12-2013

Full friendly index sets of slender and flat cylinder graphs

Wai Chee Shiu
Hong Kong Baptist University, wcshiu@hkbu.edu.hk

Man-Ho Ho
Hong Kong Baptist University

www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 2 No. 4 (2013), pp. 63-80.

www.ui.ac.ir

FULL FRIENDLY INDEX SETS OF SLENDER AND FLAT CYLINDER GRAPHS

W. C. SHIU* AND M.-H. HO

Communicated by Ivan Gutman

Abstract. Let $G = (V, E)$ be a connected simple graph. A labeling $f : V \to \mathbb{Z}_2$ induces an edge labeling $f^* : E \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |f^{*-1}(i)|$. A labeling f is called friendly if $|v_f(1) - v_f(0)| \le 1$. The full friendly index set of G consists all possible differences between the number of edges labeled by 1 and the number of edges labeled by 0. In recent years, full friendly index sets for certain graphs were studied, such as tori, grids $P_2 \times P_n$, and cylinders $C_m \times P_n$ for some n and m. In this paper we study the full friendly index sets of cylinder graphs $C_m \times P_2$ for $m \ge 3$, $C_m \times P_3$ for $m \ge 4$ and $C_3 \times P_n$ for $n \ge 4$. The results in this paper complement the existing results in literature, so the full friendly index set of cylinder graphs are completely determined.

1. **Introduction**

Let $G = (V, E)$ be a simple connected graph. A vertex labeling $f : V \to \mathbb{Z}_2$ induces an edge labeling $f^* : E \to \mathbb{Z}_2$, given by

$$f^*(xy) := f(x) + f(y),$$

where $xy \in E$. For $i \in \mathbb{Z}_2$, define $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |(f^*)^{-1}(i)|$, i.e., $v_f(i)$ is the number of vertices labeled by i and $e_f(i)$ is the number of edges labeled by i. A vertex labeling f is said to be *friendly* if

$$|v_f(1) - v_f(0)| \le 1.$$

MSC(2010): Primary: 05C78; Secondary: 05C15.

Keywords: Full friendly index sets, cylinder graphs.

Received: 4 September 2013, Accepted: 21 October 2013.

*Corresponding author.

For a friendly labeling f of a graph G the *friendly index* of G with respect to f, denoted by $i_f(G)$, is defined to be

$$i_f(G) := e_f(1) - e_f(0).$$

The *friendly index set* [2] $\mathrm{FI}(G)$ of G is defined to be

$$\mathrm{FI}(G) = \{|i_f(G)|\,|f \text{ is a friendly labeling of } G\}.$$

In [7] Shiu-Kwong generalize the friendly index set to the *full friendly index set* $\mathrm{FFI}(G)$:

$$\mathrm{FFI}(G) = \{i_f(G)|f \text{ is a friendly labeling of } G\}.$$

Friendly index of some graphs are studied in [4, 3, 5, 6]. Let $m \geq 3$ and $n \geq 2$. Denote by C_m an m-cycle and P_n an n-path. The full friendly index sets are studied in the case of a torus $C_m \times C_n$ [8,9], a cylinder $C_m \times P_n$ for $m, n \geq 4$ [10,11] and a grid $P_2 \times P_n$ [7]. In this paper we study the full friendly index sets of cylinder graphs $C_m \times P_2$ for $m \geq 3$, $C_m \times P_3$ for $m \geq 4$ and $C_3 \times P_n$ for $n \geq 4$. Together with [10,11] the full friendly index sets of cylinder graphs $C_m \times P_n$ for arbitrary m and n are completely determined.

Henceforth the term "labeling" on a graph G means a vertex labeling from $V(G)$ to $\mathbb{Z}_2$.

2. Notation and preliminary results

We refer to [1] for general notions of graphs. Let $m \geq 3$ and $n \geq 2$. Denote by C_m an m-cycle and P_n an n-path. The Cartesian product $C_m \times P_n$ is a cylinder graph with mn vertices labeled by u_{ij} (or $u_{i,j}$), where $1 \leq i \leq m$ and $1 \leq j \leq n$. The size of $C_m \times P_n$ is $2mn - m$. Two vertices u_{ij} and u_{hk} of $C_m \times P_n$ are adjacent if either

$$i = h \text{ and } j = k \pm 1, \text{ or}$$
$$j = k \text{ and } i \equiv h \pm 1 \pmod{m}$$

We recall some results of the extremely friendly index of $C_m \times P_n$ in [10].

Theorem 2.1. [10, Theorem 2.4] *If f is a friendly labeling of $C_m \times P_n$, then*

$$i_f(C_m \times P_n) \leq \begin{cases} 2mn - m - 2n, & \text{if } m \text{ is odd;} \\ 2mn - m, & \text{if } m \text{ is even.} \end{cases}$$

Theorem 2.2. [10, Theorems 3.2–3.5] *Let f be a friendly labeling of $C_m \times P_n$.*

(1) Suppose n is even.

(a) If $m \leq 2n$, then $i_f(C_m \times P_n) \geq 3m - 2mn$.

(b) If $m \geq 2n$, then

$$i_f(C_m \times P_n) \geq \begin{cases} 4n + m + 2 - 2mn, & \text{if } m \text{ is odd;} \\ 4n + m - 2mn, & \text{if } m \text{ is even.} \end{cases}$$

(2) Suppose n is odd.

(a) If $m \le 2n-1$, then $i_f(C_m \times P_n) \ge 3m+4-2mn$.

(b) If $m \ge 2n-2$, then

$$i_f(C_m \times P_n) \ge \begin{cases} 4n+m+2-2mn, & \text{if } m \text{ is odd;} \\ 4n+m-2mn, & \text{if } m \text{ is even.} \end{cases}$$

3. Non-existence of friendly indices of $C_m \times P_n$

In the previous section we recall the upper bound and the lower bound of the friendly index of the graph $C_m \times P_n$. In this section we prove that some integers lying between the upper bound and the lower bound cannot be the friendly index of $C_m \times P_n$.

We begin with some elementary observations.

