbb{R}^3$ with $g = g_e$ in Ω_e (cf. [32]).

Lemma 2.1. *There exists $\Phi_g(x, y, k)$ satisfying*

$$(-\Delta_g - k^2)\Phi_g(x, y, k) = \delta(x - y), \quad x, y \in \mathbb{R}^3,$$

$$\lim_{|x|\to+\infty} |x| \left(\frac{\partial}{\partial |x|} \Phi_g(x, y, k) - ik\Phi_g(x, y, k) \right) = 0.$$

Moreover, $\Phi_g(x, y)$ has a singularity of order 1 at $x = y$ and in a small open neighborhood of y,

$$\Phi_g(x, y) \sim C_1(x)d_g(x, y)^{-1} + C_2(x)d_g(x, y) + \cdots, \tag{2.1}$$

and

$$\nabla_x \Phi_g(x, y) \sim C_1'(x)d_g(x, y)^{-2}V_{x,y} + C_2'(x)d_g(x, y)^{-1}V_{x,y} + \cdots, \tag{2.2}$$

where $\sim$ indicates equality modulo C^∞, $C_j(x)$ and $C_j'(x)$ are smooth for all $j \in \mathbb{N}$, and $C_1(x), C_1'(x)$ are not vanishing. In (2.1) and (2.2), $d_g(x, y)$ is the Riemannian distance function and $V(x, y)$ is the unit vector at y in the direction of the geodesic from x to y.

We introduce the single- and double-layer potential operators, defined as follows,

$$SL_g\psi(x) := \int_{\partial D} \Phi_g(x, y)\psi(y)\, ds_y, \qquad x \in \mathbb{R}^3\backslash\partial D, \tag{2.3}$$

$$DL_g\psi(x) := \int_{\partial D} \frac{\partial \Phi_g(x, y)}{\partial \nu_g(y)} \psi(y)\, ds_y, \qquad x \in \mathbb{R}^3\backslash\partial D, \tag{2.4}$$

where ν_g is the outward unit normal vector to ∂D under the metric g. We also introduce the following volume potential operator,

$$V_{g,q}\psi(x) := k^2 \int_M \Phi_g(x, y)(1 - q(y))\psi(y)dy. \tag{2.5}$$

We refer to [3, 12, 28, 32, 40] for the mapping and jumping properties of the layer potential operators introduced above. For the existence of solutions to the direct scattering problem, we have the following theorem whose proof follows from a similar argument to that for Theorem 2.2 in [18]. For the readers' convenience, we include it in the following.

Theorem 2.2. *Suppose that g is smooth in $\mathbb{R}^3$ with $g = g_e$ in Ω_e, and $q \in L^\infty(D_e)$ with $q = 1$ in Ω_e. Then $u(x) \in H^1_{loc}(\mathbb{R}^3\backslash\overline{D}) \cap C(\mathbb{R}^3\backslash D)$ is a solution of (1.4) if $u(x)|_{\overline{M}}$ is given by*

$$u(x) = r(x) - V_{g,q}u(x) + (DL_g + ikSL_g)\psi(x) \quad x \in \overline{M}. \tag{2.6}$$

In (2.6), $\psi(x) \in C(\partial D)$ *satisfies*

$$\psi(x) = t(x) + 2\mathcal{T}V_{g,q}u(x) - 2(\mathcal{T}DL_g + ik\mathcal{T}SL_g)\psi(x) \quad x \in \partial D. \tag{2.7}$$

In (2.6) *and* (2.7), $r(x) = -\int_M (u^i(y)\Delta_g\Phi_g(x,y) - \Delta_g u^i(y)\Phi_g(x,y))dy$ *with* $u^i(x) = e^{ikx\cdot\theta'}$ *and* $t(x) = -2\mathcal{T}r(x)$, *where* $\mathcal{T}$ *is the one-sided trace operator when one approaches* ∂D *from* D_e. *Moreover, we have*

i) *The system (2.6)-(2.7) of integral equations is uniquely solvable in* $C(\overline{M}) \times C(\partial D)$ *for* $(r,t) \in C(\overline{M}) \times C(\partial D)$ *and depends continuously on* r *and* t.
ii) *The system (2.6)-(2.7) of integral equations is uniquely solvable in* $L^2(M) \times C(\partial D)$ *for* $(r,t) \in L^2(M) \times C(\partial D)$ *and depends continuously on* r *and* t.

