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Wai Chee Shiu
Hong Kong Baptist University, wcshiu@hkbu.edu.hk

Man-Ho Ho
Hong Kong Baptist University

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FULL FRIENDLY INDEX SETS OF SLENDER AND FLAT CYLINDER GRAPHS

W. C. SHIU* AND M.-H. HO

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ABSTRACT. Let $G = (V, E)$ be a connected simple graph. A labeling $f : V \to \mathbb{Z}_2$ induces an edge labeling $f^* : E \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |f^*-1(i)|$. A labeling $f$ is called friendly if $|v_f(1) - v_f(0)| \leq 1$. The full friendly index set of $G$ consists all possible differences between the number of edges labeled by 1 and the number of edges labeled by 0. In recent years, full friendly index sets for certain graphs were studied, such as tori, grids $P_2 \times P_n$, and cylinders $C_m \times P_n$ for some $n$ and $m$. In this paper we study the full friendly index sets of cylinder graphs $C_m \times P_2$ for $m \geq 3$, $C_m \times P_3$ for $m \geq 4$ and $C_3 \times P_n$ for $n \geq 4$. The results in this paper complement the existing results in literature, so the full friendly index set of cylinder graphs are completely determined.

1. Introduction

Let $G = (V, E)$ be a simple connected graph. A vertex labeling $f : V \to \mathbb{Z}_2$ induces an edge labeling $f^* : E \to \mathbb{Z}_2$, given by

$$f^*(xy) := f(x) + f(y),$$

where $xy \in E$. For $i \in \mathbb{Z}_2$, define $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |(f^*)^{-1}(i)|$, i.e., $v_f(i)$ is the number of vertices labeled by $i$ and $e_f(i)$ is the number of edges labeled by $i$. A vertex labeling $f$ is said to be friendly if

$$|v_f(1) - v_f(0)| \leq 1.$$
For a friendly labeling $f$ of a graph $G$ the friendly index of $G$ with respect to $f$, denoted by $i_f(G)$, is defined to be
\[ i_f(G) := e_f(1) - e_f(0). \]
The friendly index set $\text{FI}(G)$ of $G$ is defined to be
\[ \text{FI}(G) = \{ |i_f(G)| | f \text{ is a friendly labeling of } G \}. \]
In [7] Shiu-Kwong generalize the friendly index set to the full friendly index set $\text{FFI}(G)$:
\[ \text{FFI}(G) = \{ |i_f(G)| | f \text{ is a friendly labeling of } G \}. \]
Friendly index of some graphs are studied in [4, 3, 5, 6]. Let $m \geq 3$ and $n \geq 2$. Denote by $C_m$ an $m$-cycle and $P_n$ an $n$-path. The full friendly index sets are studied in the case of a torus $C_m \times C_n$ [8,9], a cylinder $C_m \times P_n$ for $m,n \geq 4$ [10,11] and a grid $P_2 \times P_n$ [7]. In this paper we study the full friendly index sets of cylinder graphs $C_m \times P_n$ for arbitrary $m$ and $n$ are completely determined.
Henceforth the term “labeling” on a graph $G$ means a vertex labeling from $V(G)$ to $\mathbb{Z}_2$.

2. Notation and preliminary results

We refer to [1] for general notions of graphs. Let $m \geq 3$ and $n \geq 2$. Denote by $C_m$ an $m$-cycle and $P_n$ an $n$-path. The Cartesian product $C_m \times P_n$ is a cylinder graph with $mn$ vertices labeled by $u_{ij}$ (or $u_{i,j}$), where $1 \leq i \leq m$ and $1 \leq j \leq n$. The size of $C_m \times P_n$ is $2mn - m$. Two vertices $u_{ij}$ and $u_{hk}$ of $C_m \times P_n$ are adjacent if either
\[ i = h \text{ and } j = k \pm 1, \quad \text{or} \]
\[ j = k \text{ and } i \equiv h \pm 1 \pmod{m} \]
We recall some results of the extremely friendly index of $C_m \times P_n$ in [10].

**Theorem 2.1.** [10] Theorem 2.4] If $f$ is a friendly labeling of $C_m \times P_n$, then
\[ i_f(C_m \times P_n) \leq \begin{cases} 2mn - m - 2n, & \text{if } m \text{ is odd;} \\ 2mn - m, & \text{if } m \text{ is even.} \end{cases} \]

**Theorem 2.2.** [10] Theorems 3.2–3.5] Let $f$ be a friendly labeling of $C_m \times P_n$.

(1) Suppose $n$ is even.
(a) If $m \leq 2n$, then $i_f(C_m \times P_n) \geq 3m - 2mn$.
(b) If $m \geq 2n$, then
\[ i_f(C_m \times P_n) \geq \begin{cases} 4n + m + 2 - 2mn, & \text{if } m \text{ is odd;} \\ 4n + m - 2mn, & \text{if } m \text{ is even.} \end{cases} \]

(2) Suppose $n$ is odd.
(a) If \( m \leq 2n - 1 \), then \( i_f(C_m \times P_n) \geq 3m + 4 - 2mn \).

(b) If \( m \geq 2n - 2 \), then

\[
i_f(C_m \times P_n) \geq \begin{cases} 
4n + m + 2 - 2mn, & \text{if } m \text{ is odd}; \\
4n + m - 2mn, & \text{if } m \text{ is even}.
\end{cases}
\]

3. Non-existence of friendly indices of \( C_m \times P_n \)

In the previous section we recall the upper bound and the lower bound of the friendly index of the graph \( C_m \times P_n \). In this section we prove that some integers lying between the upper bound and the lower bound cannot be the friendly index of \( C_m \times P_n \).

We begin with some elementary observations.

**Lemma 3.1.** Let \( f \) be a friendly labeling of \( C_m \times P_2 = (V,E) \). Then

\[
v_f(1) \equiv m \pmod{2}.
\]

**Proof.** Since the degree of each of the vertices of \( C_m \times P_2 \) is 3, it follows that

\[
e_f(1) \equiv \sum_{e \in E} f^*(e) = \sum_{v \in V} \deg(v)f(v) = \sum_{v \in V} 3f(v) \equiv 3v_f(1) \equiv v_f(1) \pmod{2}.
\]

Since \( f \) is a friendly labeling, it follows that \( v_f(0) = v_f(1) = m \). Thus \( v_f(1) \equiv m \pmod{2} \). \( \square \)

**Theorem 3.2.** \([11], \text{Theorem 2.1}\) For even \( m \) with \( m \geq 4 \) and \( n \geq 2 \), there is no friendly labeling \( f \) of \( C_m \times P_n \) such that \( e_f(1) = 2mn - m - p \), where \( p = 1, 2, 3 \).