Lemma 3.1. *Let f be a friendly labeling of $C_m \times P_2 = (V, E)$. Then*

$$v_f(1) \equiv m \pmod 2.$$

Proof. Since the degree of each of the vertices of $C_m \times P_2$ is 3, it follows that

$$e_f(1) \equiv \sum_{e \in E} f^*(e) = \sum_{v \in V} \deg(v) f(v) = \sum_{v \in V} 3f(v) \equiv 3v_f(1) \equiv v_f(1) \pmod 2.$$

Since f is a friendly labeling, it follows that $v_f(0) = v_f(1) = m$. Thus $v_f(1) \equiv m \pmod 2$. □

Theorem 3.2. [11, Theorem 2.1] *For even m with $m \ge 4$ and $n \ge 2$, there is no friendly labeling f of $C_m \times P_n$ such that $e_f(1) = 2mn - m - p$, where $p = 1, 2, 3$.*

Let G be a graph and $f : V \to \mathbb{Z}_2$ a vertex labeling of G. A subgraph H of G is said be to *mixed* with respect to f if there are two vertices $u, v \in V(H)$ such that $f(u) = 1$ and $f(v) = 0$. An edge $e \in E(G)$ is called an k-edge if $f^*(e) = k$, where $k \in \mathbb{Z}_2$.

Clearly, mixed cycles and mixed paths contain at least two 1-edges and one 1-edge, respectively. Let $k \in \mathbb{Z}_2$. A cycle C is called an *k-pure cycle*, where $k \in \mathbb{Z}_2$, with respect to f if $f(u) = k$ for all $u \in V(C)$. We define *k-pure path* in a similar fashion.

A path in $C_m \times P_n$ of the form $u_{i1}u_{i2} \cdots u_{in}$ is called a *vertical path* for each fixed $1 \le i \le m$. A cycle in $C_m \times P_n$ of the form $u_{1j}u_{2j} \cdots u_{mj}u_{1j}$ is called a *horizontal cycle* for each fixed $1 \le j \le n$.

Lemma 3.3. [11, Lemma 2.2] *For even m, if $C_m \times P_n$ contains a vertical mixed path under a friendly labeling f, then the number of vertical mixed paths is at least two.*

Lemma 3.4. [7, Corollary 5] *Let f be a labeling of a graph G that contains a cycle C as its subgraph. If C contains a 1-edge, then the number of 1-edges in C is a positive even number.*

Lemma 3.5. *Let $m \ge 6$ be even. If $C_m \times P_3$ contains a horizontal pure cycle (either a 1-pure cycle or a 0-pure cycle) and a horizontal mixed cycle with respect to a friendly labeling f, then $e_f(1) \ge 8$.*

Proof. Let r be the number of horizontal 1-pure cycles and s the number of horizontal 0-pure cycles. Since f is a friendly labeling, it follows that $0 \le r, s \le 1$. There are two cases.

(1) Suppose $r = 1$ and $s = 0$. Then there are two horizontal mixed cycles, each of which has at least two 1-edges. Since $v_f(0) = \frac{3m}{2}$, there are at least $\frac{3m}{4}$ mixed vertical paths. Thus $e_f(1) \ge 4 + \frac{3m}{4} > 8$. Hence $e_f(1) \ge 9$. The case $r = 0$ and $s = 1$ is similar.

(2) Suppose $r = 1 = s$. Then there is one horizontal mixed cycle, and all vertical paths are mixed. Thus
$$e_f(1) \ge 2 + m \ge 2 + 6 = 8.$$
□

Proposition 3.6. *Let $m \ge 6$ be even. There is no friendly labeling f of $C_m \times P_3$ such that $e_f(1) = 7$.*

Proof. Let a be the number of horizontal mixed cycles and b the number of vertical mixed paths of $C_m \times P_3$. Note that $a \ne 0$ by friendliness, and $b \ne 1$ by Lemma 3.3. If $a = 1$ or 2, then $e_f(1) \ge 8$ by Lemma 3.5.

Suppose $a = 3$. If $b = 0$, then by Lemma 3.4 each horizontal mixed cycle contains at least two 1-edges. Thus $e_f(1) \ge 2 + 2 + 2 = 6$. Note that in this case $e_f(1)$ cannot be an odd integer by the same reason. If $b \ge 2$, then $e_f(1) \ge 2 + 2 + 2 + b \ge 8$, where the 2's follows from the reason as above.

Combining all these cases together we conclude that $e_f(1) \ne 7$. □

Lemma 3.7. [11, Lemma 2.3] *Let n be even. If $C_m \times P_n$ contains a horizontal mixed cycle with respect to a friendly labeling f, then the number of horizontal mixed cycles is at least two.*

Lemma 3.8. [11, Lemma 2.4] *Let $n \ge 4$ be even and $3 \le m \le 2n$. If $C_m \times P_n$ contains a horizontal pure cycle and a horizontal mixed cycle with respect to a friendly labeling f, then*
$$e_f(1) \ge \begin{cases} m+4, & \text{if } m \text{ is odd;} \\ m+3, & \text{if } m \text{ is even and } m = 2n; \\ m+4, & \text{if } m \text{ is even and } m \le 2n-2. \end{cases}$$

Lemma 3.9. *Let $n \ge 4$ be even. There is no friendly labeling f of $C_3 \times P_n$ such that $e_f(1) = 4, 5$.*

Proof. Let a be the number of horizontal mixed cycles and b the number of vertical mixed paths. If $b = 0$, then all three vertical paths are pure and therefore $|v_f(1) - v_f(0)| \ge n \ge 4$, contradicting to the assumption that f is a friendly labeling. Thus $b \ne 0$. We consider the following three cases for a.

(1) Suppose $a = 0$. Then all three vertical paths are identical. Thus $e_f(1)$ is a multiple of 3, so $e_f(1) \ne 4, 5$.

(2) Suppose $1 \le a < n$. Then $C_3 \times P_n$ contains a horizontal mixed cycles and at least one pure cycle. By Lemma 3.8 we have $e_f(1) \ge 3 + 4 = 7$.

(3) Suppose $a = n$. Since $b \ge 1$, it follows from Lemma 3.4 that $e_f(1) \ge 2n + b \ge 8 + 1 = 9$.

Combining all these cases together we conclude that $e_f(1) \ne 4, 5$. □

4. Elementary operations on vertex labeling

In this section we prove some results that will be useful in studying the full friendly index set of $C_m \times P_n$.

Let f be a labeling of $C_m \times P_n$. An $n \times m$ matrix A_f, whose (j,i)-entry is defined by $(A_f)_{ji} = f(u_{ij})$, is called the labeling matrix of $C_m \times P_n$ under f. For convenience, we write f for A_f. Let $[a,b] = \{i \in \mathbb{Z} \mid a \le i \le b\}$. We denote by $O_{p,q}$ and $J_{p,q}$ the $p \times q$ zero matrix and the $p \times q$ matrix whose entries are 1 respectively.

For a given matrix A, define a row operation σ_i on A by shifting the i-th row of A to the right by 1 entry (the last entry of the i-th row shifts to the first entry). Denote by $\sigma_i(A)$ the resulting matrix.