Proof. First, by Lemma 2.4 of [33], we know that the system (1.4) admits at most one solution. Let $(u,\psi) \in C(\overline{M}) \times C(\partial D)$ be a pair of solutions to (2.6) and (2.7) with $r(x) = -\int_M (u^i(y)\Delta_g\Phi_g(x,y) - \Delta_g u^i(y)\Phi_g(x,y))dy$ and $t(x) = -2\mathcal{T}r(x)$. Using the mapping properties of $V_{g,q}$, SL_g and DL_g, we know that $u \in C^{2,\alpha}(\overline{M})$ and if $x \in M$,

$$r(x) = u^i(x) + \int_M \Phi_g(x,y)(\Delta_g + k^2)u^i(y)dy.$$

Hence, we have

$$\begin{aligned}(\Delta_g + k^2)u(x) &= (\Delta_g + k^2)u^i(x) + (\Delta_g + k^2)\int_M \Phi_g(x,y)(\Delta_g + k^2)u^i(y)dy \\ &\quad - (\Delta_g + k^2)V_{g,q}u(x) \\ &= (\Delta_g + k^2)u^i(x) - (\Delta_g + k^2)u^i(x) + k^2(1 - q(x))u(x),\end{aligned}$$

which readily gives

$$\Delta_g u(x) + k^2 q(x)u(x) = 0, \quad x \in M.$$

Extending u into D_e by solving (2.6), it is easy to verify that $\Delta u(x) + k^2u(x) = 0$ for $x \in \Omega_e \cap D_e$. From the jumping properties of the single and double layer potential operators, one has $u(x) = 0$ for $x \in \partial D$. Furthermore, $u(x) - u^i(x)$ satisfies the radiation condition since the fundamental solution Φ_g satisfies the radiation condition. This proves the first part of Thoerem 2.2.

Next, we reformulate the system (2.6)-(2.7) as follows

$$\begin{bmatrix} I & -DL_g \\ 0 & I \end{bmatrix}\begin{bmatrix} u \\ \psi \end{bmatrix} + \begin{bmatrix} V_{g,q} & -ikSL_g \\ -2\mathcal{T}V_{g,q} & 2(\mathcal{T}DL_g + ik\mathcal{T}SL_g) \end{bmatrix}\begin{bmatrix} u \\ \psi \end{bmatrix} = \begin{bmatrix} r \\ t \end{bmatrix}. \tag{2.8}$$

We consider (2.8) in the product spaces $C(\overline{M})\times C(\partial D)$ and $L^2(M)\times C(\partial D)$, respectively, with their canonical norms. In both spaces, the system is of Fredholm type since

$$\begin{bmatrix} I & -DL_g \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & DL_g \\ 0 & I \end{bmatrix}$$

and all entries of

$$\begin{bmatrix} V_{g,q} & -ikSL_g \\ -2\mathcal{T}V_{g,q} & 2(\mathcal{T}DL_g + ik\mathcal{T}SL_g) \end{bmatrix}$$

are compact operators. Hence, it sufficies for us to study the uniqueness of the solution to the system (2.6)-(2.7) in $L^2(M)\times C(\partial D)$. Let (u,ψ) be a solution for $r=0$ and $t=0$. We know $u\in C(\overline{M})$ from the mapping property of $V_{g,q}$. From the first part of the present proof, we further know that u solves the scattering problem (1.4) for $u^i=0$. Therefore, we have $u=0$ for $x\in D_e$ from the uniqueness result. Hence, $0=u=(DL_g+ikSL_g)\psi$ in D_e. The jumping properties of the layer potential operators then imply that $\frac{1}{2}\psi+(DL_g+ikSL_g)\psi=0$ on $\partial\Omega$. Now we define the function v by

$$v(x) := \int_{\partial D}\left[\frac{\partial}{\partial\nu(y)}\Phi_g(x,y)+ik\Phi_g(x,y)\right]\psi(y)ds(y). \tag{2.9}$$

We have $v=0$ for $x\in D_e$. By using the jumping properties again,

$$-v^- = v^+ - v^- = \psi, \quad -\frac{\partial v^-}{\partial\nu} = \frac{\partial v^+}{\partial\nu} - \frac{\partial v^-}{\partial\nu} = -2ik\psi \tag{2.10}$$

By eliminating ψ, we have that $\frac{\partial v^-}{\partial\nu}+ikv^-=0$ on ∂D. By Green's theorem , we have $v=0$ in Ω which readily implies $\psi=0$. The proof is complete. □

3. Recovering the support of a complex scatterer

In this section, we shall establish the uniqueness in determining the scattering support of a complex scatterer disregarding its contents provided there is singularity attached to q. Throughout the present section, we assume that $g\in C^\infty(\mathbb{R}^n)$, whereas there exists an open neighborhood of $\partial\Omega$, $neigh(\partial\Omega)\Subset\Omega$, and a positive constant $\epsilon_0\in\mathbb{R}_+$ such that

$$|q(x)-1|\geq\epsilon_0 \quad \text{for a.e. } x\in neigh(\partial\Omega). \tag{3.1}$$

Next, we introduce a more singular point source than $\Phi(x,y)$ which is given for every fixed $x_0\in\mathbb{R}^3$ by

$$\Psi(y,x_0) = h_1^{(1)}(k\rho)P_1(\cos(\varphi)), \tag{3.2}$$

where $h_1^{(1)}$ is the spherical Hankel function of the first kind of order one and P_1 is the Legendre polynomial of order one, and (ρ, ϕ, φ) is the spherical coordinate of $y - x_0$. $\Psi(y, x)$ is known as the spherical wave function and we refer to [3] for related background. It is noted that $\Psi(y, x_0)$ has a quadratic singularity only at the point $y = x_0$ which comes from that of the spherical Hankel function; that is, $(y - x_0)^2\Psi(y, x_0)$ is smooth in $\mathbb{R}^3$. For the subsequent use, we first present an approximation property of point sources by linear combination of plane waves.

