2013

Full friendly index sets and full product-cordial index sets of some permutation petersen graphs

Wai Chee Shiu  
*Hong Kong Baptist University, wcshiu@hkbu.edu.hk*

Man-Ho Ho  
*Hong Kong Baptist University, homanho@math.hkbu.edu.hk*

---

This document is the authors' final version of the published article.  
Link to published article: https://www.novapublishers.com/catalog/product_info.php?products_id=48175

---

**Recommended Citation**  
Shiu, Wai Chee, and Man-Ho Ho. "Full friendly index sets and full product-cordial index sets of some permutation petersen graphs."  
*Journal of Combinatorics and Number Theory* 5.3 (2013): 227-244.
FULL FRIENDLY INDEX SETS AND FULL PRODUCT-CORDIAL INDEX SETS OF SOME PERMUTATION PETERSEN GRAPHS

MAN-HO HO AND WAI CHEE SHIU

Abstract. Let $G = (V, E)$ be a connected graph without loops. A vertex labeling $g : V \to \mathbb{Z}_2$ induces two edge labelings $f^+, f^* : E \to \mathbb{Z}_2$, given by $f^+(uv) = f(u) + f(v)$ and $f^*(uv) = f(u)f(v)$ for each $uv \in E$ respectively. For $j \in \mathbb{Z}_2$, let $v_f(j) = |f^1(j)|$, $e_{f^+}(j) = |(f^+)^{-1}(j)|$ and $e_{f^*}(j) = |(f^*)^{-1}(j)|$. A vertex labeling $f$ is called friendly if $|v_f(1) - v_f(0)| \leq 1$. For a friendly labeling $f$ of $G$, the friendly index of $G$ with respect to $f$ is defined to be $i^+_f(G) = e_{f^+}(1) - e_{f^+}(0)$, and the product-cordial index is defined to be $i^*_f(G) = e_{f^*}(1) - e_{f^*}(0)$. The full friendly index set (FFI) and the full product-cordial index set (FPCI) of $G$ contain precisely all the values $i^+_f(G)$ and $i^*_f(G)$ taken over all friendly labelings of $G$, respectively. In this paper, we study the FFI and the FPCI of odd twisted cylinder and two permutation Petersen graphs.

Contents

1. Notation and preliminary results 2
2. FFI and FPCI of odd twisted cylinder 3
3. FFI and FPCI of two permutation Petersen graphs 6
3.1. Full friendly index set of $P(3n; \sigma_2)$ 7
3.2. Full friendly index set of $P(4n; \sigma_3)$ 11
References 15
Appendix 15

In this paper all graphs $G = (V, E)$ are assumed to be loopless and connected. A vertex labeling $f : V \to \mathbb{Z}_2$ induces two edge labelings $f^+, f^* : E \to \mathbb{Z}_2$, given by

$$f^+(uv) = f(u) + f(v),$$
$$f^*(uv) = f(u)f(v),$$

where $uv \in E$. For $j \in \mathbb{Z}_2$, let $v_f(j) = |f^{-1}(j)|$, $e_{f^+}(j) = |(f^+)^{-1}(j)|$ and $e_{f^*}(j) = |(f^*)^{-1}(j)|$, i.e., $v_f(i)$ is the number of vertices labeled by $i$, and

2010 Mathematics Subject Classification. Primary 05C78; Secondary 05C25.
$e_{f^+}(i), e_{f^*}(i)$ are the number of edges labeled by $i$ with respect to $f^+$ and $f^*$ respectively. A vertex labeling $f$ is said to be friendly if

$$|v_f(1) - v_f(0)| \leq 1.$$ 

The friendly index $i_f^+(G)$ of $G$ with respect to a friendly labeling $f$ is defined to be

$$i_f^+(G) := e_{f^+}(1) - e_{f^+}(0).$$

The friendly index set $\text{FI}(G)$ of $G$ [1] contains exactly the absolute value of friendly indices of all possible friendly labelings. In [10] Shiu–Kwong generalize the friendly index set to the full friendly index set $\text{FFI}(G)$ of $G$:

$$\text{FFI}(G) = \{ i_f(G) \mid f \text{ is a friendly labeling of } G \}.$$ 

The product-cordial index $i_f^*(G)$ of $G$ [6] with respect to a friendly labeling $f$ is defined to be

$$i_f^*(G) := e_{f^*}(1) - e_{f^*}(0).$$


$$\text{FPCI}(G) = \{ i_f^*(G) \mid f \text{ is a friendly labeling of } G \}.$$ 

Friendly index of some graphs are studied in [2, 3, 5, 7]. Let $m \geq 3$ and $n \geq 2$. Denote by $C_m$ an $m$-cycle and $P_n$ an $n$-path. The full friendly index sets are studied in the case of a torus $C_m \times C_n$ [13, 14], a cylinder $C_m \times P_n$ [15, 16, 9], a grid $P_2 \times P_n$ [10] and twisted cylinders [12]. Product-cordial index set is defined and studied for paths $P_n$, complete graphs $K_n$, $K_n - e$, bipartite graph $K_{m,n}$, double stars $DS(m,n)$, cycles $C_n$ and wheels $W_n$ in [9]. The product-cordial index sets of cylinders $C_m \times P_n$ is studied in [4]. In [11] Shiu–Kwong study the full product-cordial index sets of cycles $C_n$ and torus $C_m \times C_n$, and establish the relationship between the friendly index and the product-cordial index of regular graphs. In [12] Shiu–Lee study the full product-cordial index sets of twisted cylinders.

In this paper we study the full friendly index set and the full product-cordial index set of odd twisted cylinder and two permutation Petersen graphs.

1. Notation and preliminary results

For any vertex labeling $f$, a vertex $x$ is said to be a $k$-vertex if $f(x) = k$, and an edge is said to be an $(a,b)$-edge if it is incident with an $a$-vertex and a $b$-vertex. An edge $e$ is said to be a $k$-edge if $f^*(e) = k$, where $f^*$ is the edge labeling induced by the vertex labeling $f$. The number of $(a,b)$-edges is denoted by $E_f(a,b)$. 
Lemma 1. [5, Corollary 2.3] Let $G$ be an $r$-regular graph of even order and of size $q$. If $f$ is a friendly of $G$, then $i_f^+(G) = -\frac{q + i_f^+(G)}{2}$.

Since $e_f(1)$ is always positive for any connected graph $G$, $i_f^+(G) > -q$. Thus, the product-cordial index of friendly labeling of any regular connected graph $G$ of even order is always negative, so $\text{FPCI}(G) = -PCI(G)$.

Lemma 2. [14, Corollary 2.3] Let $G$ be a regular graph of even order and of size $q$. If $f$ is a friendly labeling of $G$, then $E_f(1, 1) = E_f(0, 0)$ and $i_f^+(G) = q - 4E_f(1, 1)$.

In this paper we consider cubic graphs of even order. By Lemma 2, to determine the full friendly index sets of cubic graphs it suffices to find the range and the non-existing values of $E_f(1, 1)$. Define a set $E_f(G)(1, 1) = \{E_f(1, 1) \mid f \text{ is a friendly labeling of } G\}$.

Then $\text{FFI}(G) = q - 4E(G)(1, 1)$. Lemma 1 enables us to determine the full product-cordial index set of the cubic graphs once its full friendly index set is known.