Proposition 4.1. *Consider $C_m \times P_2$ with a labeling f represented by the matrix*

$$f = \begin{pmatrix} J_{2,\lfloor m/2\rfloor} & O_{2,\lceil m/2\rceil} \end{pmatrix}.$$

For $0 \le j \le \lfloor m/2\rfloor$, let $f_j = \sigma_1^j(f)$, where $\sigma_1^j := \overbrace{\sigma_1 \circ \cdots \circ \sigma_1}^{j}$. Then $e_{f_j}(1) = 4 + 2j$.

Proof. Note that f is friendly for even m but not for odd m. Note also that shifting the vertex labeling of first horizontal cycle will not change the number of 1-edges in the horizontal cycle; it will only change the number of 1-edges in the vertical paths.

Note that $f = f_0$ and $e_f(1) = 4$. Clearly

$$f_1 = \begin{pmatrix} 0 & 1 \cdots 1 & 1 & 0 \cdots 0 \\ 1 & \underbrace{1 \cdots 1}_{\lfloor m/2\rfloor -1} & 0 & \underbrace{0 \cdots 0}_{\lceil m/2\rceil -1} \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} & J_{2,\lfloor m/2\rfloor -1} & \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} & O_{2,\lceil m/2\rceil -1} \end{pmatrix}.$$

Thus there are two more 1-edges in the vertical paths. It is easy to see that $e_{f_j}(1) - e_{f_{j-1}}(1) = 2$ for each $1 \le j \le \lfloor m/2\rfloor + 1$. Thus $e_{f_j}(1) = 4 + 2j$.

Proposition 4.2. *Consider $C_m \times P_2$.*

(1) *Let f be a friendly labeling of $C_m \times P_2$ represented by*

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Interchange the labeling of the $2j$-th column of the above matrix for all $1 \le j \le k$, and denote by f_k the resulting labeling with $f_0 := f$. Then $e_f(1) = m$ and $e_{f_k}(1) = m + 4k$ for each $0 \le k \le \lfloor m/2\rfloor$.

(2) *Let g be a friendly labeling of $C_m \times P_2$ represented by*

$$\begin{pmatrix} 1 & \cdots & 1 & \boxed{\begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{matrix}} \\ 0 & \cdots & 0 & \end{pmatrix}.$$

Interchange the labeling of the $2j$-th column of the above matrix for all $1 \le j \le k$, and denote by g_k the resulting labeling with $g_0 := g$. Then $e_g(1) = m + 2$ and $e_{g_k}(1) = m + 2 + 4k$ for each $0 \le k \le \lceil m/2\rceil - 2$.

Proof. Note that interchanging the labeling of the columns will only change the number of 1-edges in the horizontal cycles and will not change the number of 1-edges of the vertical paths.

(1) It is obvious that $e_f(1) = m$. Note that f_1 is obtained by interchanging the second column of the labeling f, and the resulting matrix is

$$\begin{pmatrix} 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus four more 1-edges are obtained from the horizontal cycles. It is easy to see that $e_{f_k}(1) - e_{f_{k-1}}(1) = 4$ for all $1 \le k \le \lfloor m/2 \rfloor$. Thus $e_{f_k}(1) = m + 4k$.

(2) Similar to the above proof. $\square$

For a friendly labeling f of a graph G, we have

$$e_f(1) - e_f(0) = 2e_f(1) - |E(G)|.$$

To compute $\mathrm{FFI}(G)$, it suffices to compute the set

$$a(G) = \{e_f(1) \mid f \text{ is a friendly labeling of } G\}.$$

Then $\mathrm{FFI}(G) = \{2i - |E(G)| \mid i \in a(G)\}$.

By substituting m by $2m$ and $2m+1$ in Proposition 4.2 we have

Corollary 4.3. *For $m \ge 2$*

$$\{2m + 2i \mid i \in [0, 2m] \setminus \{2m-1\}\} = \{2i \mid i \in [m, 3m] \setminus \{3m-1\}\} \subseteq a(C_{2m} \times P_2),$$

and for $m \ge 1$

$$\{2m + 1 + 2i \mid i \in [0, 2m]\} = \{2i + 1 \mid i \in [m, 3m]\} \subseteq a(C_{2m+1} \times P_2).$$

The following lemma is obvious.

Lemma 4.4. *Let f be a friendly labeling on $C_4 \times P_3$ represented by*

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ * & 1 & 0 & * \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

where $*$ is either 1 or 0. Interchange the $(1,2)$-entry with $(1,3)$-entry of f (or the $(3,2)$-entry with $(3,3)$-entry, not both) decreases $e_f(1)$ by 4. Interchange the $(1,2)$-entry with $(1,3)$-entry and the $(3,2)$-entry with $(3,3)$-entry decreases $e_f(1)$ by 8.

Proposition 4.5. *Consider the labeling $f = \begin{pmatrix} J_{\lfloor n/2 \rfloor, 3} \\ O_{\lceil n/2 \rceil, 3} \end{pmatrix}$ on $C_3 \times P_n$. Interchange the $(\lfloor n/2 \rfloor - i + 1, 3)$-entry with the $(\lfloor n/2 \rfloor + i, 1)$-entry of f for each $1 \le i \le k$, where $k \le \lfloor n/2 \rfloor - 1$, and denote by f_k the resulting labeling. Then $e_f(1) = 3$ and $e_{f_k}(1) = 3 + 4k$.*

Proof. Note that f is friendly for even n but not for odd n. Note also that $e_f(1) = 3$. After interchanging the $(\lfloor n/2\rfloor, 3)$-entry with the $(\lfloor n/2\rfloor + 1, 1)$-entry from f, we have the following matrix

$$f_1 = \begin{pmatrix} & J_{\lfloor n/2\rfloor-1,3} & \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ & O_{\lceil n/2\rceil-1,3} & \end{pmatrix}.$$

¿From the above matrix we see that $e_{f_1}(1) = 7 = 3 + 4$. After interchanging the $(\lfloor n/2\rfloor - 1, 3)$-entry with the $(\lfloor n/2\rfloor + 2, 1)$-entry from f_1, we have

$$f_2 = \begin{pmatrix} & J_{\lfloor n/2\rfloor-2,3} & \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ & O_{\lceil n/2\rceil-2,3} & \end{pmatrix}.$$

¿From the above matrix we see that $e_{f_2}(1) = 11 = e_{f_1}(1)+4$. It is easy to see that $e_{f_k}(1)-e_{f_{k-1}}(1) = 4$ for each $k \le \lfloor n/2\rfloor - 1$, and therefore $e_{f_k}(1) = 3 + 4k$. □

5. Realizing the full friendly index set

In this section we realize all the potential friendly indices of $C_m \times P_n$ for some n and m.