Lemma 3.1. *Let $E \subset \mathbb{R}^3$ be a compact set and $x_0 \in \mathbb{R}^3 \backslash E$ be fixed. Then there exist sequences $v_n(y)$ and $\omega_n(y)$ in the span of plane waves*

$$\mathscr{E} := \mathrm{Span}\{e^{iky\cdot\theta} : \theta \in \mathbb{S}^2\}$$

such that

$$\|v_n - \Phi(\cdot, x_0)\|_{C^1(E)} \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

and

$$\|\omega_n - \Psi(\cdot, x_0)\|_{C^1(E)} \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Proof. This follows from Lemma 3.2 in [3] by noting that $\Phi(\cdot, x_0)$ and $\Psi(\cdot, x_0)$ are (real) analytic solutions of the Helmholtz equation in any domain that does not contain x_0. See also Lemma 5 in [35]. $\square$

Theorem 3.2. *Let*

$$\Sigma = (M; g, q) \oplus D$$

be a complex scatterer with a medium (g, q) supported in M and an obstacle D, and let $\mathbf{G} = \mathbb{R}^3 \backslash (M \cup D)$. Then $\mathbf{G}$ is uniquely determined by the corresponding scattering amplitude $a_k(\theta, \theta')$ with $\theta, \theta' \in \mathbb{S}^2$ and a fixed $k \in \mathbb{R}_+$.

Proof. We shall prove the theorem by contradiction. Let

$$\widetilde{\Sigma} = (\widetilde{M}; \widetilde{g}, \widetilde{q}) \oplus \widetilde{D}$$

be a complex scatterer with the corresponding scattering amplitude $\widetilde{a}_k(\theta, \theta')$ and support $\widetilde{\mathbf{G}} := \mathbb{R}^3 \backslash (\widetilde{M} \cup \widetilde{D})$ such that

$$\widetilde{\mathbf{G}} \neq \mathbf{G} \qquad \text{and} \qquad a_k(\theta, \theta') = \widetilde{a}_k(\theta, \theta'). \tag{3.5}$$

In the sequel, we let Λ be the (unique) unbounded connected component of $\mathbb{R}^3 \backslash \overline{\Sigma \cup \widetilde{\Sigma}}$. We denote by $u^s(x)$ and $\widetilde{u}^s(x)$, respectively, the scattered wave fields corresponding to Σ and $\widetilde{\Sigma}$ for $x \notin \Sigma$ and $x \notin \widetilde{\Sigma}$ respectively. By the Rellich uniqueness theorem (cf. [33]), we know from (3.5) that $u^s(x) = \widetilde{u}^s(x)$ for all $x \in \Lambda$ and all $\theta' \in \mathbb{S}^2$. Since $\mathbf{G} \neq \widetilde{\mathbf{G}}$ and both are connected, it is easily seen that either $(\mathbb{R}^3 \backslash \overline{\Lambda}) \backslash \mathbf{G} \neq \emptyset$ or

$(\mathbb{R}^3\backslash\overline{\Lambda})\backslash\widetilde{\mathbf{G}} \neq \emptyset$. Without loss of generality, we assume the former case and set $\Lambda^* = (\mathbb{R}^3\backslash\overline{\Lambda})\backslash\mathbf{G} \neq \emptyset$. It is obvious that

$$\partial\Lambda^* \subset \partial\Lambda \cup \partial\mathbf{G} \subset \partial\widetilde{\mathbf{G}} \cup \partial\mathbf{G} \quad \text{and} \quad \partial\Lambda^*\backslash\partial\mathbf{G} \neq \emptyset.$$

We also note the fact that $\partial\widetilde{\mathbf{G}} \subset \partial\widetilde{M} \cup \partial\widetilde{D}$. Let $z_0 \in \partial\Lambda^*\backslash\partial\mathbf{G} \subset (\partial\widetilde{M} \cup \partial\widetilde{D})\backslash\partial\mathbf{G}$. We next distinguish two cases that $z_0 \in (\partial\widetilde{D}\backslash\partial\mathbf{G}) \cap \partial\Lambda^*$ and $z_0 \in (\partial\widetilde{M}\backslash\partial\mathbf{G}) \cap \partial\Lambda^*$. In the following, we fix $\rho_0 > 0$ be sufficiently large such that $\Sigma \cup \widetilde{\Sigma} \subset B_{\rho_0}(0)$, where and in the following $B_r(x)$ denote a ball centered at x with radius r. Let $\mathbf{G}_{\rho_0}$ and $\widetilde{\mathbf{G}}_{\rho_0}$, respectively, denote $D_e \cap B_{\rho_0}(0)$ and $\widetilde{D}_e \cap B_{\rho_0}(0)$.