Let \( G \) be a graph and \( f : V \rightarrow \mathbb{Z}_2 \) a vertex labeling of \( G \). A subgraph \( H \) of \( G \) is said be to mixed with respect to \( f \) if there are two vertices \( u, v \in V(H) \) such that \( f(u) = 1 \) and \( f(v) = 0 \). An edge \( e \in E(G) \) is called an \( k \)-edge if \( f^*(e) = k \), where \( k \in \mathbb{Z}_2 \).

Clearly, mixed cycles and mixed paths contain at least two 1-edges and one 1-edge, respectively. Let \( k \in \mathbb{Z}_2 \). A cycle \( C \) is called an \( k \)-pure cycle, where \( k \in \mathbb{Z}_2 \), with respect to \( f \) if \( f(u) = k \) for all \( u \in V(C) \). We define \( k \)-pure path in a similar fashion.

A path in \( C_m \times P_n \) of the form \( u_{i_1}u_{i_2} \cdots u_{i_m} \) is called a vertical path for each fixed \( 1 \leq i \leq m \). A cycle in \( C_m \times P_n \) of the form \( u_{1j}u_{2j} \cdots u_{mj}u_{1j} \) is called a horizontal cycle for each fixed \( 1 \leq j \leq n \).

**Lemma 3.3.** \([11], \text{Lemma 2.2}\) For even \( m \), if \( C_m \times P_n \) contains a vertical mixed path under a friendly labeling \( f \), then the number of vertical mixed paths is at least two.

**Lemma 3.4.** \([7], \text{Corollary 5}\) Let \( f \) be a labeling of a graph \( G \) that contains a cycle \( C \) as its subgraph. If \( C \) contains a 1-edge, then the number of 1-edges in \( C \) is a positive even number.

**Lemma 3.5.** Let \( m \geq 6 \) be even. If \( C_m \times P_3 \) contains a horizontal pure cycle (either a 1-pure cycle or a 0-pure cycle) and a horizontal mixed cycle with respect to a friendly labeling \( f \), then \( e_f(1) \geq 8 \).
Proof. Let \( r \) be the number of horizontal 1-pure cycles and \( s \) the number of horizontal 0-pure cycles. Since \( f \) is a friendly labeling, it follows that \( 0 \leq r, s \leq 1 \). There are two cases.

1. Suppose \( r = 1 \) and \( s = 0 \). Then there are two horizontal mixed cycles, each of which has at least two 1-edges. Since \( v_f(0) = \frac{3m}{2} \), there are at least \( \frac{3m}{4} \) mixed vertical paths. Thus \( e_f(1) \geq 4 + \frac{3m}{4} > 8 \). Hence \( e_f(1) \geq 9 \). The case \( r = 0 \) and \( s = 1 \) is similar.

2. Suppose \( r = 1 = s \). Then there is one horizontal mixed cycle, and all vertical paths are mixed. Thus \( e_f(1) \geq 2 + m \geq 2 + 6 = 8 \).

\( \square \)

Proposition 3.6. Let \( m \geq 6 \) be even. There is no friendly labeling \( f \) of \( C_m \times P_3 \) such that \( e_f(1) = 7 \).

Proof. Let \( a \) be the number of horizontal mixed cycles and \( b \) the number of vertical mixed paths of \( C_m \times P_3 \). Note that \( a \neq 0 \) by friendliness, and \( b \neq 1 \) by Lemma 3.3. If \( a = 1 \) or 2, then \( e_f(1) \geq 8 \) by Lemma 3.5.

Suppose \( a = 3 \). If \( b = 0 \), then by Lemma 3.4 each horizontal mixed cycle contains at least two 1-edges. Thus \( e_f(1) \geq 2 + 2 + 2 = 6 \). Note that in this case \( e_f(1) \) cannot be an odd integer by the same reason. If \( b \geq 2 \), then \( e_f(1) \geq 2 + 2 + 2 + b \geq 8 \), where the 2’s follows from the reason as above.

Combining all these cases together we conclude that \( e_f(1) \neq 7 \). \( \square \)

Lemma 3.7. [11, Lemma 2.3] Let \( n \) be even. If \( C_m \times P_n \) contains a horizontal mixed cycle with respect to a friendly labeling \( f \), then the number of horizontal mixed cycles is at least two.

Lemma 3.8. [11, Lemma 2.4] Let \( n \geq 4 \) be even and \( 3 \leq m \leq 2n \). If \( C_m \times P_n \) contains a horizontal pure cycle and a horizontal mixed cycle with respect to a friendly labeling \( f \), then

\[
e_f(1) \geq \begin{cases} m + 4, & \text{if } m \text{ is odd;} \\ m + 3, & \text{if } m \text{ is even and } m = 2n; \\ m + 4, & \text{if } m \text{ is even and } m \leq 2n - 2. \end{cases}
\]

Lemma 3.9. Let \( n \geq 4 \) be even. There is no friendly labeling \( f \) of \( C_3 \times P_n \) such that \( e_f(1) = 4, 5 \).

Proof. Let \( a \) be the number of horizontal mixed cycles and \( b \) the number of vertical mixed paths. If \( b = 0 \), then all three vertical paths are pure and therefore \( |v_f(1) - v_f(0)| \geq n \geq 4 \), contradicting to the assumption that \( f \) is a friendly labeling. Thus \( b \neq 0 \). We consider the following three cases for \( a \).

1. Suppose \( a = 0 \). Then all three vertical paths are identical. Thus \( e_f(1) \) is a multiple of 3, so \( e_f(1) \neq 4, 5 \).

2. Suppose \( 1 \leq a < n \). Then \( C_3 \times P_n \) contains a horizontal mixed cycles and at least one pure cycle. By Lemma 3.8 we have \( e_f(1) \geq 3 + 4 = 7 \).

