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## EXTREME EDGE-FRIENDLY INDICES OF COMPLETE BIPARTITE GRAPHS

WAI CHEE SHIU

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ABSTRACT. Let  $G = (V, E)$  be a simple graph. An edge labeling  $f : E \rightarrow \{0, 1\}$  induces a vertex labeling  $f^+ : V \rightarrow \mathbb{Z}_2$  defined by  $f^+(v) \equiv \sum_{uv \in E} f(uv) \pmod{2}$  for each  $v \in V$ , where  $\mathbb{Z}_2 = \{0, 1\}$  is the additive group of order 2. For  $i \in \{0, 1\}$ , let  $e_f(i) = |f^{-1}(i)|$  and  $v_f(i) = |(f^+)^{-1}(i)|$ . A labeling  $f$  is called edge-friendly if  $|e_f(1) - e_f(0)| \leq 1$ .  $I_f(G) = v_f(1) - v_f(0)$  is called the edge-friendly index of  $G$  under an edge-friendly labeling  $f$ . Extreme values of edge-friendly index of complete bipartite graphs will be determined.

### 1. Introduction

Let  $G = (V, E)$  be a simple graph. An edge labeling  $f : E \rightarrow \{0, 1\} \subset \mathbb{N}$  induces a vertex labeling  $f^+ : V \rightarrow \mathbb{Z}_2$  defined by  $f^+(v) \equiv \sum_{uv \in E} f(uv) \pmod{2}$  for each  $v \in V$ , where  $\mathbb{Z}_2 = \{0, 1\}$  is the additive group of order 2. We sometimes view the value of  $f^+(v)$  as an integer. For  $i \in \{0, 1\}$ , let  $e_f(i) = |f^{-1}(i)|$  and  $v_f(i) = |(f^+)^{-1}(i)|$ . Let  $I_f(G) = v_f(1) - v_f(0)$ . An edge labeling  $f$  is *edge-friendly* if  $|e_f(1) - e_f(0)| \leq 1$ . The concept of edge-friendly index maybe first introduced by Lee and Ng [4] on considering edge cordial labeling. Unfortunately, we cannot find this paper even through we have asked the authors. Readers are referred to [1] for detail about edge cordial.

The number  $I_f(G)$  is called the *edge-friendly index* of  $G$  under  $f$  if  $f$  is an edge-friendly labeling of  $G$ . The set  $\text{FEFI}(G) = \{I_f(G) \mid f \text{ is edge-friendly}\}$  is called the *full edge-friendly index set* of  $G$ . This is a dual concept of full friendly index set which was first introduced by the author and H. Kwong [9]. Readers who are interested on friendly index or friendly index set may refer to [2, 3, 5–15].

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In this paper, we shall study the extreme values of edge-friendly index of complete bipartite graphs  $K_{m,n}$ .

## 2. Some Basic Properties

In the following, all graphs are simple and connected. The codomain of any edge labeling is  $\mathbb{Z}_2$ . Suppose  $f$  is an edge labeling. A vertex (resp. an edge) is called an  $i$ -vertex (resp.  $i$ -edge) under  $f$  if it is labeled by  $i \in \{0, 1\}$ . Notation and concepts not defined here are referred to [16].

Suppose  $G$  is a graph of order  $p$ . Since  $v_f(1) + v_f(0) = p$  for any edge labeling  $f$  of  $G$ ,  $I_f(G) = 2v_f(1) - p$ . Thus, it suffices to study the number of 1-vertices instead of studying the edge-friendly index of  $G$  under  $f$ .