In the following we determine for $a(C_m \times P_2)$ for $m \ge 4$, $a(C_m \times P_3)$ for $m \ge 4$ and $a(C_3 \times P_n)$ for $n \ge 4$.

Theorem 5.1. *For $m \ge 2$, we have*

$$a(C_{2m} \times P_2) = \{2i \mid i \in [2, 3m] \setminus \{3m - 1\}\}.$$

Proof. Let ϕ be any friendly labeling of $C_{2m} \times P_2$. By Theorem 2.1 and (1)(b) of Theorem 2.2, we have

$$6m \ge i_\phi(C_{2m} \times P_2) \ge 8 - 6m.$$

On the other hand, we have

$$i_\phi(C_{2m} \times P_2) = 2e_\phi(1) - |E(C_{2m} \times P_2)| = 2e_\phi(1) - 6m.$$

It follows that $6m \ge e_\phi(1) \ge 4$.

Let f be the labeling of $C_{2m} \times P_2$ of Proposition 4.1. Note that f is friendly. By Proposition 4.1 we have $\{2i \mid i \in [2, m + 2]\} \subseteq a(C_{2m} \times P_2)$. The result follows from Corollary 4.3. □

Theorem 5.2. *For $m \ge 2$, we have*

$$a(C_{2m+1} \times P_2) = \{2i + 1 \mid i \in [2, 3m]\}.$$

Proof. Let ϕ be any friendly labeling of $C_{2m+1} \times P_2$. By Theorem 2.1 and (1)(b) of Theorem 2.2, we have

$$6m - 1 \geq i_\phi(C_{2m} \times P_2) \geq 7 - 6m.$$

It follows that $6m + 1 \geq e_\phi(1) \geq 5$.

Let f be a labeling on $C_{2m+1} \times P_2$ represented by the matrix

$$\left(\begin{array}{c|c|c} J_{2,m} & \begin{matrix} 0 \\ 1 \end{matrix} & O_{2,m} \end{array} \right).$$

Note that f is a friendly labeling of $C_{2m+1} \times P_2$ and $e_f(1) = 5$. By applying the procedure in Proposition 4.1 we have $e_{f_j}(1) = 5 + 2j$ for $0 \leq j \leq m$. Thus $\{2i + 1 \mid i \in [2, m+2]\} \subseteq a(C_{2m+1} \times P_2)$. The result follows from Corollary 4.3. □

Theorem 5.3. *For $m \geq 3$, we have*

$$a(C_{2m} \times P_3) = \{6, 10m\} \cup [8, 10m - 4].$$

Proof. Let ϕ be any friendly labeling of $C_{2m} \times P_3$. By Theorem 2.1 and (2)(b) of Theorem 2.2, we have

$$10m \geq i_\phi(C_{2m} \times P_3) \geq 12 - 10m.$$

That means $10m \geq e_\phi(1) \geq 6$. By Theorem 3.2, $e_\phi(1) \notin \{10m - 1, 10m - 2, 10m - 3\}$.

Let f be a labeling on $C_{2m} \times P_3$ represented by the matrix $\begin{pmatrix} J_{3,m} & O_{3,m} \end{pmatrix}$. Note that f is a friendly labeling of $C_{2m} \times P_2$ and $e_f(1) = 6$. Let $f_j = \sigma_3^j(f)$ for $0 \leq j \leq m$. Similar to the proof of Proposition 4.1, we have $e_{f_j}(1) = 6 + 2j$ for $0 \leq j \leq m$. Thus $\{2i \mid i \in [3, m+3]\} \subseteq a(C_{2m} \times P_3)$. The matrix representing f_m is given by

$$\begin{pmatrix} J_{2,m} & O_{2,m} \\ O_{1,m} & J_{1,m} \end{pmatrix}.$$

Consider $\sigma_1^j(f_m)$ for $0 \leq j \leq m$. Similar to the proof of Proposition 4.1 we see that $\{2i \mid i \in [m+3, 2m+3]\} \subseteq a(C_{2m} \times P_3)$.

Consider another labeling g of $C_{2m} \times P_3$ represented by the matrix

$$\left(\begin{array}{c|cc|c} & 1 & 1 & \\ J_{3,m-1} & 0 & 0 & O_{3,m-1} \\ & 1 & 0 & \end{array} \right).$$

Note that g is a friendly labeling and $e_g(1) = 9$. Consider $\sigma_1^j(g)$ for $0 \leq j \leq m - 1$. Similar to the proof of Proposition 4.1, we have $\{2i + 1 \mid i \in [4, m+3]\} \subseteq a(C_{2m} \times P_3)$. Let $\widetilde{g} = \sigma_1^{m-1}(g)$. Consider $\sigma_3^j(\widetilde{g})$ for $0 \leq j \leq m - 1$. Similarly we have $\{2i + 1 \mid i \in [m+3, 2m+2]\} \subseteq a(C_{2m} \times P_3)$.

Combining the above cases, we have $\{6\} \cup [8, 4m + 6] \subseteq a(C_{2m} \times P_3)$.

By Theorem 2.1 we have $e_\phi(1) \le 10m$ for any friendly labeling ϕ. Let h be a labeling of $C_{2m} \times P_3$ whose matrix representation is given by

$$\begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Then h is a friendly labeling and $e_h(1) = 10m$.

Case 1: Suppose $m = 2k$ for some $k \ge 2$. Then we can subdivide the above matrix into k submatrices (blocks) of size 3×4 starting from the first column. Apply the procedure in Lemma 4.4 to the first row and the third row in each of these blocks consecutively, we see that $\{4i \mid i \in [3k, 5k]\} \subseteq a(C_{2m} \times P_3)$.

Consider the labelings p, q and r whose matrix representations are of the form

$$p = \left(\boxed{\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}} \begin{array}{c} \cdots \\ \cdots \\ \cdots \end{array} \boxed{\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}} \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right),$$

$$q = \left(\boxed{\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}} \begin{array}{c} \cdots \\ \cdots \\ \cdots \end{array} \boxed{\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}} \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{array}\right),$$

$$r = \left(\boxed{\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}} \begin{array}{c} \cdots \\ \cdots \\ \cdots \end{array} \boxed{\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}} \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right).$$

Note that p, q and r are friendly labelings of $C_{2m} \times P_3$ and $e_p(1) = 20k - 5$, $e_q(1) = 20k - 6$ and $e_r(1) = 20k - 7$. By applying the procedure of Lemma 4.4 to the first $k-1$ blocks of all these matrices consecutively, we see that $\{4i+3 \mid i \in [3k, 5k-2]\}$, $\{4i+2 \mid i \in [3k, 5k-2]\}$ and $\{4i+1 \mid i \in [3k, 5k-2]\}$ are subsets of $a(C_{2m} \times P_3)$.