3. Suppose \( a = n \). Since \( b \geq 1 \), it follows from Lemma 3.4 that \( e_f(1) \geq 2n + b \geq 8 + 1 = 9 \).

Combining all these cases together we conclude that \( e_f(1) \neq 4, 5 \). \( \square \)
4. Elementary operations on vertex labeling

In this section we prove some results that will be useful in studying the full friendly index set of $C_m \times P_n$.

Let $f$ be a labeling of $C_m \times P_n$. An $n \times m$ matrix $A_f$, whose $(j,i)$-entry is defined by $(A_f)_{ji} = f(u_{ij})$, is called the labeling matrix of $C_m \times P_n$ under $f$. For convenience, we write $f$ for $A_f$. Let $[a,b] = \{i \in \mathbb{Z} \mid a \leq i \leq b\}$. We denote by $O_{p,q}$ and $J_{p,q}$ the $p \times q$ zero matrix and the $p \times q$ matrix whose entries are 1 respectively.

For a given matrix $A$, define a row operation $\sigma_i$ on $A$ by shifting the $i$-th row of $A$ to the right by 1 entry (the last entry of the $i$-th row shifts to the first entry). Denote by $\sigma_i(A)$ the resulting matrix.

**Proposition 4.1.** Consider $C_m \times P_2$ with a labeling $f$ represented by the matrix

$$f = \begin{pmatrix} J_{2,\lfloor m/2 \rfloor} & O_{2,\lfloor m/2 \rfloor} \end{pmatrix}.$$ 

For $0 \leq j \leq \lfloor m/2 \rfloor$, let $f_j = \sigma_1^j(f)$, where $\sigma_1^j := \sigma_1 \circ \cdots \circ \sigma_1$. Then $e_{f_j}(1) = 4 + 2j$.

**Proof.** Note that $f$ is friendly for even $m$ but not for odd $m$. Note also that shifting the vertex labeling of first horizontal cycle will not change the number of 1-edges in the horizontal cycle; it will only change the number of 1-edges in the vertical paths.

Thus, there are two more 1-edges in the vertical paths. It is easy to see that $e_{f_j}(1) - e_{f_{j-1}}(1) = 2$ for each $1 \leq j \leq \lfloor m/2 \rfloor + 1$. Thus $e_{f_j}(1) = 4 + 2j$.

**Proposition 4.2.** Consider $C_m \times P_2$.

(1) Let $f$ be a friendly labeling of $C_m \times P_2$ represented by

$$f_1 = \begin{pmatrix} 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lfloor m/2 \rfloor - 1 & \lfloor m/2 \rfloor - 1 & \cdots & \lfloor m/2 \rfloor - 1 & \lfloor m/2 \rfloor - 1 & \cdots & \lfloor m/2 \rfloor - 1 \\ \end{pmatrix}.$$ 

Interchange the labeling of the $2j$-th column of the above matrix for all $1 \leq j \leq k$, and denote by $f_k$ the resulting labeling with $f_0 := f$. Then $e_f(1) = m$ and $e_{f_k}(1) = m + 4k$ for each $0 \leq k \leq \lfloor m/2 \rfloor$.

(2) Let $g$ be a friendly labeling of $C_m \times P_2$ represented by

$$g_1 = \begin{pmatrix} 1 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 1 \\ \end{pmatrix}.$$ 

Interchange the labeling of the $2j$-th column of the above matrix for all $1 \leq j \leq k$, and denote by $g_k$ the resulting labeling with $g_0 := g$. Then $e_g(1) = m + 2$ and $e_{g_k}(1) = m + 2 + 4k$ for each $0 \leq k \leq \lfloor m/2 \rfloor - 2$. 
Proof. Note that interchanging the labeling of the columns will only change the number of 1-edges in the horizontal cycles and will not change the number of 1-edges of the vertical paths.

(1) It is obvious that $e_f(1) = m$. Note that $f_1$ is obtained by interchanging the second column of the labeling $f$, and the resulting matrix is

$$
\begin{pmatrix}
1 & 0 & 1 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0
\end{pmatrix}.
$$

Thus four more 1-edges are obtained from the horizontal cycles. It is easy to see that $e_{f_k}(1) - e_{f_{k-1}}(1) = 4$ for all $1 \leq k \leq \lfloor m/2 \rfloor$. Thus $e_{f_k}(1) = m + 4k$.

(2) Similar to the above proof. □

For a friendly labeling $f$ of a graph $G$, we have

$$
e_f(1) - e_f(0) = 2e_f(1) - |E(G)|.
$$

To compute FFI($G$), it suffices to compute the set

$$
a(G) = \{ e_f(1) \mid f \text{ is a friendly labeling of } G \}.
$$

Then FFI($G$) = \{ $2i - |E(G)| \mid i \in a(G)$\}.

By substituting $m$ by $2m$ and $2m + 1$ in Proposition 4.2 we have

**Corollary 4.3.** For $m \geq 2$

$$
\{2m + 2i \mid i \in [0, 2m] \setminus \{2m - 1\}\} = \{2i \mid i \in [m, 3m] \setminus \{3m - 1\}\} \subseteq a(C_{2m} \times P_2),
$$

and for $m \geq 1$

$$
\{2m + 1 + 2i \mid i \in [0, 2m]\} = \{2i + 1 \mid i \in [m, 3m]\} \subseteq a(C_{2m+1} \times P_2).
$$

The following lemma is obvious.

**Lemma 4.4.** Let $f$ be a friendly labeling on $C_4 \times P_3$ represented by

$$
\begin{pmatrix}
1 & 0 & 1 & 0 \\
* & 1 & 0 & * \\
1 & 0 & 1 & 0
\end{pmatrix},
$$

where $*$ is either 1 or 0. Interchange the (1,2)-entry with (1,3)-entry of $f$ (or the (3,2)-entry with (3,3)-entry, not both) decreases $e_f(1)$ by 4. Interchange the (1,2)-entry with (1,3)-entry and the (3,2)-entry with (3,3)-entry decreases $e_f(1)$ by 8.