Case 1. $z_0 \in (\partial\widetilde{D}\backslash\partial\mathbf{G}) \cap \partial\Lambda^*$. Let $\tau_0 > 0$ be sufficiently small such that $B_{\tau_0}(z_0) \subset \mathbf{G}$ and $B_{\tau_0}(z_0) \cap \widetilde{M} = \emptyset$. Set $S := \partial\widetilde{D} \cap B_{\tau_0}(z_0)$ and without loss of generality we assume that $S \subset \partial\widetilde{D} \cap \partial\Lambda$. Obviously, $B_{\tau_0}(z_0)$ is divided by S into two parts and we denote by $B_{\tau_0}^+$ the one contained in Λ. We now consider the two scattering problems corresponding to Σ and $\widetilde{\Sigma}$ with the incident field $u^i(x)$ being a point source $\Psi(x,z)$ for $z \in B_{\tau_0}^+$ and, let $w^s(x,z)$ and $\widetilde{w}^s(x,z)$ denote, respectively, the scattered wave fields. Since the scattered wave coincide in Λ for all plane waves, by Lemma 3.1, it is straightforward to show that $w^s(x,z) = \widetilde{w}^s(x,z)$ for $x \in \Lambda$ and $z \in B_{\tau_0}^+$. Next, it is observed that $\Psi(x,z)$ with $z \in B_{\tau_0}^+$ is smooth for $x \in \overline{M}$ and it is directly verified that $\|\Psi(\cdot,z)\|_{C^1(M)} \leq C$ for all $z \in B_{\tau_0}^+$. From the well-posedness of the forward scattering problem, we see $\|w^s(\cdot,z)\|_{C(\mathbf{G}_{\rho_0})} \leq C$ with C a constant independent of z.

Next, we choose $h > 0$ such that the sequence

$$z_n := z_0 + \frac{h}{n}\nu(z_0), \quad n = 1, 2, \ldots \tag{3.6}$$

is contained in $B_{\tau_0}^+$, where $\nu(z_0)$ is the outward unit normal vector to $\partial\widetilde{D}$ at z_0. By our earlier discussion, $|w^s(z_0, z_n)| \leq C$ uniformly for $n \in \mathbb{N}$. On the other hand, by using the Dirichlet boundary condition of $\widetilde{w}^s$ on $\partial\widetilde{D}$, we have

$$|w^s(z_0, z_n)| = |\widetilde{w}^s(z_0, z_n)| = |-\Psi(z_0, z_n)| \to \infty \quad \text{as} \quad n \to \infty. \tag{3.7}$$

We obviously have a contradiction.

Case 2. $z_0 \in (\partial\widetilde{M}\backslash\partial\mathbf{G}) \cap \partial\Lambda^*$. Similar to Case 1, we let $B_{\tau_0}(z_0)$ be a sufficiently small ball such that $B_{\tau_0}(z_0) \subset \mathbf{G}$ and $B_{\tau_0}(x_0) \cap \overline{\widetilde{D}} = \emptyset$. Moreover, let $S := \partial\widetilde{M} \cap B_{\tau_0}(z_0)$ which is assumed to lie entirely on $\partial\Lambda$, and let $B_{\tau_0}^+$ denote the part of $B_{\tau_0}(z_0)$ contained in Λ. By a completely similar argument as that for Case 1, we know $|w^s(\cdot,z)|_{C(\mathbf{G}_{\rho_0})} \leq C$ for

$x \in B^+_{\tau_0}$, and we shall derive a contradiction by showing that $\widetilde{w}^s(x,z)$ reveals singular behavior near z_0. To this end, let $z_n, n = 1, 2, \dots$ be the sequence given in (3.6) with $\nu(z_0)$ the outward unit normal vector to $\partial\widetilde{M}$ at z_0 for the present case. By Lemma 2.1, $\Phi_g(x,z)$ has the singular order 1 at x = z. So similar to Lemma 4 in [35], we observe that $V_{\widetilde{g},\widetilde{q}}\Psi(x,z_n) \leq C/|x-z_n|$. Hence $\|V_{\widetilde{g},\widetilde{q}}\Psi(\cdot,z_n)\|_{L^2(\mathbf{G}_{\rho_0})} \leq C$ uniformly for $n \in \mathbb{N}$. Moreover, noting z_n's are contained in $B^+_{\tau_0}$ which is away from $\overline{\widetilde{D}}$, $\|\Psi(\cdot,z_n) + \mathcal{T}V_{\widetilde{g},\widetilde{q}}\Psi(\cdot,z_n)\|_{C(\partial\widetilde{D})} \leq C$. By Theorem 2.2, ii), we see $\|\widetilde{w}^s(\cdot,z_n)\|_{L^2(\widetilde{M})} \leq C$ and $\|\psi(\cdot,z_n)\|_{C(\partial\widetilde{D})} \leq C$, where $\psi(\cdot,z_n)$ is the density function in (2.7) corresponding to the incident waves $\Psi(\cdot,z_n)$. Next, using the mapping properties that $V_{\widetilde{g},\widetilde{q}}$ maps $L^2(\widetilde{M})$ continuously into $C(\widetilde{\mathbf{G}}_{\rho_0})$, and $SL_{\widetilde{g}}$ and $DL_{\widetilde{g}}$ maps $C(\partial\widetilde{D})$ continuously into $C(\widetilde{\mathbf{G}}_{\rho_0})$ (cf. [3]), we see $|V_{\widetilde{g},\widetilde{q}}\widetilde{w}^s(z_0,z_n)| \leq C$ and $|(\mathrm{DL}_{\widetilde{g}} + ik\mathrm{SL}_{\widetilde{g}})\psi(z_0,z_n)| \leq C$ uniformly for $n \in \mathbb{N}$.