Combining the above four cases, we have $[6m, 10m-4] \cup \{10m\} \subseteq a(C_{2m} \times P_3)$.

Let s be a friendly labeling of $C_{2m} \times P_3$ whose matrix representation is given by

$$s = \left(\boxed{\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}} \begin{array}{c} \cdots \\ \cdots \\ \cdots \end{array} \boxed{\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}}\right).$$

Note that $e_s(1) = 6m$. By applying a similar procedure in Lemma 4.4 to the first row of each block of s consecutively, we see that $\{4i \mid i \in [2k, 3k]\} \subseteq a(C_{2m} \times P_3)$.

Consider the labelings t, u and v of $C_{2m} \times P_3$ whose matrix representations are given by

$$t = \left(\begin{array}{|cccc|c|cccc|cccc} \cline{1-4}\cline{6-9} 1&0&1&0&\cdots&1&0&1&0&1&0&1&0\\ 1&1&0&0&\cdots&1&1&0&0&1&0&0&1\\ 0&1&1&0&\cdots&0&1&1&0&0&1&1&0\\ \cline{1-4}\cline{6-9} \end{array}\right),$$

$$u = \left(\begin{array}{|cccc|c|cccc|cccc} \cline{1-4}\cline{6-9} 1&0&1&0&\cdots&1&0&1&0&1&0&1&0\\ 1&1&0&0&\cdots&1&1&0&0&1&0&1&1\\ 0&1&1&0&\cdots&0&1&1&0&0&1&0&0\\ \cline{1-4}\cline{6-9} \end{array}\right),$$

$$v = \left(\begin{array}{|cccc|c|cccc|cccc} \cline{1-4}\cline{6-9} 1&0&1&0&\cdots&1&0&1&0&1&0&1&0\\ 1&1&0&0&\cdots&1&1&0&0&1&0&1&1\\ 0&1&1&0&\cdots&0&1&1&0&0&0&1&0\\ \cline{1-4}\cline{6-9} \end{array}\right).$$

Note that t, u and v are friendly labelings of $C_{2m} \times P_3$ and $e_t(1) = 6m + 2$, $e_u(1) = 6m + 1$ and $e_v(1) = 6m - 1$. By applying a similar procedure of Lemma 4.4 to the first row of each boxed block of these matrices, we see that $\{4i+2 \mid i \in [2k+1, 3k]\}$, $\{4i+1 \mid i \in [2k+1, 3k]\}$ and $\{4i+3 \mid i \in [2k, 3k-1]\}$ are subsets of $a(C_{2m} \times P_3)$.

Combining the above four cases, we have $[4m + 3, 6m + 2] \subseteq a(C_{2m} \times P_3)$. The theorem holds for even m by considering all the above cases.

Case 2: Suppose $m = 2k + 1$ for some $k \geq 1$. We shall keep the labelings h, p, q, r, s, t, u and v for $m = 2k$. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

We construct a labeling $\overline{h}$ similar to h in Case 1 by inserting the sub-matrix A into h as the last two columns. Then $e_{\overline{h}}(1) = 20k + 10 = 10m$. Similar to Case 1 (i.e., apply the procedure in Lemma 4.4 to the first k blocks consecutively), we have $\{4i + 10 \mid i \in [3k, 5k]\}$.

Construct labelings $\overline{p}$, $\overline{q}$ and $\overline{r}$ by inserting the sub-matrix A into p, q and r between the last fifth and the last fourth column, respectively. Then $e_{\overline{p}}(1) = 20k+5$, $e_{\overline{q}}(1) = 20k+4$ and $e_{\overline{r}}(1) = 20k+3$. Similar to Case 1, after combining the above four cases, we have $[6m + 4, 10m - 4] \cup \{10m\} \subseteq a(C_{2m} \times P_3)$. Denote by $\overline{p}_{k-1}$ the labeling after the procedure in Lemma 4.4 is applied $k-1$ times. Then $e_{\overline{p}_{k-1}}(1) = 20k + 5 - 8(k - 1) = 12k + 13$. By swapping the entries of the first row of A in $\overline{p}_{k-1}$, we see that $e(1) = 12k + 9 = 6m + 3$.

Similarly, let $\overline{s}$ be obtained from s by inserting the sub-matrix B as the last two columns. We also construct labelings $\overline{t}$, $\overline{u}$ and $\overline{v}$ by inserting the sub-matrix B into t, u and v between the last fifth and last the fourth column, respectively. Then $e_{\overline{s}}(1) = 12k + 6$, $e_{\overline{t}}(1) = 12k + 8$, $e_{\overline{u}}(1) = 12k + 7$ and $e_{\overline{v}}(1) = 12k + 5$. Similar to Case 1, we will obtain $\{4m + 5\} \cup [4m + 7, 6m + 2] \subseteq a(C_{2m} \times P_3)$. Note that $4m + 6$ is covered before defining the labeling h.

The theorem now holds for odd m. $\square$

Theorem 5.4. *For $m \geq 2$, we have*

$$a(C_{2m+1} \times P_3) = [7, 10m+2].$$

Proof. For $m = 2k$, let f_j be the labelings of $C_{2m} \times P_3$ defined in the proof of Theorem 5.3, $0 \leq j \leq m$. Let $\overline{f}_j$ be the labeling obtained from f_j by inserting the sub-matrix $A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ as the last column. Note that $\overline{f}_j$ is friendly. Similar to the proof of Theorem 5.3 we have $\{7 + 2i \mid 0 \leq i \leq 2m\} \setminus \{9\} \subseteq a(C_{2m+1} \times P_3)$. If we replace the sub-matrix A in $\overline{f}_j$ by B, where $B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then it is easy to see that $\{8 + 2i \mid 0 \leq i \leq 2m\} \subseteq a(C_{2m+1} \times P_3)$. On the other hand, it is easy to see that $e_{\sigma_1^1(\overline{f}_0)}(1) = 9$. Combining all these cases we have $[7, 4m+8] \subseteq a(C_{2m+1} \times P_3)$.