2. FFI and FPCI of odd twisted cylinder

Let $m \geq 3$. A permutation cubic graph of order $2m$ is defined by taking two vertex-disjoint $m$-cycles and equip a perfect matching between the vertices of these two cycles. More precisely, let $C = u_1u_2\cdots u_m$ and $C^* = v_1v_2\cdots v_m$ be two $m$-cycles, and $\sigma \in S_m$, where $S_m$ is the permutation group on the set $\{1, \ldots, m\}$. The permutation cubic graph of order $2m$, denoted by $P(m; \sigma) = (V, E)$, is a simple graph with $V = \{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ and $E = E(C) \cup E(C^*) \cup \{u_iv_{\sigma(i)} \mid 1 \leq i \leq m\}$.

A special case of permutation cubic graph is twisted cylinder $P(2n; \sigma)$, where $\sigma \in S_{2n}$ is given by $\sigma = (1, 2)(3, 4)\cdots(2n - 1, 2n)$. The full friendly index set and the full product-cordial index set of $P(2n; \sigma)$ are determined in [12].

In this section we study the full friendly index set of odd twisted cylinder $P(2n + 1; \sigma_1)$, where $\sigma_1 \in S_{2n+1}$ is given

$$\sigma_1 = (1, 2)(3, 4)\cdots(2n - 1, 2n).$$

Since the size of $P(2n + 1; \sigma_1)$ is $6n + 3$, it follows from Lemma 2 that

$$i_f^+(P(2n + 1; \sigma_1)) = 6n + 3 - 4E_f(1, 1)$$

for any friendly labeling $f$ of $P(2n + 1; \sigma_1)$.

Lemma 3. [10, Corollary 5] Let $f$ be a labeling of a graph $G$ that contains a cycle $C$. If $C$ contains an 1-edge, then the number of 1-edges in $C$ is a positive even integer.

Corollary 1. The number of 0-edges of any odd cycle is odd with respect to any labeling. Thus, there is at least one 0-edge in any odd cycle.
First of all we find the extreme friendly indices of $\mathcal{P}(2n + 1; \sigma_1)$.

**Theorem 1.** If $f$ is friendly labeling of $\mathcal{P}(2n + 1; \sigma_1)$, then the minimum value of $E_f(1, 1)$ is 1 and the maximum value of $i_f^+(\mathcal{P}(2n + 1; \sigma_1))$ is $6n - 1$.

**Proof.** Note that $C$ and $C^*$ are two odd cycles contained in $\mathcal{P}(2n + 1; \sigma_1)$. Since the number of 0-edges is odd for any odd cycle, by Corollary 1 the number of 0-edges of $\mathcal{P}(2n + 1; \sigma_1)$ is at least 2. Since $\mathcal{P}(2n + 1; \sigma_1)$ is a 3-regular graph of even order, it follows from Lemma 2 that $E_f(1, 1) = E_f(0, 0) \geq 1$ for any friendly labeling $f$. Define a labeling $f_{\max}$ on $\mathcal{P}(2n + 1; \sigma_1)$ by

$$f_{\max}(u) = \begin{cases} 1, & \text{for } u = u_{2n+1}, \text{ or } u_{2i}, v_{2i}, \text{ for } 1 \leq i \leq n; \\ 0, & \text{otherwise}. \end{cases}$$

Note that $f_{\max}$ is friendly and $E_{f_{\max}}(1, 1) = 1$. Therefore, the minimum value of $E_f(1, 1) = 1$. It follows from Lemma 2 that the maximum value of $i_{f_{\max}}^+(\mathcal{P}(2n + 1; \sigma_1))$ is $i_{f_{\max}}^+(\mathcal{P}(2n + 1; \sigma_1)) = 6n + 3 - 4E_{f_{\max}}(1, 1) = 6n + 3 - 4 = 6n - 1$. □

**Theorem 2.** If $f$ is a friendly labeling of $\mathcal{P}(2n + 1; \sigma_1)$, then the maximum value of $E_f(1, 1)$ is $3n - 1$ and therefore the minimum value of $i_f^+(G)$ is $-6n + 7$.

**Proof.** Define a labeling $f_{\min}$ of $\mathcal{P}(2n + 1; \sigma_1)$ by

$$f_{\min}(u) = \begin{cases} 1, & \text{for } u = u_i, \text{ for } 1 \leq i \leq n; \\ 1, & \text{for } u = v_i, \text{ for } 1 \leq i \leq n + 1; \\ 0, & \text{otherwise}. \end{cases}$$

Note that $f_{\min}$ is friendly, and $E_{f_{\min}}(1, 1) = 3n - 1$.

Let $f$ be a friendly labeling of $\mathcal{P}(2n + 1; \sigma_1)$. Let $G$ be the subgraph of $\mathcal{P}(2n + 1; \sigma_1)$ induced by all the 1-vertices. Thus the order of $G$ is $2n + 1$. For each $0 \leq j \leq 3$, denote by $a_j$ the number of vertices of degree $j$ in $G$. By the Handshaking Lemma, we have

$$2E_f(1, 1) = a_1 + 2a_2 + 3a_3$$

$$= (a_1 + a_2 + a_3) + a_2 + 2a_3$$

$$= 2n + 1 - a_0 + a_2 + 2a_3.$$  

If all the 1-vertices lie on either $C$ or $C^*$ but not both, then $G$ is $C$ or $C^*$ and $E_f(1, 1) = 2n + 1 \leq E_{f_{\min}}(1, 1)$; where equality holds only if $n = 2$. Since $n \geq 3$ by assumption, to maximize $E_f(1, 1)$ we can assume both $C$ and $C^*$ have at least one 1-vertex. Thus $C$ or $C^*$ contains at least two 1-vertices of degree less than 3 in $G$.

Without loss of generality, we assume $C$ contains $2n$ 1-vertices and $C^*$ contains one 1-vertex with respect to a friendly labeling $h$. Then $E_h(1, 1) = 2n$ or $2n - 1$, so $E_h(1, 1) < E_{f_{\min}}(1, 1)$. Therefore, to maximize $E_f(1, 1)$ we
can assume both $C$ and $C^*$ contain at least two 1-vertices of degree less than 3 in $G$. Thus $a_3 \leq 2n + 1 - 4 = 2n - 3$. Thus we have to maximize

$$E_f(1, 1) = n + \frac{1 - a_0 + a_2}{2} + a_3$$

subject to

$$0 \leq a_0 + a_2 + a_3 \leq 2n + 1 \text{ and } a_3 \leq 2n - 3.$$ 

By the simplex method, the algebraic maximum value is 3$n - 1$ when $(a_0, a_1, a_2, a_3) = (0, 0, 5, 2n - 4)$ or $(a_0, a_1, a_2, a_3) = (0, 1, 3, 2n - 3)$. Combining with the friendly labeling $f_{\text{min}}$, the maximum value of $E_f(1, 1)$ is $3n - 1$. The minimum value of $i_f^+(P(2n + 1; \sigma_1))$ is $i_f^+(P(2n + 1; \sigma_1)) = 6n + 3 - 4E_{f_{\text{min}}}(1, 1) = -6n + 7$. \hfill \Box

We realize all the values lying in the extremely friendly indices of $P(2n + 1; \sigma_1)$ as friendly indices. The following two lemmas will be used repeatedly throughout this paper.