By the notation in Theorem 2.2,

$$r(x,z) = \Psi(x,z) + \int_{\widetilde{M}} \Phi_g(x,y)(\Delta_g - \Delta)\Psi(y,z)dy. \tag{3.8}$$

We note that

$$(\Delta_g - \Delta)\Psi(y,z) = \nabla\cdot(g^{-1} - I)\nabla\Psi(y,z) + \frac{1}{\sqrt{\det g}}(\nabla\sqrt{\det g})\cdot(g^{-1}\nabla\Psi(y,z)). \tag{3.9}$$

Since $g \in C^\infty(\mathbb{R}^3)$ and $g = g_e$ in $\widetilde{\Omega}_e$, there exists an open neighborhood of $\partial\widetilde{\Omega}$, $neigh(\partial\widetilde{\Omega}) \Subset \widetilde{\Omega}$, such that for any $m \in \mathbb{N}$

$$g_{ij}(x) = \delta_{ij} + (d(x,\partial\widetilde{\Omega}))^m (P_m)_{ij}(x), \quad x \in neigh(\partial\widetilde{\Omega}), \tag{3.10}$$

where $d(x,\partial\widetilde{\Omega})$ is the distance from x to $\partial\widetilde{\Omega}$, and P_m is a matrix-valued function with smooth entries. By (3.8)–(3.10), it is readily seen that the volume integral term in $r(x,z)$ is smooth and uniformly bounded (independent of z) in $\widetilde{\mathbf{G}}_{\rho_0}$ since the singularity of $\nabla\Psi(y,z)$ could be canceled.

On the other hand, similar to Lemma 3 in [35], we could prove that

$$|V_{\widetilde{g},\widetilde{q}}\Psi(z_0,z_n)| \to \infty \quad \text{as } \ n \to \infty. \tag{3.11}$$

Hence, by the integral equation given in (2.6),

$$\begin{aligned} |\widetilde{w}^s(z_0,z_n)| \geq{}& |V_{\widetilde{g},\widetilde{q}}\Psi(z_0,z_n)| - |V_{\widetilde{g},\widetilde{q}}\widetilde{w}^s(z_0,z_n)| - |(\mathrm{DL}_{\widetilde{g}} + ik\mathrm{SL}_{\widetilde{g}})\psi(z_0,z_n)| \\ & - \Big|\int_{\widetilde{M}} \Phi_g(z_0,y)(\Delta_g - \Delta)\Psi(y,z_n)dy\Big| \to \infty \end{aligned} \tag{3.12}$$

as $n \to \infty$, which yields a contradiction and completes the proof.

□

Next, we show that the medium part and the obstacle part of the support of a complex scatterer could be distinguished.

Theorem 3.3. *Let*

$$\Sigma = (M; g, q) \oplus D$$

be a complex scatterer with a medium (g, q) *supported in* M *and an obstacle* D*, and let* $\mathbf{G} = \mathbb{R}^3 \backslash (\overline{M \cup D})$*. Then* $\partial\mathbf{G} \cap \partial M$ *and* $\partial\mathbf{G} \cap \partial D$ *are uniquely determined by the corresponding scattering amplitude* $a_k(\theta, \theta')$ *with* $\theta, \theta' \in \mathbb{S}^2$ *and a fixed* $k \in \mathbb{R}_+$*.*

Proof. Similar to the proof of Theorem 3.2, we let $\widetilde{\Sigma} = (\widetilde{M}; \widetilde{g}, \widetilde{q}) \oplus \widetilde{D}$ be a different complex scatterer such that $a_k(\theta, \theta') = \widetilde{a}_k(\theta, \theta')$. By Theorem 3.2, we know

$$\widetilde{\mathbf{G}} = \mathbb{R}^3 \backslash (\overline{\widetilde{M} \cup \widetilde{D}}) = \mathbb{R}^3 \backslash (\overline{M \cup D}) = \mathbf{G}.$$