**Proposition 4.5.** Consider the labeling $f = \left( J_{\lfloor n/2 \rfloor, 3} \atop O_{\lfloor n/2 \rfloor, 3} \right)$ on $C_3 \times P_n$. Interchange the ($\lfloor n/2 \rfloor - i + 1, 3$)-entry with the ($\lfloor n/2 \rfloor + i, 1$)-entry of $f$ for each $1 \leq i \leq k$, where $k \leq \lfloor n/2 \rfloor - 1$, and denote by $f_k$ the resulting labeling. Then $e_f(1) = 3$ and $e_{f_k}(1) = 3 + 4k$. 
Proof. Note that $f$ is friendly for even $n$ but not for odd $n$. Note also that $e_f(1) = 3$. After interchanging the $([n/2], 3)$-entry with the $([n/2] + 1, 1)$-entry from $f$, we have the following matrix

$$f_1 = \begin{pmatrix} J_{[n/2]-1,3} \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ O_{[n/2]-1,3} \end{pmatrix}. $$

From the above matrix we see that $e_{f_1}(1) = 7 = 3 + 4$. After interchanging the $([n/2] - 1, 3)$-entry with the $([n/2] + 2, 1)$-entry from $f_1$, we have

$$f_2 = \begin{pmatrix} J_{[n/2]-2,3} \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ O_{[n/2]-2,3} \end{pmatrix}. $$

From the above matrix we see that $e_{f_2}(1) = 11 = e_{f_1}(1) + 4$. It is easy to see that $e_{f_k}(1) - e_{f_{k-1}}(1) = 4$ for each $k \leq [n/2] - 1$, and therefore $e_{f_k}(1) = 3 + 4k$. \[\Box\]

5. Realizing the full friendly index set

In this section we realize all the potential friendly indices of $C_m \times P_n$ for some $n$ and $m$.

In the following we determine for $a(C_m \times P_2)$ for $m \geq 4$, $a(C_m \times P_3)$ for $m \geq 4$ and $a(C_3 \times P_n)$ for $n \geq 4$.

Theorem 5.1. For $m \geq 2$, we have

$$a(C_{2m} \times P_2) = \{2i \mid i \in [2, 3m] \setminus \{3m - 1\}\}. $$

Proof. Let $\phi$ be any friendly labeling of $C_{2m} \times P_2$. By Theorem 2.1 and (1)(b) of Theorem 2.2 we have

$$6m \geq i_{\phi}(C_{2m} \times P_2) \geq 8 - 6m. $$

On the other hand, we have

$$i_{\phi}(C_{2m} \times P_2) = 2e_{\phi}(1) - |E(C_{2m} \times P_2)| = 2e_{\phi}(1) - 6m. $$

It follows that $6m \geq e_{\phi}(1) \geq 4$.

Let $f$ be the labeling of $C_{2m} \times P_2$ of Proposition 4.1. Note that $f$ is friendly. By Proposition 4.1 we have $\{2i \mid i \in [2, m + 2]\} \subseteq a(C_{2m} \times P_2)$. The result follows from Corollary 4.3. \[\Box\]

Theorem 5.2. For $m \geq 2$, we have

$$a(C_{2m+1} \times P_2) = \{2i + 1 \mid i \in [2, 3m]\}. $$
Proof. Let \( \phi \) be any friendly labeling of \( C_{2m+1} \times P_2 \). By Theorem 2.1 and (1)(b) of Theorem 2.2 we have

\[
6m - 1 \geq i_\phi(C_{2m} \times P_2) \geq 7 - 6m.
\]

It follows that \( 6m + 1 \geq e_\phi(1) \geq 5. \)

Let \( f \) be a labeling on \( C_{2m+1} \times P_2 \) represented by the matrix

\[
\begin{pmatrix}
J_{2,m} & 0 \\
0 & O_{2,m}
\end{pmatrix}.
\]

Note that \( f \) is a friendly labeling of \( C_{2m+1} \times P_2 \) and \( e_f(1) = 5 + 2j \) for \( 0 \leq j \leq m \). Thus \( \{2i + 1 \mid i \in [2, m + 2]\} \subseteq a(C_{2m+1} \times P_2) \).

The result follows from Corollary 4.3. \( \square \)

**Theorem 5.3.** For \( m \geq 3 \), we have

\[
a(C_{2m} \times P_3) = \{6, 10m\} \cup \{8, 10m - 4\}.
\]

**Proof.** Let \( \phi \) be any friendly labeling of \( C_{2m} \times P_3 \). By Theorem 2.1 and (2)(b) of Theorem 2.2 we have

\[
10m \geq i_\phi(C_{2m} \times P_3) \geq 12 - 10m.
\]

That means \( 10m \geq e_\phi(1) \geq 6. \) By Theorem 3.2, \( e_\phi(1) \notin \{10m - 1, 10m - 2, 10m - 3\} \).

Let \( f \) be a labeling on \( C_{2m} \times P_3 \) represented by the matrix \( (J_{3,m} \ O_{3,m}) \). Note that \( f \) is a friendly labeling of \( C_{2m} \times P_2 \) and \( e_f(1) = 6. \) Let \( f_j = \sigma_3^j(f) \) for \( 0 \leq j \leq m \). Similar to the proof of Proposition 4.1 we have \( e_f(1) = 6 + 2j \) for \( 0 \leq j \leq m \). Thus \( \{2i \mid i \in [3, m + 3]\} \subseteq a(C_{2m} \times P_3) \).

The matrix representing \( f_m \) is given by

\[
\begin{pmatrix}
J_{2,m} & O_{2,m} \\
O_{1,m} & J_{1,m}
\end{pmatrix}.
\]

Consider \( \sigma_3^j(f_m) \) for \( 0 \leq j \leq m \). Similar to the proof of Proposition 4.1 we see that \( \{2i \mid i \in [m + 3, 2m + 3]\} \subseteq a(C_{2m} \times P_3) \).

Consider another labeling \( g \) of \( C_{2m} \times P_3 \) represented by the matrix

\[
\begin{pmatrix}
J_{3,m-1} & 1 & 1 \\
0 & 0 & 0 \\
1 & 0 & O_{3,m-1}
\end{pmatrix}.
\]

Note that \( g \) is a friendly labeling and \( e_g(1) = 9 \). Consider \( \sigma_3^j(g) \) for \( 0 \leq j \leq m - 1 \). Similar to the proof of Proposition 4.1 we have \( \{2i + 1 \mid i \in [4, m + 3]\} \subseteq a(C_{2m} \times P_3) \). Let \( \bar{g} = \sigma_1^{m-1}(g) \). Consider \( \sigma_3^j(\bar{g}) \) for \( 0 \leq j \leq m - 1 \). Similarly we have \( \{2i + 1 \mid i \in [m + 3, 2m + 2]\} \subseteq a(C_{2m} \times P_3) \).