**Lemma 4.** [14, Lemma 2.7] Let $f$ be a labeling of a graph $G$ such that $E_f(1, 1) = k$. Suppose there are two non-adjacent vertices $u, v$ such that $f(u) = 1$, $f(v) = 0$, the vertex $u$ is adjacent to $x$ 1-vertices, and the vertex $v$ is adjacent to $y$ 1-vertices. If $g$ is a labeling on $G$ defined by $g(u) = 0$ and $g(v) = 1$, and $g(w) = f(w)$ for all $w \in V(G) \setminus \{u, v\}$, then $E_g(1, 1) = k - x + y$, and the numbers of 1-vertices and 0-vertices with respect to the labeling $g$ are the same as those with respect to $f$.

**Lemma 5.** [14, Lemma 2.8] Let $f$ be a labeling of a graph $G$ such that $E_f(1, 1) = k$. Suppose there are two adjacent vertices $u, v$ such that $f(u) = 1$, $f(v) = 0$, the vertex $u$ is adjacent to $x$ 1-vertices, and the vertex $v$ is adjacent to $y$ 1-vertices. If $g$ is a labeling on $G$ defined by $g(u) = 0$ and $g(v) = 1$, and $g(w) = f(w)$ for all $w \in V(G) \setminus \{u, v\}$, then $E_g(1, 1) = k - x + y - 1$, and the numbers of 1-vertices and 0-vertices with respect to the labeling $g$ are the same as those with respect to $f$.

Let $[a, b] = \{i \in \mathbb{Z} \mid a \leq i \leq b\}$.

**Theorem 3.** $E(P(2n + 1; \sigma_1))(1, 1) = [1, 3n - 1]$. Thus

$$\text{FFI}(P(2n + 1; \sigma_1)) = \{6n + 3 - 4i \mid 1 \leq i \leq 3n - 1\}.$$ 

**Proof.** Recall that $P(2n + 1; \sigma_1)$ consists of two odd cycles $C$ and $C^*$ whose vertices are labeled by $u_i$’s and $v_i$’s respectively, for $1 \leq i \leq 2n + 1$.

Consider the friendly labeling $f_{\text{max}}$ given in Theorem 4. Interchange the labelings of $v_{2j - 1}$ with $v_{2j}$ consecutively for $1 \leq j \leq n$, and denote the resulting labeling by $f_j$ with $f_0 = f_{\text{max}}$. By Lemma 5, we have $E_{f_1}(1, 1) = E_{f_0}(1, 1) - 0 + 2 - 1 = E_{f_0}(1, 1) + 1$. It is easy to see that $E_{f_j}(1, 1) = E_{f_{j-1}}(1, 1) + 1$ for each $1 \leq j \leq n$. Thus $[1, n + 1] \subseteq E(P(2n + 1; \sigma_1))(1, 1)$.

After performing the above procedure, the labeling on $P(2n + 1; \sigma_1)$ is $f_n$ and $E_{f_n}(1, 1) = n + 1$. Interchange the labelings of $u_{2j - 1}$ and $v_{2j - 1}$.
consecutively for $1 \leq j \leq n$, and denote the resulting labeling by $g_j$ with $g_0 = f_n$. By Lemma 4, $E_{g_1} = n + 2$. It is easy to see that $E_{g_1} = E_{g_{n-1}} = 1 + 1$ for each $1 \leq j \leq n$. Thus $[n + 2, 2n + 1] \subseteq E(\mathcal{P}(2n + 1; \sigma_1))(1, 1)$.

After performing the above procedure, the labeling on $\mathcal{P}(2n + 1; \sigma_1)$ is $g_n$ and $E_{g_n}(1, 1) = 2n + 1$. Interchange the labelings of $u_{n+j}$ and $v_{j}$ consecutively for $1 \leq j \leq n + 1$, and denote the resulting labeling by $h_j$ with $h_0 = g_n$. By Lemma 4, $E_{h_1} = 2n$. It is easy to see that $E_{h_1}(1, 1) = E_{h_{n-1}}(1, 1) + 1$ for each $2 \leq j \leq n - 1$. Note that $E_{h_{n-1}}(1, 1) = 3n - 2$. If $n$ is even, then $E_{h_n}(1, 1) = 3n - 1$. If $n$ is odd, then $E_{h_n}(1, 1) = 3n - 2$ and $E_{h_{n+1}}(1, 1) = 3n - 1$. Thus $[2n, 3n - 1] \subseteq E(\mathcal{P}(2n + 1; \sigma_1))(1, 1)$.

The theorem follows by combining all the above cases. 

Corollary 2. $\text{FI}(\mathcal{P}(2n + 1; \sigma_1)) = \{2i + 1 \mid 1 \leq i \leq 3n - 1\} \setminus \{6n - 3\}$.

By Lemma 1 we have the following corollary.

Corollary 3. For $n \geq 1$,

$$\text{FPCI}(\mathcal{P}(2n+1; \sigma_1)) = \{2i - 6n - 3 \mid 1 \leq i \leq 3n - 1\} = \{-2j - 1 \mid j \in [2, 3n]\}.$$

Thus $\text{PCI}(\mathcal{P}(2n + 1; \sigma_1)) = \{2j + 1 \mid j \in [2, 3n]\}$.

3. FFI and FPCI of two permutation Petersen graphs

Let $n \geq 3$. A permutation Petersen graph $P(n; \sigma)$ of order $2n$, where $\sigma \in S_n$ has no fixed point, is the graph with vertex set \{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\} and edge set \{x_ix_{i+1}, x_iy_i, y_\sigma(i) \mid 1 \leq i \leq n\}, where the addition is taken modulo $n$. The cycle $C = x_1x_2 \cdots x_nx_1$ is called the outer cycle of $P(n; \sigma)$. If $\sigma = \sigma_1\sigma_2 \cdots \sigma_r$ is the cycle decomposition of $\sigma$ and $\sigma_i = (i_1, i_2, \ldots, i_s)$, we write $C^i = y_{i_1}y_{i_2} \cdots y_{i_s}y_{i_1}$, and let $H$ be the disjoint union of all $C^i$'s. This is a generalization of permutation cubic graph defined in [8], namely, $\mathcal{P}(m; \sigma) = \mathcal{P}(m; \sigma^{-1}\tau\sigma)$, where $\tau$ is the $m$-cycle $(1, 2, \ldots, m)$.

We study the full friendly index set of two permutation Petersen graphs: $P(3n; \sigma_2)$, where

$$\sigma_2 = (1, 2, 3)(4, 5, 6) \cdots (3n - 2, 3n - 1, 3n),$$

and $P(4n; \sigma_3)$, where

$$\sigma_3 = (1, 2, 3, 4)(5, 6, 7, 8) \cdots (4n - 3, 4n - 2, 4n - 1, 4n).$$

The size of $P(3n; \sigma_2)$ and $P(4n; \sigma_3)$ are $9n$ and $12n$, respectively. It follows from Lemma 2 that

$$i^+_f(P(3n; \sigma_2)) = 9n - 4E_f(1, 1),$$

$$i^+_f(P(4n; \sigma_3)) = 12n - 4E_f(1, 1),$$

where $f$ is a friendly labeling.
3.1. **Full friendly index set of** \( P(3n; \sigma_2) \): First of all we find the extreme friendly indices of \( P(3n; \sigma_2) \).