Consider the labeling h represented by the following matrix

$$\left(\begin{array}{|cccc|c|cccc|c|} 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 \end{array}\right).$$

Note that h is a friendly labeling of $C_{2m+1} \times P_3$ and $e_h(1) = 10m+2$. Apply the procedure in Lemma 4.4 to the first row and the third row in each of first k blocks consecutively, we see that $\{10m + 2 - 4i \mid 0 \leq i \leq 2k\} \subseteq a(C_{2m+1} \times P_3)$. Consider the labelings p, q and r represented by the matrices

$$p = \left(\begin{array}{|cccc|c|cccc|c|} 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 \end{array}\right),$$

$$q = \left(\begin{array}{|cccc|c|cccc|c|} 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \end{array}\right),$$

$$r = \left(\begin{array}{|cccc|c|cccc|c|} 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 \end{array}\right).$$

They are friendly and $e_p(1) = 10m+1$, $e_q(1) = 10m$, and $e_r(1) = 10m-1$. Similarly, apply the procedure in Lemma 4.4 to p, q and r we see that $\{10m + 1 - 4i \mid 0 \leq i \leq 2k\}$ and $\{10m - 4i \mid 0 \leq i \leq 2k\}$ and $\{10m - 1 - 4i \mid 0 \leq i \leq 2k\}$ are subsets of $a(C_{2m+1} \times P_3)$. Combining these four cases we see that $[6m - 1, 10m + 2] \subseteq a(C_{2m+1} \times P_3)$.

Consider the labelings

$$t = \left(\begin{array}{|cccc|c|cccc|c|}\hline 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 \\ \hline\end{array}\right),$$

$$u = \left(\begin{array}{|cccc|c|cccc|c|}\hline 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 \\ \hline\end{array}\right),$$

$$v = \left(\begin{array}{|cccc|c|cccc|c|}\hline 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\ \hline\end{array}\right),$$

$$w = \left(\begin{array}{|cccc|c|cccc|c|}\hline 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\ \hline\end{array}\right).$$

These are friendly labelings and $e_t(1) = 6m+4$, $e_u(1) = 6m+3$, $e_v(1) = 6m+2$ and $e_w(1) = 6m+1$. Similar to the proof of Case 1 of Theorem 5.3, we see that $[4m+1, 6m+4] \subseteq a(C_{2m+1} \times P_3)$.

By considering all the above cases, the theorem holds when m is even. When m is odd, one can prove the theorem similar to the proof of Case 2 of Theorem 5.3. Thus the theorem holds for all $m \geq 2$. □

Theorem 5.5. *For $n \geq 3$, we have*

$$a(C_3 \times P_{2n}) = \{3\} \cup [6, 10n-3].$$

Proof. Let ϕ be any friendly labeling of $C_3 \times P_{2n}$. By Lemma 3.9, Theorem 2.1 and (1)(a) of Theorem 2.2, we have $10n-3 \geq e_\phi(1) \geq 3$ and $e_\phi(1) \neq 4, 5$.

Obviously, $q = \begin{pmatrix} O_{1,3} \\ J_{n,3} \\ O_{n-1,3} \end{pmatrix}$ is a friendly labeling of $C_3 \times P_{2n}$ and $e_q(1) = 6$.

Let f be the labeling of $C_3 \times P_{2n}$ in Proposition 4.5. It is a friendly labeling and $e_f(1) = 3$. By applying the procedure in Proposition 4.5 to f we see that $\{4k+3 \mid k \in [0, n-1]\} \subseteq a(C_3 \times P_{2n})$.

Consider the labelings g, h and ℓ of $C_3 \times P_{2n}$ represented by the matrices

$$g = \begin{pmatrix} 1 \ \ 1 \ \ 0 \\ J_{n-1,3} \\ 1 \ \ 0 \ \ 0 \\ O_{n-1,3} \end{pmatrix}, \qquad h = \begin{pmatrix} 1 \ \ 1 \ \ 1 \\ 1 \ \ 1 \ \ 0 \\ J_{n-2,3} \\ 1 \ \ 0 \ \ 0 \\ O_{n-1,3} \end{pmatrix}, \qquad \ell = \begin{pmatrix} 1 \ \ 1 \ \ 0 \\ 1 \ \ 1 \ \ 0 \\ J_{n-2,3} \\ 1 \ \ 1 \ \ 0 \\ O_{n-1,3} \end{pmatrix}.$$

Note that g, h and ℓ are friendly and $e_g(1) = 8$, $e_h(1) = 9$ and $e_\ell(1) = 10$ respectively. For the labelings g and h, interchange the $(n-i+1, 3)$-entry with the $(n+i, 3)$-entry for $1 \le i \le k$ if $n \ge 4$, where $k \le n-3$. The resulting labelings are denoted by g_k and h_k, respectively.

For the labeling g, we have

$$\{8 + 4k \mid 0 \le k \le n-3\} \subseteq a(C_3 \times P_{2n}).$$

Extend the above procedure to the labeling g to $k = n-2$ and $k = n-1$. It is easy to see that $e_{g_{n-2}}(1) = 8 + 4(n-2) = 4n$ and $e_{g_{n-1}}(1) = 8 + 4(n-2) + 2 = 4n+2$. Thus

$$\{8 + 4k \mid 0 \le k \le n-2\} \cup \{4n+2\} \subseteq a(C_3 \times P_{2n}).$$

For the labeling h, we have

$$\{9 + 4k \mid 0 \le k \le n-3\} \subseteq a(C_3 \times P_{2n}).$$

For the labeling ℓ, first interchange the $(n, 3)$-entry with the $(n+2, 1)$-entry, and then interchange the $(n+1-i, 3)$-entry with the $(n+i, 3)$-entry consecutively, for $2 \le i \le n-3$ if $n \ge 5$. Then we see that

$$\{10 + 4i \mid 0 \le i \le n-3\} \subseteq a(C_3 \times P_{2n}).$$

Combining all the above cases, we see that $\{3\} \cup [6, 4n] \cup \{4n+2\} \subseteq a(C_3 \times P_{2n})$.

The matrix representing the labeling f_{n-1} is given by

$$\begin{pmatrix} J_{1,3} \\ A \\ B \\ O_{1,3} \end{pmatrix}, \text{ where } A = \begin{pmatrix} J_{n-1,2} & O_{n-1,1} \end{pmatrix} \text{ and } B = \begin{pmatrix} J_{n-1,1} & O_{n-1,2} \end{pmatrix}.$$

For $2 \le k \le n-1$ and $n+1 \le k \le 2n-2$, shift consecutively the k-th row to the right by one unit if k is even, and to the left by one unit if k odd. Applying this procedure we get

$$\{2i - 1 \mid i \in [2n, 4n-4]\} \subseteq a(C_3 \times P_{2n}) \text{ when } n \text{ is odd;}$$
$$\{2i - 1 \mid i \in [2n, 4n-3]\} \setminus \{6n-3\} \subseteq a(C_3 \times P_{2n}) \text{ when } n \text{ is even.}$$

To realize the value $6n-3$ for even n, we make a special labeling as follows. Apply the above procedure up to shifting the $(n-1)$-th row, and then shift the $(n+1)$-th row to the right by 1 unit.