We assume that

$$\partial D \cap \partial\mathbf{G} \neq \partial\widetilde{D} \cap \partial\mathbf{G}.$$

Without loss of generality, we assume $(\partial\widetilde{D} \cap \partial\mathbf{G}) \backslash (\partial D \cap \partial\mathbf{G}) \neq \emptyset$ and fix $z_0 \in (\partial\widetilde{D} \cap \partial\mathbf{G}) \backslash (\partial D \cap \partial\mathbf{G})$. Clearly, $z_0 \in \partial M \cap \partial\mathbf{G}$. Let $B_{\tau_0}(z_0)$ be sufficiently small such that $B_{\tau_0}(z_0) \cap \overline{D} = \emptyset$ and let $B_{\tau_0}^+$ denote the part of $B_{\tau_0}(x_0)$ lying in $\mathbf{G}$. Let z_n be given as that in (3.6) with $\nu(z_0)$ the outward unit normal vector to $\partial\mathbf{G}$ at z_0. As in the proof of Theorem 3.2, we consider the scattering problems corresponding to the point sources $\Phi(x, z_n)$ and denote by $w^s(x, z_n)$ and $\widetilde{w}^s(x, z_n)$ the scattered wave fields corresponding to Σ and $\widetilde{\Sigma}$, respectively. Clearly, by Lemma 3.1, we see $w^s(x, z_n) = \widetilde{w}^s(x, z_n)$ for $x \in \overline{\mathbf{G}}^c$. Since $\|\Phi(\cdot, z_n)\|_{L^2(M)} \leq C$ uniformly for $n \in \mathbb{N}$, we know $\|V_{g,q}\Phi(\cdot, z_n)\|_{C(\overline{G}_{\rho_0})} \leq C$ uniformly for $n \in \mathbb{N}$. Hence by Theorem 2.2 and the proof of Theorem 3.2, $|w^s(z_0, z_n)| \leq C$ uniformly for $n \in \mathbb{N}$. But on the other hand, noting $z_0 \in \partial\widetilde{D}$, we have from the homogeneous Dirichlet boundary on $\partial\widetilde{D}$ that $\widetilde{w}^s(z_0, z_n) = -\Phi(z_0, z_n)$. Finally, we have a contradiction by observing that $|\Phi(z_0, z_n)| \to \infty$ as $n \to \infty$.

The proof is complete.

□

4. Recovering the support of an isotropic inhomogeneity

In Section 3, we consider the recovery of the singularity attached to the lower order term q. This is achieved by showing that the scattered wave field blows up if a second order singular point source Ψ approaches the exterior boundary of an inhomogeneous medium or the exterior boundary of an obstacle. We also show that if one lets a first order singular point source Φ approach the exterior boundary of a complex scatters, the scattered wave blows up if the boundary part belongs to the obstacle component and remains bounded if the boundary part belongs to the medium component. The latter blow-up behavior can be used to distinguish the medium boundary and the obstacle boundary of the support of a complex scatterer.

In this section, we consider recovering the support of an inhomogeneous medium when there is singularity presented to the leading order term g. We shall be mainly concerned with the isotropic case by assuming that

$$(g_{ij}) = \gamma(\delta_{ij}) \text{ in } \Omega \text{ with } \gamma = \gamma(x) \in C^1(\overline{\Omega}), \tag{4.1}$$

where it is further assumed that there exists an open neighborhood of $\partial\Omega$, $\mathcal{N} := neigh(\partial\Omega) \Subset \Omega$, and a positive constant $\epsilon_0 \in \mathbb{R}_+$ such that

$$|\gamma(x) - 1| \geq \epsilon_0 \quad \text{for a.e. } x \in \mathcal{N}. \tag{4.2}$$

Next, we show that one can recover an isotropic inhomogeneity $(\Omega; \gamma, q)$ with γ described in (4.1)–(4.2) and $q \in L^\infty(\Omega)$, by using a first order singular point source. To that end, we first present two lemmas.

Lemma 4.1 ([23]). *Consider the following transmission problem,*

$$\nabla \cdot \gamma(x)\nabla u^-(x) + k^2 q(x) u^-(x) = 0 \qquad x \in \Omega, \tag{4.3}$$

$$\Delta u^s(x) + k^2 u^s(x) = 0 \qquad x \in \Omega_e, \tag{4.4}$$

$$u^-(x) = u^s(x) + f(x) \qquad x \in \partial\Omega \tag{4.5}$$

$$\gamma(x)\frac{\partial u^-}{\partial \nu}(x) = \frac{\partial u^s}{\partial \nu}(x) + p(x) \qquad x \in \partial\Omega, \tag{4.6}$$

$$\lim_{|x|\to+\infty} |x|^{\frac{n-1}{2}} \left(\frac{\partial u^s}{\partial |x|}(x) - iku^s(x) \right) = 0. \tag{4.7}$$

where $f(x) \in H^{1/2}(\partial\Omega)$ and $p(x) \in H^{-1/2}(\partial\Omega)$. There exists a unique solution $(u^-, u^s) \in H^1(\Omega) \times H^1_{loc}(\mathbb{R}^n\backslash\overline{\Omega})$ to the above problem. Moreover, the solution

$$\|u^-\|_{H^1(\Omega)} + \|u^s\|_{H^1(B_R\backslash\overline{\Omega})} \leq C(\|f\|_{H^{1/2}(\partial\Omega)} + \|p\|_{H^{-1/2}(\partial\Omega)}), \tag{4.8}$$

where $B_R := B_R(0)$ contains Ω and C is a positive constant depending only on Ω, R and k, γ, q,