**Theorem 1.** Let \( f \) be a friendly labeling of \( P(3n; \sigma_2) \). The minimum value of \( E_f(1,1) \) is \( n/2 \) if \( n \) is even and \((n+1)/2 \) if \( n \) is odd. Thus the maximum value of \( i^+_f(P(3n; \sigma_2)) \) is \( 7n \) if \( n \) is even and \( 7n - 2 \) if \( n \) is odd.

**Proof.** Note that all the \( C_i \)'s are \( C_3 \) and \( P(3n; \sigma_2) \) contains \( n \) \( C_3 \)'s. Since \( C_3 \) is an odd cycle, it follows from Corollary 1 that there are at least \( n \) 0-edges in \( P(3n; \sigma_2) \). Since \( E_f(1,1) = E_f(0,0) \) by Lemma 2 it follows that, if \( n \) is even, then \( E_f(1,1) \geq \frac{n}{2} \); if \( n \) is odd, \( E_f(1,1) \geq \frac{n+1}{2} \).

For even \( n \), define a labeling \( f_{\text{max}} \) on \( P(3n; \sigma_2) \) by
\[
    f_{\text{max}}(v) = \begin{cases} 
1, & \text{if } v = x_{2i} \text{ or } y_{2i-1} \text{ for } 1 \leq i \leq \frac{3n}{2}; \\
0, & \text{otherwise.}
\end{cases}
\]

Note that \( f_{\text{max}} \) is friendly. Since only \( \frac{n}{2} \) of the cycles of \( \sigma_2 \) contain two odd integers, it follows that \( E_{f_{\text{max}}}(1,1) = \frac{n}{2} \). Thus the maximum value of \( i^+_f(P(3n; \sigma_2)) \) is \( i^+_{f_{\text{max}}}(P(3n; \sigma_2)) = 9n - 4\left(\frac{n}{2}\right) = 7n \) by Lemma 2.

For odd \( n \), define a labeling \( g_{\text{max}} \) on \( P(3n; \sigma_2) \) by
\[
    g_{\text{max}}(v) = \begin{cases} 
1, & \text{if } v = x_{2i} \text{ or } y_{2i-1} \text{ for } 1 \leq i \leq \frac{3n+1}{2}; \\
0, & \text{otherwise.}
\end{cases}
\]

Note that \( g_{\text{max}} \) is friendly. Since only \( \frac{n+1}{2} \) of the cycles of \( \sigma_2 \) contain two odd integers, it follows that \( E_{g_{\text{max}}}(1,1) = \frac{n+1}{2} \). Thus the maximum value of \( i^+_f(P(3n; \sigma_2)) \) is \( i^+_{g_{\text{max}}}(P(3n; \sigma_2)) = 9n - 4\left(\frac{n+1}{2}\right) = 7n - 2 \) by Lemma 2. \( \square \)

**Theorem 2.** Let \( f \) be a friendly labeling of \( P(3n; \sigma_2) \). The maximum value of \( E_f(1,1) \) is \( 4n - 1 + n/2 \) if \( n \) is even and \( 4n - 2 + (n-1)/2 \) if \( n \) is odd. Thus the minimum value of \( i^+_f(P(3n; \sigma_2)) \) is \(-9n + 4 \) if \( n \) is even and \(-9n + 10 \) if \( n \) is odd.

**Proof.** For even \( n \), define a labeling \( f_{\text{min}} \) on \( P(3n; \sigma_2) \) by
\[
    f_{\text{min}}(v) = \begin{cases} 
1, & \text{if } v = x_i \text{ or } y_i \text{ for } 1 \leq i \leq \frac{3n}{2}; \\
0, & \text{otherwise.}
\end{cases}
\]

Note that \( f_{\text{min}} \) is friendly. There are \( \frac{3n}{2} - 1 \) \((1,1)\)-edges on the outer circle, \( \frac{3n}{2} \) \((1,1)\)-edges on the edges \( x_iy_i \) and \( \frac{3n}{2} \) \((1,1)\)-edges on the \( C_i \)'s, so \( E_{f_{\text{min}}}(1,1) = \frac{3n}{2} - 1 + \frac{3n}{2} + \frac{3n}{2} = 4n - 1 + \frac{n}{2} \).

For odd \( n \), define a labeling \( g_{\text{min}} \) on \( P(3n; \sigma_2) \) by
\[
    g_{\text{min}}(v) = \begin{cases} 
1, & \text{if } v = x_i \text{ for } 1 \leq i \leq \frac{3(n-1)}{2}; \\
1, & \text{if } v = y_i \text{ for } 1 \leq i \leq \frac{3(n-1)}{2} \text{ or } 3n - 2 \leq i \leq 3n; \\
0, & \text{otherwise.}
\end{cases}
\]

Note that \( g_{\text{min}} \) is friendly. There are \( \frac{3(n-1)}{2} - 1 \) \((1,1)\)-edges on the outer circle, \( \frac{3(n-1)}{2} \) \((1,1)\)-edges on the \( x_iy_i \) edges and \( \frac{3(n-1)}{2} + 3 \) \((1,1)\)-edges on the \( C_i \)'s, so \( E_{g_{\text{min}}}(1,1) = 4n - 2 + \frac{n-1}{2} \).
Let $f$ be a friendly labeling of $P(3n; \sigma_2)$, and $G$ the subgraph of $P(3n; \sigma_2)$ induced by all the 1-vertices. Then $G$ is of order $3n$. For each $0 \leq j \leq 3$, denote by $a_j$ the number of vertices of degree $j$ in $G$. By Handshaking Lemma, we have

$$2E_f(1,1) = a_1 + 2a_2 + 3a_3$$

$$= (a_1 + a_2 + a_3) + 2a_2$$

$$= 3n - a_0 + a_2 + 2a_3.$$

If all the 1-vertices lie on the outer cycle $C$, then $G = C_{3n}$ and therefore $E_f(1,1) = 3n < E_{g_{\text{min}}}(1,1)$. Thus, to maximize $E_f(1,1)$ we can assume the outer cycle $C$ contains at least one 0-vertex. Let $k$ be the number of 1-vertices contained in $C$.

**Case 1:** If $k \geq 2$, then $G \cap C$ is a forest of order greater than 1. Then $G$ contains at least two vertices of degree less than 3.

**Case 2:** If $k = 1$, then $H$ contains only one 0-vertex, which is adjacent to two 1-vertices in $H$. Thus $G$ contains at least three vertices of degree less than 3.

In either case, we have $a_3 \leq 3n - 2$. Thus we have to maximize

$$E_f(1,1) = \frac{3n - a_0 + a_2}{2} + a_3$$

subject to

$$0 \leq a_0 + a_2 + a_3 \leq 3n \text{ and } a_3 \leq 3n - 2.$$
Suppose one of the vertices of degree 2 in $G$ lies in $H$, say $y_i$. Since each vertex in $G \cap C$ is adjacent to a vertex in $H$, $y_i$ is adjacent to a 0-vertex contained in $C$; otherwise there is a 0-vertex in $H$, which will produce one more vertex of degree less than 3. Thus there are $(3k - 1)$ 1-vertices of degree 3 in $H$ for some $k \in \mathbb{N}$. Each of these $(3k - 1)$ 1-vertices is adjacent to a 1-vertex in $C$ of degree 3 except two 1-vertices of degree 2. Therefore $3n - 3 = a_3 = 3k - 1 + 3k - 1 - 2 = 6k - 4$, which is impossible.