The matrix representing the labeling g_{n-1} is given by

$$g_{n-1} = \begin{pmatrix} & A & \\ 1 & 0 & 1 \\ & B & \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } A = \begin{pmatrix} J_{n,2} & O_{n,1} \end{pmatrix}, B = \begin{pmatrix} O_{n-2,2} & J_{n-2,1} \end{pmatrix}.$$

For $1 \le k \le n-1$, shift consecutively the k-th row to the right by 1 unit if k is odd, and to the left by 1 unit if k is even. It is easy to see that each operation increases $e(1)$ by 2. After these procedures, if $n \ge 5$, then we interchange the $(n+2, 3)$-entry with the $(n+3, 2)$-entry, the $(n+3, 3)$-entry with the

$(n+4,1)$-entry, the $(n+4,3)$-entry with the $(n+5,2)$-entry, the $(n+5,3)$-entry with the $(n+6,1)$-entry, etc., up to interchanging the entry in the $(2n-3)$-th row with the entry in the $(2n-2)$-th row. Again, it is easy to see that each interchange increases $e(1)$ by 2. Thus

$$\{4n+2+2k \mid 1 \le k \le 2n-5\} \subseteq a(C_3 \times P_{2n}).$$

Let p be the labeling whose matrix representation is given by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ & \vdots & \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $1 \le k \le n+2$, shift consecutively the k-th row to the right by 1 unit if k is odd, and to the left by 1 unit if k is even. It is easy to see that each shift decreases $e(1)$ by 2. The resulting labeling is denoted by p_k and let $p_0 = p$. Thus

$$\{10n-3-2k \mid 0 \le k \le n+2\} \subseteq a(C_3 \times P_{2n}).$$

By swapping the $(2n-1,3)$-entry and $(2n,3)$-entry of p_k for $0 \le k \le n+2$, it decreases $e(1)$ by 1. So we get

$$\{10n-4-2k \mid 0 \le k \le n+2\} \subseteq a(C_3 \times P_{2n}).$$

The theorem follows from considering all the above cases. $\square$

Theorem 5.6. *For $n \ge 2$, $a(C_3 \times P_{2n+1}) = [5, 10n+2]$.*

Proof. By Theorem 2.1 and (2) of Theorem 2.2 we have $10n+2 \ge e_\phi(1) \ge 5$ for any friendly labeling ϕ of $C_3 \times P_{2n+1}$.

Let $f = \begin{pmatrix} & J_{n,3} & \\ 1 & 0 & 0 \\ & O_{n,3} & \end{pmatrix}$. Then f is friendly and $e_f(1) = 5$. Interchanging the $(n-i+1,3)$-entry with the $(n+i+1,1)$-entry for $1 \le i \le k$ for each k $(1 \le k \le n-1)$. The resulting labeling is denoted by f_k. We see that $\{4k+1 \mid k \in [1,n]\} \subseteq a(C_3 \times P_{2n+1})$.

Let g, h and ℓ be labelings of $C_3 \times P_{2n+1}$ whose matrix representations are given by

$$g = \begin{pmatrix} 1 & 1 & 0 \\ & J_{n,3} & \\ & O_{n,3} & \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ & J_{n-1,3} & \\ & O_{n,3} & \end{pmatrix}, \qquad \ell = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ & J_{n-1,3} & \\ & O_{n,3} & \end{pmatrix}.$$

Note that g, h and ℓ are friendly, and $e_g(1) = 6$, $e_h(1) = 7$ and $e_\ell(1) = 8$. For the labeling g, interchange the $(k, 3)$-entry with the $(n + k, 3)$-entry consecutively for $2 \le k \le n$. The resulting labeling is denoted by g_k. It is easy to see that each interchange increases $e(1)$ by 4. Thus

$$\{6 + 4k \mid 0 \le k \le n - 1\} \subseteq a(C_3 \times P_{2n+1}).$$

For the labelings h and ℓ, interchange the $(k, 3)$-entry with the $(n - 1 + k, 3)$-entry consecutively for $3 \le k \le n$ if $n \ge 3$. It is easy to see that each interchange increases $e(1)$ by 4. Thus

$$\{7 + 4k \mid 0 \le k \le n - 2\} \subseteq a(C_3 \times P_{2n+1}).$$

$$\{8 + 4k \mid 0 \le k \le n - 2\} \subseteq a(C_3 \times P_{2n+1}).$$

Combining the above results, we have $[5, 4n + 2] \subseteq a(C_3 \times P_{2n+1})$.

Consider

$$f_n = \begin{pmatrix} J_{1,3} \\ A \\ B \\ O_{1,3} \end{pmatrix}, \text{ where } A = \begin{pmatrix} J_{n-1,2} & O_{n-1,1} \end{pmatrix}, B = \begin{pmatrix} J_{n,1} & O_{n,2} \end{pmatrix}.$$

For $k \in [2, 2n - 1] \setminus \{n\}$, shift consecutively the k-th row to the left by 1 unit if k is odd, and to the right by 1 unit if k is even. Applying this procedure we get

$$\{4n + 1 + 2i \mid i \in [0, 2n - 3]\} \subseteq a(C_3 \times P_{2n+1}) \text{ when } n \text{ is odd;}$$
$$\{4n + 1 + 2i \mid i \in [0, 2n - 2]\} \setminus \{6n - 1\} \subseteq a(C_3 \times P_{2n+1}) \text{ when } n \text{ is even.}$$

To realize the value $6n-1$ for even n, we make a special labeling as follows. Apply the above procedure up to shifting the $(n - 1)$-th row, and then shift the n-th row to the right by 1 unit.