Combining the above cases, we have \( \{6\} \cup \{8, 4m + 6\} \subseteq a(C_{2m} \times P_3) \).
By Theorem 2.1 we have \( e_\phi(1) \leq 10m \) for any friendly labeling \( \phi \). Let \( h \) be a labeling of \( C_{2m} \times P_3 \) whose matrix representation is given by

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0
\end{pmatrix}.
\]

Then \( h \) is a friendly labeling and \( e_h(1) = 10m \).

**Case 1:** Suppose \( m = 2k \) for some \( k \geq 2 \). Then we can subdivide the above matrix into \( k \) submatrices (blocks) of size \( 3 \times 4 \) starting from the first column. Apply the procedure in Lemma 4.4 to the first row and the third row in each of these blocks consecutively, we see that \( \{ 4i \mid i \in [3k, 5k] \} \subseteq a(C_{2m} \times P_3) \).

Consider the labelings \( p, q \) and \( r \) whose matrix representations are of the form

\[
p = \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0
\end{pmatrix},
q = \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix},
r = \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}.
\]

Note that \( p, q \) and \( r \) are friendly labelings of \( C_{2m} \times P_3 \) and \( e_p(1) = 20k - 5 \), \( e_q(1) = 20k - 6 \) and \( e_r(1) = 20k - 7 \). By applying the procedure of Lemma 4.4 to the first \( k - 1 \) blocks of all these matrices consecutively, we see that \( \{ 4i + 3 \mid i \in [3k, 5k - 2] \} \), \( \{ 4i + 2 \mid i \in [3k, 5k - 2] \} \) and \( \{ 4i + 1 \mid i \in [3k, 5k - 2] \} \) are subsets of \( a(C_{2m} \times P_3) \).

Combining the above four cases, we have \([6m, 10m - 4] \cup \{10m \} \subseteq a(C_{2m} \times P_3)\).

Let \( s \) be a friendly labeling of \( C_{2m} \times P_3 \) whose matrix representation is given by

\[
s = \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0
\end{pmatrix}.
\]

Note that \( e_s(1) = 6m \). By applying a similar procedure in Lemma 4.4 to the first row of each block of \( s \) consecutively, we see that \( \{ 4i \mid i \in [2k, 3k] \} \subseteq a(C_{2m} \times P_3) \).
Consider the labelings \( t, u \) and \( v \) of \( C_{2m} \times P_3 \) whose matrix representations are given by

\[
\begin{align*}
t &= \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0
\end{pmatrix}, \\
u &= \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0
\end{pmatrix}, \\
v &= \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0
\end{pmatrix}.
\end{align*}
\]

Note that \( t, u \) and \( v \) are friendly labelings of \( C_{2m} \times P_3 \) and \( e_t(1) = 6m + 2 \), \( e_u(1) = 6m + 1 \) and \( e_v(1) = 6m - 1 \). By applying a similar procedure of Lemma 4.4 to the first row of each boxed block of these matrices, we see that \( \{4i+2 \mid i \in [2k+1, 3k]\} \), \( \{4i+1 \mid i \in [2k+1, 3k]\} \) and \( \{4i+3 \mid i \in [2k, 3k-1]\} \) are subsets of \( a(C_{2m} \times P_3) \).

Combining the above four cases, we have \( [4m+3, 6m+2] \subseteq a(C_{2m} \times P_3) \). The theorem holds for even \( m \) by considering all the above cases.

**Case 2:** Suppose \( m = 2k + 1 \) for some \( k \geq 1 \). We shall keep the labelings \( h, p, q, r, s, t, u \) and \( v \) for \( m = 2k \). Let

\[
A = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}.
\]

We construct a labeling \( \overline{h} \) similar to \( h \) in Case 1 by inserting the sub-matrix \( A \) into \( h \) as the last two columns. Then \( e_{\overline{h}}(1) = 20k + 10 = 10m \). Similar to Case 1 (i.e., apply the procedure in Lemma 4.4 to the first \( k \) blocks consecutively), we have \( \{4i + 10 \mid i \in [3k, 5k]\} \).

Construct labelings \( p, q \) and \( r \) by inserting the sub-matrix \( A \) into \( p, q \) and \( r \) between the last fifth and the last fourth column, respectively. Then \( e_p(1) = 20k + 5 \), \( e_q(1) = 20k + 4 \) and \( e_r(1) = 20k + 3 \). Similar to Case 1, after combining the above four cases, we have \( [6m + 4, 10m - 4] \cup \{10m\} \subseteq a(C_{2m} \times P_3) \). Denote by \( \overline{p}_{k-1} \) the labeling after the procedure in Lemma 4.4 is applied \( k-1 \) times. Then \( e_{\overline{p}_{k-1}}(1) = 20k + 5 - 8(k - 1) = 12k + 13 \). By swapping the entries of the first row of \( A \) in \( \overline{p}_{k-1} \), we see that \( e(1) = 12k + 9 = 6m + 3 \).

Similarly, let \( \overline{r} \) be obtained from \( s \) by inserting the sub-matrix \( B \) as the last two columns. We also construct labelings \( \overline{t}, \overline{u} \) and \( \overline{v} \) by inserting the sub-matrix \( B \) into \( t, u \) and \( v \) between the last fifth and last the fourth column, respectively. Then \( e_{\overline{t}}(1) = 12k + 6 \), \( e_{\overline{u}}(1) = 12k + 8 \), \( e_{\overline{v}}(1) = 12k + 7 \) and \( e_{\overline{r}}(1) = 12k + 5 \). Similar to Case 1, we will obtain \( \{4m+5\} \cup [4m+7, 6m+2] \subseteq a(C_{2m} \times P_3) \). Note that \( 4m+6 \) is covered before defining the labeling \( h \).

The theorem now holds for odd \( m \). □
Theorem 5.4. For \( m \geq 2 \), we have

\[ a(C_{2m+1} \times P_3) = [7, 10m + 2]. \]

Proof. For \( m = 2k \), let \( f_j \) be the labelings of \( C_{2m} \times P_3 \) defined in the proof of Theorem 5.3, \( 0 \leq j \leq m \).

Let \( \tilde{f}_j \) be the labeling obtained from \( f_j \) by inserting the sub-matrix \( A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \) as the last column.