Suppose one of the vertices of degree 2 in $G$ lies in $C$, say $x_i$. Since each vertex in $G \cap C$ except $x_i$ is adjacent to a vertex in $H$, and all the vertices in $H$ are of degree 3 in $G$, there are $3k$ such vertices for some $k \in \mathbb{N}$, and each of them is adjacent to a vertex in $G \cap C$. Therefore, $3n - 3 = a_3 = 3k + 3k - 2 = 6k - 2$, which is impossible.

Thus $E_f(1,1) \neq 4n - 1 + \frac{n - 1}{2}$, and therefore $E_f(1,1) \leq 4n - 2 + \frac{n - 1}{2}$ when $n$ is odd. By considering the friendly labeling $g_{	ext{min}}$, the maximum value of $E_f(1,1)$ is $E_{g_{	ext{min}}}(1,1) = 4n - 2 + \frac{n - 1}{2}$, and therefore the minimum value of $i_{g_{	ext{min}}}^+(P(3n; \sigma_2))$ is

$$i_{g_{	ext{min}}}^+(P(3n; \sigma_2)) = 9n - 4E_{g_{	ext{min}}}(1,1) = 12n - 4 \left(4n - 2 + \frac{n - 1}{2}\right) = -9n + 10$$

by Lemma 2.

**Theorem 3.** For $n \geq 1$,

$$E(P(3n; \sigma_2)) = \begin{cases} \left\{\frac{n}{2}, 4n - 1 + n/2\right\}, & \text{if } n \text{ is even;} \\ \left\{n + 1/2, 4n - 2 + (n - 1)/2\right\}, & \text{if } n \text{ is odd.} \end{cases}$$

Thus

$$\text{FFI}(P(3n; \sigma_2)) = \begin{cases} \left\{9n - 4i \mid i \in \left\{\frac{n}{2}, 4n - 1 + n/2\right\}\right\}, & \text{if } n \text{ is even;} \\ \left\{9n - 4i \mid i \in \left\{(n + 1)/2, 4n - 2 + (n - 1)/2\right\}\right\}, & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** Suppose $n$ is even. Consider the friendly labeling $f_{\text{max}}$ defined in the proof of Theorem 1. Then $E_{f_{\text{max}}}(1,1) = \frac{n}{2}$. Interchange the labelings of $y_{6j-1}$ with $y_{6j}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f_j^1$ with $f_0^1 = f_{\text{max}}$. By Lemma 5, we have $E_{f_j^1} = E_{f_0} + 1$. It is easy to see that $E_{f_j^1}(1,1) = E_{f_j^1_{j-1}}(1,1) + 1$ for each $1 \leq j \leq n/2$. Thus $\left\{\frac{n}{2}, n\right\} \subseteq E(P(3n; \sigma_2))$.

The current labeling on $P(3n; \sigma_2)$ is $f_{n/2}^1$, and $E_{f_{n/2}^1}(1,1) = n$. Interchange the labelings of $y_{6j-5}$ with $y_{6j-4}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f_j^2$ with $f_0^2 = f_{n/2}^1$. By Lemma 5, we have $E_{f_j^2}(1,1) = E_{f_0^2} + 1 = n + 1$. It is easy to see that $E_{f_j^2}(1,1) = E_{f_j^2_{j-1}}(1,1) + 1$ for each $1 \leq j \leq n/2$. Thus $\left\{n, n + \frac{n}{2}\right\} \subseteq E(P(3n; \sigma_2))$.

The current labeling on $P(3n; \sigma_2)$ is $f_{n/2}^2$, and $E_{f_{n/2}^2}(1,1) = n + \frac{n}{2}$. Interchange the labelings of $y_{6j}$ with $y_{6j-5}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f_j^3$ with $f_0^3 = f_{n/2}^2$. By Lemma 4, we
have \( E_{f_1}(1, 1) = E_{f_6} + 1 = n + \frac{n}{2} + 1 \). It is easy to see that \( E_{f_3}(1, 1) = E_{f_{j-1}}(1, 1) + 1 \) for each \( 1 \leq j \leq n/2 \). Thus \([n + \frac{n}{2}, 2n] \subseteq E(P(3n; \sigma_2))\).

The current labeling on \( P(3n; \sigma_2) \) is \( f_{n/2}^3(1, 1) = \sigma_2 \). Interchange the labelings of \( y_{0j-1} \) with \( y_{0j-2} \) consecutively for \( 1 \leq j \leq n/2 \), and denote the resulting labeling by \( f_j^3 \) with \( f_0^3 = f_{n/2}^3 \). By Lemma 5, we have \( E_{f_1}(1, 1) = E_{f_4} + 1 = 2n + 1 \). It is easy to see that \( E_{f_3}(1, 1) = E_{f_{j-1}}(1, 1) + 1 \) for each \( 1 \leq j \leq n/2 \). Thus \([2n, 2n + \frac{n}{2}] \subseteq E(P(3n; \sigma_2))\).

The current labeling on \( P(3n; \sigma_2) \) is \( f_{n/2}^3(1, 1) = \sigma_2 \). Interchange the labelings of \( y_{0j} \) with \( y_{0j-3} \) consecutively for \( 1 \leq j \leq n/2 \), and denote the resulting labeling by \( f_j^5 \) with \( f_0^5 = f_{n/2}^5 \). By Lemma 4, we have \( E_{f_1}(1, 1) = E_{f_5} + 1 = 2n + \frac{n}{2} + 1 \). It is easy to see that \( E_{f_3}(1, 1) = E_{f_{j-1}}(1, 1) + 1 \) for each \( 1 \leq j \leq n/2 \). Thus \([2n + \frac{n}{2}, 3n] \subseteq E(P(3n; \sigma_2))\).

The current labeling on \( P(3n; \sigma_2) \) is \( f_{n/2}^5(1, 1) = \sigma_2 \). Interchange the labelings of \( y_{0j-1} \) with \( y_{0j-2} \) consecutively for \( 1 \leq j \leq n/2 \), and denote the resulting labeling by \( f_j^6 \) with \( f_0^6 = f_{n/2}^6 \). By Lemma 5, we have \( E_{f_1}(1, 1) = E_{f_6} + 1 = 3n + 1 \). It is easy to see that \( E_{f_3}(1, 1) = E_{f_{j-1}}(1, 1) + 1 \) for each \( 1 \leq j \leq n/2 \). Thus \([3n, 3n + \frac{n}{2}] \subseteq E(P(3n; \sigma_2))\).

The current labeling on \( P(3n; \sigma_2) \) is \( f_{n/2}^6(1, 1) = \sigma_2 \). Interchange the labelings of \( y_{0j-2} \) with \( y_{0j-5} \) consecutively for \( 1 \leq j \leq n/2 \), and denote the resulting labeling by \( f_j^7 \) with \( f_0^7 = f_{n/2}^7 \). By Lemma 4, we have \( E_{f_1}(1, 1) = E_{f_7} + 1 = 3n + \frac{n}{2} + 1 \). It is easy to see that \( E_{f_3}(1, 1) = E_{f_{j-1}}(1, 1) + 1 \) for each \( 1 \leq j \leq n/2 \). Thus \([3n + \frac{n}{2}, 4n] \subseteq E(P(3n; \sigma_2))\).