Consider the labeling

$$g_n = \begin{pmatrix} A \\ B \end{pmatrix}, \text{ where } A = \begin{pmatrix} J_{n,2} & O_{n,1} \end{pmatrix}, B = \begin{pmatrix} & J_{1,3} & \\ O_{n-1,2} & & J_{n-1,2} \\ & O_{1,3} & \end{pmatrix}.$$

For $1 \le k \le 2n - 1$, shift consecutively the k-th row to the right by 1 unit if k is odd, and to the left by 1 unit if k is even. It is easy to see that each shift increases $e(1)$ by 2, except shifting the n-th row and the $(n + 1)$-th row which preserve $e(1)$. Thus

$$\{4n + 2 + 2i \mid i \in [0, 2n - 3]\} \subseteq a(C_3 \times P_{2n+1}).$$

Let p be the labeling whose matrix representation is given by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ & \vdots & \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Similar to the procedure for the matrix p in Theorem 5.5, we have

$$[8n-3, 10n+2] \subseteq a(C_3 \times P_{2n+1}).$$

The theorem follows from considering all the above cases. □

By constructing labelings directly, it is easy to obtain that $a(C_4 \times P_3) = [6, 16] \cup \{20\}$, $a(C_3 \times P_2) = \{3, 5, 7\}$, $a(C_3 \times P_3) = [5, 12]$ and $a(C_3 \times P_4) = \{3\} \cup [6, 17]$.

We summarize the full friendly index sets of cylinder graphs $C_m \times P_2$ for $m \geq 3$, $C_m \times P_3$ for $m \geq 3$, and $C_3 \times P_n$ for $n \geq 4$, as follows.

Theorem 5.7. *The full friendly index set of $C_m \times P_n$ is given by*

$$\mathrm{FFI}(C_m \times P_2) = \{4i - 3m \mid i \in [2, 3m/2 - 2] \cup \{3m/2\}\} \textit{ if } m \geq 4 \textit{ is even.}$$
$$\mathrm{FFI}(C_m \times P_2) = \{4i - 3m + 2 \mid i \in [2, (3m-1)/2]\} \textit{ if } m \geq 5 \textit{ is odd.}$$
$$\mathrm{FFI}(C_m \times P_3) = \{2i - 5m \mid i \in \{6, 5m\} \cup [8, 5m-3]\} \textit{ if } m \geq 6 \textit{ is even.}$$
$$\mathrm{FFI}(C_m \times P_3) = \{2i - 5m \mid i \in [7, 5m-2]\} \textit{ if } m \geq 5 \textit{ is odd.}$$
$$\mathrm{FFI}(C_3 \times P_n) = \{2i - 10n - 3 \mid i \in \{3\} \cup [6, 5n-3]\} \textit{ if } n \geq 4 \textit{ is even.}$$
$$\mathrm{FFI}(C_3 \times P_n) = \{2i - 10n - 3 \mid i \in [5, 5n+2]\} \textit{ if } n \geq 5 \textit{ is odd.}$$
$$\mathrm{FFI}(C_3 \times P_2) = \{-3, 1, 5\}.$$
$$\mathrm{FFI}(C_3 \times P_3) = \{2i - 15 \mid i \in [5, 12]\}.$$
$$\mathrm{FFI}(C_4 \times P_3) = \{2i - 20 \mid i \in [6, 16] \cup \{20\}\}.$$

Together with [10, 11] (the results are listed as follows), the full friendly index set of $C_m \times P_n$, for all m and n, are completely determined.

For $m, n \geq 4$, $\mathrm{FFI}(C_m \times P_n)$ *is given by*

$$\big\{ -2mn + m + 2i \mid i \in [2n+2, 2mn-m-4] \cup \{2n, 2mn-m\} \big\}$$

for $m \geq 2n+2$ *and* m, n *are even;*

$$\big\{ -2mn + m + 2i \mid i \in [m+4, 2mn-m-4] \cup \{m+2, 2mn-m\} \big\}$$

for $m \leq 2n-2$, m *is even and* n *is odd;*

$$\big\{ -2mn + m + 2i \mid i \in [2n+2, 2mn-m-4] \cup \{2n, 2mn-m\} \big\}$$

for $m \geq 2n$ *and* m *is even and* n *is odd;*

$$\big\{ -2mn + m + 2i \mid i \in [m+4, 2mn-m-n] \cup \{m\} \big\}$$

for $m \leq 2n-3$, m *is odd and* n *is even;*

$$\big\{ -2mn + m + 2i \mid i \in [m+2, 2mn-m-n] \cup \{m\} \big\}$$

for $m = 2n-1$ *and* n *is even;*

$$\big\{ -2mn + m + 2i \mid i \in [2n, 2mn-m-n] \big\}$$

for $m \geq 2n+1$ *and* m *is odd and* n *is even;*

$$\big\{ -2mn + m + 2i \mid i \in [m+4, 2mn-m-n] \cup \{m+2\} \big\}$$

for $m \leq 2n-3$ *and* m, n *are odd;*

$$\big\{ -2mn + m + 2i \mid i \in [2n+1, 2mn-m-n] \big\}$$

for $m \geq 2n-1$ *and* m, n *are odd.*

Acknowledgments

This work is supported by Faculty Research Grant, Hong Kong Baptist University.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, **244**, Springer, New York, 2008.

[2] G. Chartrand, S.-M. Lee and P. Zhang, Uniformly cordial graphs, *Discrete Math.*, **306** (2006) 726–737.

[3] H. Kwong, S.-M. Lee and H. K. Ng, On friendly index sets of 2-regular graphs, *Discrete Math.*, **308** (2008) 5522–5532.

[4] H. Kwong and S.-M. Lee, On friendly index sets of generalized books, *J. Combin. Math. Combin. Comput.*, **66** (2008) 43–58.

[5] S.-M. Lee and H. K. Ng, On friendly index sets of bipartite graphs, *Ars Combin.*, **86** (2008) 257–271.

[6] E. Salehi and S.-M. Lee, On friendly index sets of trees, *Congr. Numer.*, **178** (2006) 173–183.

[7] W. C. Shiu and H. Kwong, Full friendly index sets of $P_2 \times P_n$, *Discrete Math.*, **308** (2008) 3688–3693.

[8] W. C. Shiu and M. H. Ling, Extreme friendly indices of $C_m \times C_n$, *Congr. Numer.*, **188** (2007) 175–182.

[9] W. C. Shiu and M. H. Ling, Full friendly index sets of Cartesian products of two cycles, *Acta Math. Sin. (Engl. Ser.)*, **26** (2010) 1233–1244.

[10] W. C. Shiu and F. S. Wong, Extreme friendly indices of $C_m \times P_n$, *Congr. Numer.*, **197** (2009) 65–75.

[11] W. C. Shiu and F. S. Wong, Full friendly index sets of cylinder graphs, *Australas. J. Combin.*, **52** (2012) 141–162.

Wai Chee Shiu

Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Kowloon Tong, Hong Kong, China

Email: wcshiu@hkbu.edu.hk

Man-Ho Ho

Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Kowloon Tong, Hong Kong, China

Email: homanho@math.hkbu.edu.hk