Note that \( \tilde{f}_j \) is friendly. Similar to the proof of Theorem 5.3 we have \( \{7 + 2i \mid 0 \leq i \leq 2m\} \setminus \{9\} \subseteq a(C_{2m+1} \times P_3) \). If we replace the sub-matrix \( A \) in \( \tilde{f}_j \) by \( B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \), then it is easy to see that \( \{8 + 2i \mid 0 \leq i \leq 2m\} \subseteq a(C_{2m+1} \times P_3) \). On the other hand, it is easy to see that \( e_{\sigma_1(\mathcal{I}_0)}(1) = 9 \).

Consider the labeling \( h \) represented by the following matrix

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

Note that \( h \) is a friendly labeling of \( C_{2m+1} \times P_3 \) and \( e_h(1) = 10m + 2 \). Apply the procedure in Lemma 4.4 to the first row and the third row in each of first \( k \) blocks consecutively, we see that \( \{10m + 2 - 4i \mid 0 \leq i \leq 2k\} \subseteq a(C_{2m+1} \times P_3) \). Consider the labelings \( p, q \) and \( r \) represented by the matrices

\[
p = \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1
\end{pmatrix},
\]

\[
q = \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1
\end{pmatrix},
\]

\[
r = \begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1
\end{pmatrix}.
\]

They are friendly and \( e_p(1) = 10m + 1 \), \( e_q(1) = 10m \), and \( e_r(1) = 10m - 1 \). Similarly, apply the procedure in Lemma 4.4 to \( p, q \) and \( r \) we see that \( \{10m + 1 - 4i \mid 0 \leq i \leq 2k\} \) and \( \{10m - 4i \mid 0 \leq i \leq 2k\} \) and \( \{10m - 1 - 4i \mid 0 \leq i \leq 2k\} \) are subsets of \( a(C_{2m+1} \times P_3) \). Combining these four cases we see that \( [6m - 1, 10m + 2] \subseteq a(C_{2m+1} \times P_3) \).
Consider the labelings
\[
\begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 \\
\end{pmatrix},
\]
\[
\begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0 & 1 \\
\end{pmatrix},
\]
\[
\begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
\]
\[
\begin{pmatrix}
1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\
\end{pmatrix}.
\]

These are friendly labelings and \(e_t(1) = 6m + 4\), \(e_u(1) = 6m + 3\), \(e_v(1) = 6m + 2\) and \(e_w(1) = 6m + 1\).

Similar to the proof of Case 1 of Theorem 5.3, we see that \([4m + 1, 6m + 4] \subseteq a(C_{2m+1} \times P_3)\).

By considering all the above cases, the theorem holds when \(m\) is even. When \(m\) is odd, one can prove the theorem similar to the proof of Case 2 of Theorem 5.3. Thus the theorem holds for all \(m \geq 2\). □

**Theorem 5.5.** For \(n \geq 3\), we have
\[a(C_3 \times P_{2n}) = \{3\} \cup [6, 10n - 3].\]

**Proof.** Let \(\varphi\) be any friendly labeling of \(C_3 \times P_{2n}\). By Lemma 3.9, Theorem 2.1 and (1)(a) of Theorem 2.2 we have \(10n - 3 \geq e_\varphi(1) \geq 3\) and \(e_\varphi(1) \neq 4, 5\).

Obviously, \(q = \begin{pmatrix} O_{1,3} \\ J_{n,3} \\ O_{n-1,3} \end{pmatrix}\) is a friendly labeling of \(C_3 \times P_{2n}\) and \(e_q(1) = 6\).

Let \(f\) be the labeling of \(C_3 \times P_{2n}\) in Proposition 4.5. It is a friendly labeling and \(e_f(1) = 3\). By applying the procedure in Proposition 4.5 to \(f\) we see that \(\{4k + 3 \mid k \in [0, n - 1]\} \subseteq a(C_3 \times P_{2n})\).

Consider the labelings \(g, h\) and \(\ell\) of \(C_3 \times P_{2n}\) represented by the matrices
\[
g = \begin{pmatrix} 1 & 1 & 0 \\ J_{n-1,3} \\ 1 & 0 & 0 \\ O_{n-1,3} \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ J_{n-2,3} \\ 1 & 0 & 0 \\ O_{n-1,3} \end{pmatrix}, \quad \ell = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ J_{n-2,3} \\ 1 & 1 & 0 \\ O_{n-1,3} \end{pmatrix}.
\]
Note that $g$, $h$ and $\ell$ are friendly and $e_g(1) = 8$, $e_h(1) = 9$ and $e_\ell(1) = 10$ respectively. For the labelings $g$ and $h$, interchange the $(n - i + 1, 3)$-entry with the $(n + i, 3)$-entry for $1 \leq i \leq k$ if $n \geq 4$, where $k \leq n - 3$. The resulting labelings are denoted by $g_k$ and $h_k$, respectively.

For the labeling $g$, we have

$$\{8 + 4k \mid 0 \leq k \leq n - 3\} \subseteq a(C_3 \times P_{2n}).$$

Extend the above procedure to the labeling $g$ to $k = n - 2$ and $k = n - 1$. It is easy to see that $e_{g_{n-2}}(1) = 8 + 4(n - 2) = 4n$ and $e_{g_{n-1}}(1) = 8 + 4(n - 2) + 2 = 4n + 2$. Thus

$$\{8 + 4k \mid 0 \leq k \leq n - 2\} \cup \{4n + 2\} \subseteq a(C_3 \times P_{2n}).$$

For the labeling $h$, we have

$$\{9 + 4k \mid 0 \leq k \leq n - 3\} \subseteq a(C_3 \times P_{2n}).$$

For the labeling $\ell$, first interchange the $(n, 3)$-entry with the $(n + 2, 1)$-entry, and then interchange the $(n + 1 - i, 3)$-entry with the $(n + i, 3)$-entry consecutively, for $2 \leq i \leq n - 3$ if $n \geq 5$. Then we see that

$$\{10 + 4i \mid 0 \leq i \leq n - 3\} \subseteq a(C_3 \times P_{2n}).$$

Combining all the above cases, we see that $\{3\} \cup [6, 4n] \cup \{4n + 2\} \subseteq a(C_3 \times P_{2n}).$

The matrix representing the labeling $f_{n-1}$ is given by

$$\begin{pmatrix} J_{1,3} \\ A \\ B \\ O_{1,3} \end{pmatrix},$$

where $A = \begin{pmatrix} J_{n-1,2} & O_{n-1,1} \end{pmatrix}$ and $B = \begin{pmatrix} J_{n-1,1} & O_{n-1,2} \end{pmatrix}$.