Lemma 4.2. *For any $z_0 \in \partial\Omega$, let*

$$z_n := z_0 + \frac{h}{n}\nu(z_0) \in \Omega_e, \quad n = 1, 2, \ldots, \tag{4.9}$$

where $h \in \mathbb{R}_+$ and $\nu(z_0)$ is the outward unit normal vector to $\partial\Omega$ at z_0. Let $u^s(x, z_n)$ denote the scattered wave field to (4.3)–(4.7) by *taking* $f(x)=\Phi(x,z_n)$ *and* $p(x)=\frac{\partial\Phi}{\partial\nu}(x,z_n)$. *Then, we have that if* $\|u^s(\cdot,z_n)\|_{H^1(B_R\backslash\overline{\Omega})}$ *is uniformly bounded for* $n\in\mathbb{N}$, *then* $|u^s(z_n,z_n)|\rightarrow\infty$ *as* $n\rightarrow\infty$.

It is remarked that in Lemma 4.2, $u^s(x,z_n)$ is actually smooth for $x\in\Omega_e$, and hence one can consider the point values $u^s(z_n,z_n)$. Similar to our study in the previous section, we only prove the lemma in the three-dimensional case.

Proof of Lemma 4.2. First, by integrating by parts, we have

$$
\begin{aligned}
&\int_\Omega k^2(1-\frac{q}{\gamma})u^-(x)\Phi(x,z_n)\ dx-\int_\Omega\frac{1}{\gamma}\nabla\gamma(x)\cdot\nabla u^-(x)\Phi(x,z_n)\ dx\\
=&\int_{\partial\Omega}\frac{\partial}{\partial\nu}u^-(x)\Phi(x,z_n)-\frac{\partial}{\partial\nu}\Phi(x,z_n)u^-(x)\ ds(x)
\end{aligned}\tag{4.10}
$$

Let $\mathbb{I}_L$ and $\mathbb{I}_R$ denote, respectively, the LHS and RHS terms in (4.10). By using the transmission boundary condition across $\partial\Omega$, we further have

$$
\begin{aligned}
\mathbb{I}_R=&\int_{\partial\Omega}\frac{1}{\gamma}\frac{\partial}{\partial\nu}\Phi(x,z_n)\Phi(x,z_n)-\frac{\partial}{\partial\nu}\Phi(x,z_n)\Phi(x,z_n)ds(x)\\
&+\int_{\partial\Omega}\frac{1}{\gamma}\frac{\partial}{\partial\nu}u^s(x,z_n)\Phi(x,z_n)-\frac{\partial}{\partial\nu}\Phi(x,z_n)u^s(x,z_n)ds(x)\\
&:=\mathbb{I}_1+\mathbb{I}_2
\end{aligned}\tag{4.11}
$$

By using integration by parts again, one has

$$
\begin{aligned}
\mathbb{I}_1=&k^2\int_\Omega(1-\frac{1}{\gamma})\Phi^2(x,z_n)dx+\int_\Omega(\frac{1}{\gamma}-1)|\nabla\Phi(x,z_n)|^2dx\\
&+\int_\Omega(\nabla\frac{1}{\gamma}\cdot\nabla\Phi(x,z_n))\Phi(x,z_n)dx\\
:=&\int_\Omega(\frac{1}{\gamma}-1)|\nabla\Phi(x,z_n)|^2dx+\mathbb{I}_{11}
\end{aligned}\tag{4.12}
$$

On the other hand, by Green's formula, we also have that

$$
\begin{aligned}
\mathbb{I}_2=&-u^s(z_n,z_n)+\int_{\partial\Omega}(\frac{1}{\gamma}-1)\frac{\partial}{\partial\nu}u^s(x,z_n)\Phi(x,z_n)\ ds(x)\\
=&-u^s(z_n,z_n)+\mathbb{I}_{22}.
\end{aligned}\tag{4.13}
$$