The current labeling on \( P(3n; \sigma_2) \) is \( f_{n/2}^7(1, 1) = \sigma_2 \). For each \( 1 \leq i \leq n \), define a set

\[
V_i = \{ x_{3i-2}, x_{3i-1}, x_{3i}, y_{3i-2}, y_{3i-1}, y_{3i} \}.
\]

Note that all the 1-vertices are contained in \( V_{2i-1} \) for \( 1 \leq i \leq n/2 \). For each \( 1 \leq i \leq n/2 - 1 \), interchange the labelings of \( V_{i+1} \) with \( V_{i+1} \) consecutively, i.e., interchange the labelings of \( x_{3(2i+1)+k} \) with \( x_{3(2i+1)+k} \) and \( y_{3(2i+1)+k} \) with \( y_{3(2i+1)+k} \) for each \( 0 \leq k \leq 2 \) simultaneously. Denote the resulting labeling by \( f_j^8 \) with \( f_0^8 = f_{n/2}^8 \). It is easy to see that \( E_{f_1}(1, 1) = 4n + 1 \) and \( E_{f_3}(1, 1) = E_{f_{j-1}}(1, 1) + 1 \) for each \( 1 \leq j \leq n/2 - 1 \). Thus \([4n, 4n + 1 + \frac{n}{2}] \subseteq E(P(3n; \sigma_2))\).

Combining all the above cases we get the desired result for \( E(P(3n; \sigma_2)) \) for even \( n \).

Now suppose \( n \) is odd. Perform similar procedures as in the case where \( n \) is even, except we start with the friendly labeling \( g_{\text{max}} \) defined in the proof of Theorem 1 and after the 7-th step we interchange the labelings of \( x_{3n-1} \) with \( y_{3n-2} \), and denote by \( g \) the resulting labeling. Since \( E_{g_{(n-1)/2}}(1, 1) = 4n - 2 \), it follows from Lemma 4 that \( E_g(1, 1) = E_{g_{(n-1)/2}}(1, 1) + 1 = 4n - 1 \). Now
perform the last procedure in the case where \( n \) is even to get the desired result for \( E(P(3n; \sigma_2)) \) for odd \( n \).

\[ \square \]

**Corollary 4.** For \( n \geq 1 \),

\[ \text{FL}(P(3n; \sigma_2)) = \begin{cases} \{9n - 4i \mid n/2 \leq i \leq 9n/4\}, & \text{if } n \text{ is even;} \\ \{9n - 4i \mid (n + 1)/2 \leq i \leq (9n + 1)/4\}, & \text{if } n \text{ is odd.} \end{cases} \]

**Corollary 5.** For \( n \geq 1 \),

\[ \text{FFI}(P(3n; \sigma_2)) = \begin{cases} \{-2j \mid j \in [1, 4n]\}, & \text{if } n \text{ is even;} \\ \{1 - 2j \mid j \in [3, 4n]\}, & \text{if } n \text{ is odd.} \end{cases} \]

### 3.2. Full friendly index set of \( P(4n; \sigma_3) \).

First of all we find the extreme friendly indices of \( P(4n; \sigma_3) \).

**Theorem 4.** Let \( f \) be a friendly labeling of \( P(4n; \sigma_3) \). The minimum value of \( E_f(1, 1) \) is 0. Thus the maximum value of \( i_f^+(P(4n; \sigma_3)) \) is \( 12n \).

**Proof.** For any graph, \( E_f(1, 1) \geq 0 \). Define a friendly labeling \( f_{\text{max}} \) on \( P(4n; \sigma_3) \) by

\[ f_{\text{max}}(v) = \begin{cases} 1, & \text{if } v = x_{2i} \text{ or } v = y_{2i-1} \text{ for } 1 \leq i \leq 2n; \\ 0, & \text{otherwise.} \end{cases} \]

Then \( E_{f_{\text{max}}}(1, 1) = 0 \). Therefore the maximum value of \( i_f^+(P(4n; \sigma_3)) \) is \( 12n \) by Lemma \( 2 \).

**Theorem 5.** Let \( f \) be a friendly labeling of \( P(4n; \sigma_3) \). The maximum value of \( E_f(1, 1) \) is \( 6n - 1 \) for even \( n \), and is \( 6n - 2 \) for odd \( n \). Thus the minimum value of \( i_f^+(P(4n; \sigma_3)) \) is \( -12n + 4 \) for even \( n \) and is \( -12n + 8 \) for odd \( n \).

**Proof.** Define a labeling \( f_{\text{min}} \) on \( P(4n; \sigma_3) \) as follows. If \( n \) is even, then \( f_{\text{min}} \) is defined by

\[ f_{\text{min}}(v) = \begin{cases} 1, & \text{if } v = x_i \text{ or } v = y_i \text{ for } 1 \leq i \leq 2n; \\ 0, & \text{otherwise,} \end{cases} \]

if \( n \) is odd, then \( f_{\text{min}} \) is defined by

\[ f_{\text{min}}(v) = \begin{cases} 1, & \text{if } v = x_i \text{ or } v = y_i \text{ for } i \in [1, 2n - 2] \cup \{4n - 1, 4n\}; \\ 0, & \text{otherwise.} \end{cases} \]

Note that \( f_{\text{min}} \) is friendly in both cases and \( E_{f_{\text{min}}}(1, 1) = 12\left(\frac{n}{2}\right) - 1 = 6n - 1 \) if \( n \) is even, and \( E_{f_{\text{min}}}(1, 1) = 12\left(\frac{n+1}{2}\right) + 4 = 6n - 2 \) if \( n \) is odd.

Let \( f \) be a friendly labeling of \( P(4n; \sigma_3) \), and \( G \) the subgraph of \( P(4n; \sigma_3) \) induced by all the 1-vertices. Then \( G \) is of order \( 4n \). For each \( 0 \leq j \leq 3 \), denote by \( a_j \) the number of vertices of degree \( j \) in \( G \). By Handshaking Lemma, we have

\[ 2E_f(1, 1) = a_1 + 2a_2 = a_3 \]

\[ = (a_1 + a_2 + a_3) + a_2 + 2a_3 \]

\[ = 4n - a_0 + a_2 + 2a_3. \]
If all the 1-vertices lie on the outer cycle $C$, then $G$ is $C_{4n}$ and $E_f(1,1) = 4n < E_{f_{\text{min}}}(1,1)$. Thus, to maximize $E_f(1,1)$ we can assume $C$ contains at least one 0-vertices. By a similar argument in the proof of Theorem 2, we may assume $a_3 \leq 4n - 2$. We maximize

$$E_f(1,1) = 2n + \frac{-a_0 + a_2}{2} + a_3$$

subject to

$$0 \leq a_0 + a_2 + a_3 \leq 2n + 1 \text{ and } a_3 \leq 4n - 2.$$ 

By the simplex method, the algebraic maximum value is $6n - 1$ when $(a_0, a_1, a_2, a_3) = (0, 0, 2, 4n - 2)$. For even $n$, by considering the friendly labeling $f_{\text{min}}$, the maximum of $E_f(1,1)$ is $E_{f_{\text{min}}}(1,1) = 6n - 1$. Thus the minimum value of $i^+_f(P(4n;\sigma_3))$ is

$$i^+_f(P(4n;\sigma_3)) = 12n - 4E_{f_{\text{min}}}(1,1) = 12n - 4(6n - 1) = -12n + 4.$$ 