For $2 \leq k \leq n - 1$ and $n + 1 \leq k \leq 2n - 2$, shift consecutively the $k$-th row to the right by one unit if $k$ is even, and to the left by one unit if $k$ odd. Applying this procedure we get

$$\{2i - 1 \mid i \in [2n, 4n - 4]\} \subseteq a(C_3 \times P_{2n}) \text{ when } n \text{ is odd;}$$

$$\{2i - 1 \mid i \in [2n, 4n - 3]\} \setminus \{6n - 3\} \subseteq a(C_3 \times P_{2n}) \text{ when } n \text{ is even.}$$

To realize the value $6n - 3$ for even $n$, we make a special labeling as follows. Apply the above procedure up to shifting the $(n - 1)$-th row, and then shift the $(n + 1)$-th row to the right by 1 unit.

The matrix representing the labeling $g_{n-1}$ is given by

$$g_{n-1} = \begin{pmatrix} A \\ 1 & 0 & 1 \\ B \\ 0 & 0 & 0 \end{pmatrix},$$

where $A = \begin{pmatrix} J_{n,2} & O_{n,1} \end{pmatrix}$, $B = \begin{pmatrix} O_{n-2,2} & J_{n-2,1} \end{pmatrix}$.

For $1 \leq k \leq n - 1$, shift consecutively the $k$-th row to the right by 1 unit if $k$ is odd, and to the left by 1 unit if $k$ is even. It is easy to see that each operation increases $e(1)$ by 2. After these procedures, if $n \geq 5$, then we interchange the $(n + 2, 3)$-entry with the $(n + 3, 2)$-entry, the $(n + 3, 3)$-entry with the
(n + 4, 1)-entry, the (n + 4, 3)-entry with the (n + 5, 2)-entry, the (n + 5, 3)-entry with the (n + 6, 1)-entry, etc., up to interchanging the entry in the (2n − 3)-th row with the entry in the (2n − 2)-th row. Again, it is easy to see that each interchange increases e(1) by 2. Thus

\[ \{4n + 2 + 2k \mid 1 \leq k \leq 2n - 5\} \subseteq a(C_3 \times P_{2n}). \]

Let \( p \) be the labeling whose matrix representation is given by

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
& \ddots \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

For \( 1 \leq k \leq n + 2 \), shift consecutively the \( k \)-th row to the right by 1 unit if \( k \) is odd, and to the left by 1 unit if \( k \) is even. It is easy to see that each shift decreases \( e(1) \) by 2. The resulting labeling is denoted by \( p_k \) and let \( p_0 = p \). Thus

\[ \{10n - 3 - 2k \mid 0 \leq k \leq n + 2\} \subseteq a(C_3 \times P_{2n}). \]

By swapping the \((2n - 1, 3)\)-entry and \((2n, 3)\)-entry of \( p_k \) for \( 0 \leq k \leq n + 2 \), it decreases \( e(1) \) by 1. So we get

\[ \{10n - 4 - 2k \mid 0 \leq k \leq n + 2\} \subseteq a(C_3 \times P_{2n}). \]

The theorem follows from considering all the above cases. \( \Box \)

**Theorem 5.6.** For \( n \geq 2 \), \( a(C_3 \times P_{2n+1}) = [5, 10n + 2] \).

**Proof.** By Theorem 2.1 and (2) of Theorem 2.2 we have \( 10n + 2 > e_\phi(1) \geq 5 \) for any friendly labeling \( \phi \) of \( C_3 \times P_{2n+1} \).

Let \( f = \begin{pmatrix} J_{n,3} & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \). Then \( f \) is friendly and \( e_f(1) = 5 \). Interchanging the \((n - i + 1, 3)\)-entry with the \((n + i + 1, 1)\)-entry for \( 1 \leq i \leq k \) for each \( k \) \((1 \leq k \leq n - 1) \). The resulting labeling is denoted by \( f_k \). We see that \( \{4k + 1 \mid k \in [1, n]\} \subseteq a(C_3 \times P_{2n+1}) \).

Let \( g, h \) and \( \ell \) be labelings of \( C_3 \times P_{2n+1} \) whose matrix representations are given by

\[
g = \begin{pmatrix} 1 & 1 & 0 \\ J_{n,3} & 1 & 0 \\ O_{n,3} \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ J_{n-1,3} & 1 & 0 \\ O_{n,3} \end{pmatrix}, \quad \ell = \begin{pmatrix} 1 & 1 & 0 \\ J_{n-1,3} & 1 & 0 \\ O_{n,3} \end{pmatrix}.
\]
Note that $g$, $h$ and $\ell$ are friendly, and $e_g(1) = 6$, $e_h(1) = 7$ and $e_\ell(1) = 8$. For the labeling $g$, interchange the $(k,3)$-entry with the $(n+k,3)$-entry consecutively for $2 \leq k \leq n$. The resulting labeling is denoted by $g_k$. It is easy to see that each interchange increases $e(1)$ by 4. Thus

$$\{6 + 4k \mid 0 \leq k \leq n - 1\} \subseteq a(C_3 \times P_{2n+1}).$$

For the labelings $h$ and $\ell$, interchange the $(k,3)$-entry with the $(n-1+k,3)$-entry consecutively for $3 \leq k \leq n$ if $n \geq 3$. It is easy to see that each interchange increases $e(1)$ by 4. Thus

$$\{7 + 4k \mid 0 \leq k \leq n - 2\} \subseteq a(C_3 \times P_{2n+1}).$$

$$\{8 + 4k \mid 0 \leq k \leq n - 2\} \subseteq a(C_3 \times P_{2n+1}).$$

Combining the above results, we have $[5, 4n + 2] \subseteq a(C_3 \times P_{2n+1})$.