Since $\|u^s(\cdot,z_n)\|_{H^1(B_R\backslash\overline{\Omega})}$ is uniformly bounded for $n\in\mathbb{N}$, it is directly shown that

$$\begin{aligned}&\left|\int_{\partial\Omega}\frac{\partial u^s}{\partial\nu}(x,z_n)\Phi(x,z_n)\ ds(x)\right|\\ \leq& C'\|u^s(\cdot,z_n)\|_{H^1(B_R\backslash\overline{\Omega})}\|\Phi(x,z_n)\|_{H^1(\Omega)}\leq Ch_n^{-1/2},\end{aligned}\tag{4.14}$$

where and in the following, C and C' stands for some generic positive constants, and $h_n:=h/n$. On the other hand, by straightforward calculations, one can show that

$$|\mathbb{I}_L|\leq Ch_n^{-1/2};\quad |\mathbb{I}_{11}|\leq Ch_n^{-1/2},\tag{4.15}$$

but by noting (4.2)

$$\left|\int_\Omega(\frac{1}{\gamma}-1)|\nabla\Phi(x,z_n)|^2dx\right|\geq Ch_n^{-1}\tag{4.16}$$

Hence, by combining (4.11)–(4.16), we have

$$\begin{aligned}|u^s(z_n,z_n)|\geq&\left|\int_\Omega(\frac{1}{\gamma}-1)|\nabla\Phi(x,z_n)|^2dx\right|-|\mathbb{I}_{11}|-|\mathbb{I}_{22}|-|\mathbb{I}_L|\\ \geq& Ch_n^{-1}-C'h_n^{-1/2}\to+\infty\quad\text{as}\quad n\to\infty.\end{aligned}\tag{4.17}$$

The proof is complete. □

We are in a position to present the main theorem of this section.

Theorem 4.3. *Let $(\Omega;\gamma,q)$ be an isotropic inhomogeneous medium such that γ satisfies (4.1)–(4.2), and $q\in L^\infty(\Omega)$. Then, Ω is uniquely determined by the corresponding scattering amplitude $a_k(\theta,\theta')$, for all $\theta,\theta'\in\mathbb{S}^{n-1}$ and a fixed $k\in\mathbb{R}_+$.*

Proof. By making use of Lemma 4.2, the proof of the theorem follows from a similar manner to that for Theorem 3.2, which we shall sketch in the following. We shall make use of the same notations as those introduced in the proof of Theorem 3.2. Let $(\Omega;\gamma,q)$ and $(\widetilde{\Omega};\widetilde{\gamma},\widetilde{q})$ be two admissible inhomogeneous mediums with $\Omega\neq\widetilde{\Omega}$ such that $a_k(\theta,\theta')=\widetilde{a}_k(\theta,\theta')$. Hence, one has

$$u^s(x,z_n)=\widetilde{u}^s(x,z_n)\quad\text{for } x\in\Lambda,\tag{4.18}$$

where it is assumed that $\lim_{n\to\infty}z_n\to z_0\in\partial\Lambda\cap\partial\Omega$ and $\{z_n\}$ lie away from $\partial\widetilde{\Omega}$. Let B_R be a sufficiently large ball which contains both Ω and $\widetilde{\Omega}$. It is readily verified by Lemma 4.1 that $\|\widetilde{u}^s(\cdot,z_n)\|_{H^1(B_R\backslash\overline{\widetilde{\Omega}})}$ is uniformly bounded for $n\in\mathbb{N}$, and moreover by the interior estimate for elliptic partial differential equations, one has that $|\widetilde{u}^s(z_n,z_n)|$ is also uniformly bounded for $n\in\mathbb{N}$. Hence, by (4.18), $|u^s(z_n,z_n)|$ and

$\|u^s(\cdot, z_n)\|_{H^1((B_R\backslash\overline{\Omega})\cap\Lambda)}$ are both bounded uniformly for $n \in \mathbb{N}$. However, the uniform boundedness of $\|u^s(\cdot, z_n)\|_{H^1((B_R\backslash\overline{\Omega})\cap\Lambda)}$ readily implies the uniform boundedness of $\|u^s(\cdot, z_n)\|_{H^1(B_R\backslash\overline{\Omega})}$, which by Lemma 4.2 further implies that $|u^s(z_n, z_n)| \to \infty$ as $n \to \infty$. Hence, we arrive at a contradiction, thus completing the proof. □

Finally, we consider the recovery of a complex scatterer which may include both isotropic inhomogeneities and obstacles. Indeed, we have

Theorem 4.4. *Let*

$$\Sigma = (M; \gamma, q) \oplus D$$

be a complex scatterer with a medium (γ, q) *supported in* M *and an obstacle* D*, where* γ *satisfies* (4.1) *and* (4.2) *(with* Ω *replaced by* M *there), and* $q \in L^\infty(M)$*. Let* $\mathbf{G} = \mathbb{R}^3\backslash(\overline{M \cup D})$*. Then* $\partial\mathbf{G}$ *is uniquely determined by the corresponding scattering amplitude* $a_k(\theta, \theta')$ *with* $\theta, \theta' \in \mathbb{S}^2$ *and a fixed* $k \in \mathbb{R}_+$*.*

Proof. The determination of $\partial M \cap \partial\mathbf{G}$ follows from a similar argument to the proof of Theorem 4.3, and the determination of $\partial D \cap \partial\mathbf{G}$ follows from a similar argument to the proof of Theorem 3.3. □

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