For odd $n$, suppose $E_f(1,1) = 6n - 1$. Then we have $(a_0, a_1, a_2, a_3) = (0, 0, 2, 4n - 2)$. Similar to the proof of Theorem 2, we know that $G \cap C$ is a path. Since $a_2 = 2$, all the 1-vertices in $H$ are of degree 3, so there are $4k$ 1-vertices in $H$, where $k \in \mathbb{N}$, as $H$ is the disjoint union of $n C_4$’s. Each of these $4k$ 1-vertices in $H$ is adjacent to one 1-vertex in $C$ of degree 3 except two 1-vertices. Thus $4n - 2 = a_3 = 4k + 4k - 2 = 8k - 2$. It implies $n = 2k$, which contradicts $n$ being odd. Thus for odd $n$, $E_f(1,1) \neq 6n - 1$, and therefore $E_f(1,1) \leq 6n - 2$. By considering the friendly labeling $f_{\text{min}}$ in the case of odd $n$, the maximum value of $E_f(1,1)$ is $E_{f_{\text{min}}}(1,1) = 6n - 2$. So the minimum value of $i^+_f(P(4n;\sigma_3))$ is $i^+_f(P(4n;\sigma_3)) = 12n - 4E_{f_{\text{min}}}(1,1) = 12n - 4(6n - 2) = -12n + 8$ by Lemma 2. \hfill \square

Theorem 6.

$$E(P(4n;\sigma_3)) = \begin{cases} [0, 6n - 1], & \text{if } n \text{ is even}; \\ [0, 6n - 2], & \text{if } n \text{ is odd}. \end{cases}$$

Thus

$$\text{FFI}(P(4n;\sigma_3)) = \begin{cases} \{12n - 4i \mid 0 \leq i \leq 6n - 1\}, & \text{if } n \text{ is even}; \\ \{12n - 4i \mid 0 \leq i \leq 6n - 2\}, & \text{if } n \text{ is odd}. \end{cases}$$

Proof. For each $1 \leq i \leq n$, define a set

$$V_i = \{x_{4i-3}, x_{4i-2}, x_{4i-1}, x_{4i}, y_{4i-3}, y_{4i-2}, y_{4i-1}, y_{4i}\}.$$ 

Suppose $n$ is even. Consider the friendly labeling $f_{\text{max}}$ defined in the proof of Theorem 4. Then $E_{f_{\text{max}}}(1,1) = 0$. Define a labeling $f_1^i$ on $P(4n;\sigma_3)$ by

$$f_1^i(v) = \begin{cases} f_{\text{max}}(v) + 1, & \text{if } v \in V_2; \\ f_{\text{max}}(v), & \text{otherwise}. \end{cases}$$

For each $2 \leq i \leq n/2$, define a labeling $f_i^1$ on $P(4n;\sigma_3)$ by

$$f_i^1(v) = \begin{cases} f_{\text{max}}(v) + 1, & \text{if } v \in V_2; \\ f_{i-1}^1(v), & \text{otherwise}. \end{cases}$$
Note that $f^1_j$ is friendly for all $1 \leq i \leq n/2$. It is easy to see that $E_{f^1_j}(1, 1) = 1$ and $E_{f^1_j}(1, 1) = E_{f^-1_j}(1, 1) + 1$ for each $2 \leq i \leq n/2$. Thus $[0, \frac{n}{2}] \subseteq E(P(3n; \sigma_2))$.

The current labeling on $P(4n; \sigma_3)$ is $f^1_{n/2}$, and $E_{f^1_{n/2}}(1, 1) = \frac{n}{2}$. Interchange the labelings of $x_{8j-7}$ with $y_{8j-7}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f^3_j$ with $f^3_0 = f^1_{n/2}$. By Lemma 5, we have $E_{f^3_j}(1, 1) = E_{f^3_0} + 1 = \frac{n}{2} + 1$. It is easy to see that $E_{f^3_j}(1, 1) = E_{f^3_{j-1}}(1, 1) + 1$ for each $1 \leq j \leq n/2$. Thus $[\frac{n}{2}, n] \subseteq E(P(4n; \sigma_3))$.

The current labeling on $P(4n; \sigma_3)$ is $f^2_{n/2}$, and $E_{f^2_{n/2}}(1, 1) = n$. Interchange the labelings of $x_{8j-7}$ with $y_{8j-6}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f^4_j$ with $f^4_0 = f^2_{n/2}$. By Lemma 5, we have $E_{f^4_j}(1, 1) = E_{f^4_0} + 1 = n + 1$. It is easy to see that $E_{f^4_j}(1, 1) = E_{f^4_{j-1}}(1, 1) + 1$ for each $1 \leq j \leq n/2$. Thus $[n, n + \frac{n}{2}] \subseteq E(P(4n; \sigma_3))$.

The current labeling on $P(4n; \sigma_3)$ is $f^3_{n/2}$, and $E_{f^3_{n/2}}(1, 1) = n + \frac{n}{2}$. Interchange the labelings of $x_{8j}$ with $y_{8j}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f^5_j$ with $f^5_0 = f^3_{n/2}$. By Lemma 5, we have $E_{f^5_j}(1, 1) = E_{f^5_0} + 1 = n + \frac{n}{2} + 1$. It is easy to see that $E_{f^5_j}(1, 1) = E_{f^5_{j-1}}(1, 1) + 1$ for each $1 \leq j \leq n/2$. Thus $[n + \frac{n}{2}, 2n] \subseteq E(P(4n; \sigma_3))$.

The current labeling on $P(4n; \sigma_3)$ is $f^4_{n/2}$, and $E_{f^4_{n/2}}(1, 1) = 2n$. Interchange the labelings of $x_{8j-1}$ with $y_{8j-1}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f^6_j$ with $f^6_0 = f^4_{n/2}$. By Lemma 4, we have $E_{f^6_j}(1, 1) = E_{f^6_0} + 1 = 2n + 1$. It is easy to see that $E_{f^6_j}(1, 1) = E_{f^6_{j-1}}(1, 1) + 1$ for each $1 \leq j \leq n/2$. Thus $[2n, 2n + \frac{n}{2}] \subseteq E(P(4n; \sigma_3))$.

The current labeling on $P(4n; \sigma_3)$ is $f^5_{n/2}$, and $E_{f^5_{n/2}}(1, 1) = 2n + \frac{n}{2}$. Interchange the labelings of $x_{8j-2}$ with $y_{8j-6}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f^6_j$ with $f^6_0 = f^5_{n/2}$. By Lemma 4, we have $E_{f^6_j}(1, 1) = E_{f^6_0} + 1 = 2n + \frac{n}{2} + 1$. It is easy to see that $E_{f^6_j}(1, 1) = E_{f^6_{j-1}}(1, 1) + 1$ for each $1 \leq j \leq n/2$. Thus $[2n + \frac{n}{2}, 3n] \subseteq E(P(4n; \sigma_3))$.