Consider

$$f_n = \begin{pmatrix} J_{1,3} \\ A \\ B \\ O_{1,3} \end{pmatrix}, \text{ where } A = \begin{pmatrix} J_{n-1,2} & O_{n-1,1} \end{pmatrix}, B = \begin{pmatrix} J_{n,1} & O_{n,2} \end{pmatrix}. $$

For $k \in [2, 2n - 1] \setminus \{n\}$, shift consecutively the $k$-th row to the left by 1 unit if $k$ is odd, and to the right by 1 unit if $k$ is even. Applying this procedure we get

$$\{4n + 1 + 2i \mid i \in [0, 2n - 3]\} \subseteq a(C_3 \times P_{2n+1}) \text{ when } n \text{ is odd;}$$

$$\{4n + 1 + 2i \mid i \in [0, 2n - 2]\} \setminus \{6n - 1\} \subseteq a(C_3 \times P_{2n+1}) \text{ when } n \text{ is even.}$$

To realize the value $6n - 1$ for even $n$, we make a special labeling as follows. Apply the above procedure up to shifting the $(n-1)$-th row, and then shift the $n$-th row to the right by 1 unit.

Consider the labeling

$$g_n = \begin{pmatrix} A \\ B \end{pmatrix}, \text{ where } A = \begin{pmatrix} J_{n,2} & O_{n,1} \end{pmatrix}, B = \begin{pmatrix} J_{1,3} \\ O_{n-1,2} & J_{n-1,2} \\ O_{1,3} \end{pmatrix}. $$

For $1 \leq k \leq 2n - 1$, shift consecutively the $k$-th row to the right by 1 unit if $k$ is odd, and to the left by 1 unit if $k$ is even. It is easy to see that each shift increases $e(1)$ by 2, except shifting the $n$-th row and the $(n+1)$-th row which preserve $e(1)$. Thus

$$\{4n + 2 + 2i \mid i \in [0, 2n - 3]\} \subseteq a(C_3 \times P_{2n+1}).$$
Let $p$ be the labeling whose matrix representation is given by
\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\vdots \\
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}.
\]

Similar to the procedure for the matrix $p$ in Theorem 5.5, we have
\[
[8n - 3, 10n + 2] \subseteq a(C_3 \times P_{2n+1}).
\]

The theorem follows from considering all the above cases. □

By constructing labelings directly, it is easy to obtain that $a(C_4 \times P_3) = [6, 16] \cup \{20\}$, $a(C_3 \times P_2) = \{3, 5, 7\}$, $a(C_3 \times P_3) = [5, 12]$ and $a(C_3 \times P_4) = \{3\} \cup [6, 17]$.

We summarize the full friendly index sets of cylinder graphs $C_m \times P_n$ for $m \geq 3$, $C_m \times P_3$ for $m \geq 3$, and $C_3 \times P_n$ for $n \geq 4$, as follows.

**Theorem 5.7.** The full friendly index set of $C_m \times P_n$ is given by
\[
\begin{align*}
\text{FFI}(C_m \times P_2) &= \{4i - 3m \mid i \in [2, 3m/2 - 2] \cup \{3m/2\}\} \text{ if } m \geq 4 \text{ is even.} \\
\text{FFI}(C_m \times P_2) &= \{4i - 3m + 2 \mid i \in [2, (3m - 1)/2]\} \text{ if } m \geq 5 \text{ is odd.} \\
\text{FFI}(C_m \times P_3) &= \{2i - 5m \mid i \in \{6, 5m\} \cup [8, 5m - 3]\} \text{ if } m \geq 6 \text{ is even.} \\
\text{FFI}(C_m \times P_3) &= \{2i - 5m \mid i \in [7, 5m - 2]\} \text{ if } m \geq 5 \text{ is odd.} \\
\text{FFI}(C_3 \times P_n) &= \{2i - 10n - 3 \mid i \in \{3\} \cup [6, 5n - 3]\} \text{ if } n \geq 4 \text{ is even.} \\
\text{FFI}(C_3 \times P_n) &= \{2i - 10n - 3 \mid i \in [5, 5n + 2]\} \text{ if } n \geq 5 \text{ is odd.} \\
\text{FFI}(C_3 \times P_2) &= \{-3, 1, 5\}. \\
\text{FFI}(C_3 \times P_3) &= \{2i - 15 \mid i \in [5, 12]\}. \\
\text{FFI}(C_4 \times P_3) &= \{2i - 20 \mid i \in [6, 16] \cup \{20\}\}.
\end{align*}
\]

Together with [10, 11] (the results are listed as follows), the full friendly index set of $C_m \times P_n$, for all $m$ and $n$, are completely determined.
For \( m, n \geq 4 \), \( \text{FFI}(C_m \times P_n) \) is given by

\[
\begin{align*}
\{ -2mn + m + 2i \mid i \in [2n+2, 2mn-m-4] \cup \{2n, 2mn-m\} \} \\
\text{for } m \geq 2n+2 \text{ and } m, n \text{ are even;}
\{ -2mn + m + 2i \mid i \in [m+4, 2mn-m-4] \cup \{m+2, 2mn-m\} \} \\
\text{for } m \leq 2n-2, \text{ m is even and } n \text{ is odd;}
\{ -2mn + m + 2i \mid i \in [2n+2, 2mn-m-4] \cup \{2n, 2mn-m\} \} \\
\text{for } m \geq 2n \text{ and } m \text{ is even and } n \text{ is odd;}
\{ -2mn + m + 2i \mid i \in [m+4, 2mn-m-n] \cup \{m\} \} \\
\text{for } m \leq 2n-3, \text{ m is odd and } n \text{ is even;}
\{ -2mn + m + 2i \mid i \in [m+2, 2mn-m-n] \cup \{m\} \} \\
\text{for } m = 2n-1 \text{ and } n \text{ is even;}
\{ -2mn + m + 2i \mid i \in [2n, 2mn-m-n] \} \\
\text{for } m \geq 2n+1 \text{ and } m \text{ is odd and } n \text{ is even;}
\{ -2mn + m + 2i \mid i \in [m+4, 2mn-m-n] \cup \{m+2\} \} \\
\text{for } m \leq 2n-3 \text{ and } m, n \text{ are odd;}
\{ -2mn + m + 2i \mid i \in [2n+1, 2mn-m-n] \} \\
\text{for } m \geq 2n-1 \text{ and } m, n \text{ are odd.}
\end{align*}
\]

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References


Wai Chee Shiu
Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Kowloon Tong, Hong Kong, China
Email: wcshiu@hkbu.edu.hk

Man-Ho Ho
Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Kowloon Tong, Hong Kong, China
Email: homanho@math.hkbu.edu.hk