The current labeling on $P(4n; \sigma_3)$ is $f^6_{n/2}$, and $E_{f^6_{n/2}}(1, 1) = 3n$. Interchange the labelings of $x_{8j-6}$ with $y_{8j-6}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f^7_j$ with $f^7_0 = f^6_{n/2}$. By Lemma 5, we have $E_{f^7_j}(1, 1) = E_{f^7_0} + 1 = 3n + 1$. It is easy to see that $E_{f^7_j}(1, 1) = E_{f^7_{j-1}}(1, 1) + 1$ for each $1 \leq j \leq n/2$. Thus $[3n, 3n + \frac{n}{2}] \subseteq E(P(4n; \sigma_3))$.

The current labeling on $P(4n; \sigma_3)$ is $f^7_{n/2}$, and $E_{f^7_{n/2}}(1, 1) = 3n + \frac{n}{2}$. Interchange the labelings of $x_{8j-3}$ with $y_{8j-6}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f^8_j$ with $f^8_0 = f^7_{n/2}$. By Lemma 4, we have $E_{f^8_j}(1, 1) = E_{f^8_0} + 1 = 3n + \frac{n}{2} + 1$. It is easy to see that $E_{f^8_j}(1, 1) = E_{f^8_{j-1}}(1, 1) + 1$ for each $1 \leq j \leq n/2$. Thus $[3n + \frac{n}{2}, 4n] \subseteq E(P(4n; \sigma_3))$. 
The current labeling on $P(4n; \sigma_3)$ is $f_{n/2}^9$, and $E_{f_{n/2}^9}(1, 1) = 4n$. Interchange the labelings of $x_{8j-5}$ with $y_{8j-5}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f_j^9$ with $f_0^9 = f_{n/2}^9$. By Lemma 5, we have $E_{f_0^9}(1, 1) = E_{f_0^9} + 1 = 4n+1$. It is easy to see that $E_{f_{f_{j-1}^9}}(1, 1) = 4n+1$ for each $1 \leq j \leq n/2$. Thus $[4n, 4n+\frac{n}{2}] \subseteq E(P(4n; \sigma_3))$.

The current labeling on $P(4n; \sigma_3)$ is $f_{n/2}^{10}$, and $E_{f_{n/2}^{10}}(1, 1) = 4n + \frac{n}{2}$. Interchange the labelings of $x_{8j-5}$ with $y_{8j}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f_j^{10}$ with $f_0^{10} = f_{n/2}^{10}$. By Lemma 4, we have $E_{f_0^{10}}(1, 1) = E_{f_0^{10}} + 1 = 4n + \frac{n}{2} + 1$. It is easy to see that $E_{f_{f_{j-1}^{10}}}(1, 1) = 4n+1$ for each $1 \leq j \leq n/2$. Thus $[4n + \frac{n}{2}, \frac{5n}{2}] \subseteq E(P(4n; \sigma_3))$.

The current labeling on $P(4n; \sigma_3)$ is $f_{n/2}^{11}$, and $E_{f_{n/2}^{11}}(1, 1) = 5n$. Interchange the labelings of $x_{8j-4}$ with $y_{8j}$ consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by $f_j^{11}$ with $f_0^{11} = f_{n/2}^{11}$. By Lemma 4, we have $E_{f_{f_{j-1}^{11}}}(1, 1) = 5n + 1 = 5n+1$ for each $1 \leq j \leq n/2$. Thus $[5n, 5n + \frac{n}{2}] \subseteq E(P(4n; \sigma_3))$.

Combining all the above cases we get the desired result for $E(P(3n; \sigma_2))$ for even $n$.

Now suppose $n$ is odd. Perform similar procedures as in the case where $n$ is even, except the followings. After the first step, define a labeling $	ilde{f}$ on $P(4n; \sigma_3)$ by

$$f_1^1(v) = \begin{cases} 
  f_{\max}(v) + 1, & \text{if } v \in V_n \\
  f_{\max}(v), & \text{otherwise}
\end{cases}$$

$	ilde{f}$ is friendly. Note that $E_{f_1^1}(1, 1) = (n-1)/2$. Then, after the fourth step, we interchange the labelings of $x_{4n}$ and $y_{4n}$, and denote by $f_1^4$ the resulting labeling. By Lemma 5, we have $E_{f_1^4}(1, 1) = n + (n-1)/2$. Then interchange the labelings of $x_{4n-3}$ and $y_{4n}$, and denote by $f_2^4$ the resulting labeling. By Lemma 4, we have $E_{f_2^4}(1, 1) = E_{f_2^4}(1, 1) + 1 = n + (n-1)/2 + 1$. After the last step, We get the desired result for $E(P(3n; \sigma_2))$ for odd $n$. □

**Corollary 6.** $\text{FI}(P(4n; \sigma_3)) = \{48 - 4i \mid 0 \leq i \leq 3n\}$. 
Corollary 7.

\[
\text{FPCI}(P(4n; \sigma_3)) = \begin{cases} 
\{-2j \mid j \in [1, 6n]\}, & \text{if } n \text{ is even;} \\
\{-2j \mid j \in [2, 6n]\}, & \text{if } n \text{ is odd.}
\end{cases}
\]

References


Appendix

In this appendix we illustrate the labeling procedure of Theorem 3 for \(P(12; \sigma_2)\) and the one of Theorem 6 for \(P(16; \sigma_3)\).

For \(P(12; \sigma_2)\), we have
$E_{f_{\max}}(1, 1) = 2$

$E_{f_1^1}(1, 1) = 3$

$E_{f_2^1}(1, 1) = 4$

$E_{f_2^1}(1, 1) = 5$

$E_{f_2^2}(1, 1) = 6$

$E_{f_1^2}(1, 1) = 7$

$E_{f_2^2}(1, 1) = 8$

$E_{f_1^3}(1, 1) = 9$

$E_{f_2^3}(1, 1) = 10$

$E_{f_1^4}(1, 1) = 11$

$E_{f_2^4}(1, 1) = 12$

$E_{f_1^4}(1, 1) = 13$
For $P(16; \sigma_3)$.

$E_{f_2}(1, 1) = 14$  
$E_{f_1}(1, 1) = 15$  
$E_{f_2}(1, 1) = 16$

$E_{f_2}(1, 1) = 17$

$E_{f_{\max}}(1, 1) = 0$  
$E_{f_1}(1, 1) = 1$  
$E_{f_2}(1, 1) = 2$

$E_{f_2}(1, 1) = 3$  
$E_{f_2}(1, 1) = 4$  
$E_{f_1}(1, 1) = 5$
$E_{f_2^2}(1,1) = 18$  \hspace{1cm} $E_{f_2^6}(1,1) = 19$  \hspace{1cm} $E_{f_2^6}(1,1) = 20$

$E_{f_2^7}(1,1) = 21$  \hspace{1cm} $E_{f_2^6}(1,1) = 22$  \hspace{1cm} $E_{f_2^6}(1,1) = 23$

Department of Mathematics, Hong Kong Baptist University

E-mail address: homanho@hkbu.edu.hk

E-mail address: wcshiu@hkbu.edu.hk