**Lemma 2.1.** *Let  $f$  be any edge labeling of a graph  $G = (V, E)$ . Then  $v_f(1)$  must be even.*

**Proof.** Since the value of each edge contributes twice toward the sum of values of vertex,

$$v_f(1) = \sum_{u \in V} f^+(v) \equiv 2 \sum_{e \in E} f(e) \equiv 0 \pmod{2}.$$

□

By means of the above lemma, we may write  $v_f(1) = 2j$  for some  $j$  with  $0 \leq j \leq \lfloor p/2 \rfloor$ , where  $f$  is an edge labeling of a graph  $G$  of order  $p$ . So  $I_f(G) = 4j - p$  for some  $j$ ,  $0 \leq j \leq \lfloor p/2 \rfloor$ . It implies that

$$\text{FEFI}(G) \subseteq \{4j - p \mid 0 \leq j \leq \lfloor p/2 \rfloor\}.$$

A labeling matrix  $L_f(G)$  for an edge labeling  $f$  of a graph  $G$  is a matrix whose rows and columns are indexed by the vertices of  $G$  and the  $(u, v)$ -entry is  $f(uv)$  if  $uv \in E$ , and is  $*$  otherwise.

Suppose  $L_f(G)$  is a labeling matrix for the edge labeling  $f$  of  $G$ . If we view the entries of  $L_f(G)$  as elements in  $\mathbb{Z}_2$ , then  $f^+(v)$  is the  $v$ -row sum (as well as  $v$ -column sum), where entries with  $*$  will be treated as 0.

Let  $X = \{x_1, \dots, x_m\}$  and  $Y = \{y_1, \dots, y_n\}$  be the bipartition of the complete bipartite graph  $K_{m,n}$ . Under this indexing of vertices, a labeling matrix for any edge labeling  $f$  is of the form

$$\begin{pmatrix} \star_m & A \\ A^T & \star_n \end{pmatrix},$$

where  $\star_r$  is a square matrix of order  $r$  with all entries being  $*$  and  $A$  is an  $m \times n$  matrix whose entries are elements of  $\mathbb{Z}_2$ . So the multi-set of row sums and column sums of  $A$  is equal to the sequence  $\{f^+(x_1), \dots, f^+(x_m), f^+(y_1), \dots, f^+(y_n)\}$ . Thus, we shall only consider such matrix  $A$  and we shall denote it as  $A_f(G)$  when there is some ambiguity. Thus, we shall use such matrix  $A_f(G)$  (or  $A$ ) to define an edge labeling  $f$ . Let  $v_A(1)$  denote the number of 1's being row sum or column sum. Then  $v_A(1) = v_f(1)$ . Similarly, we may define  $v_A(0)$ , which will equal to  $v_f(0)$ . Also we may define  $e_A(1)$  and  $e_A(0)$  to be the number of 1 and 0 used to form the matrix  $A$ , respectively. So  $e_A(i) = e_f(i)$ ,  $i = 0, 1$ .

An  $m \times n$  matrix  $A$  satisfying the following conditions is called a *friendly matrix* of  $K_{m,n}$ :

1. Each entry of  $A$  is either 1 or 0;
2.  $|e_A(1) - e_A(0)| \leq 1$ .

In order to find an edge-friendly labeling, it suffices to find a suitable size of friendly matrix. Following is a well-known lemma.

**Lemma 2.2.** *Let  $(X, Y)$  be a bipartition of a bipartite graph  $G$ . Then*

$$\sum_{x \in X} \deg(x) = \sum_{y \in Y} \deg(y) = q(G),$$

where  $q(G)$  is the size of  $G$ .

### 3. Minimum Value of $v_f(1)$

For easy to describe some matrices, we let  $J_{m,n}$  be the  $m \times n$  matrix whose entries are 1 and  $O_{m,n}$  be the  $m \times n$  zero matrix.

For  $K_{1,n}$ , it is easy to obtain that

$$FEFI(K_{1,n}) = \begin{cases} \{-2, 2\}, & n = 4k + 1; \\ \{1\}, & n = 4k + 2; \\ \{0\}, & n = 4k + 3; \\ \{-1\}, & n = 4k + 4, \end{cases}$$

where  $k \geq 0$ . For  $K_{2,2} \cong C_4$ , it is easy to get that

$$FEFI(K_{2,2}) = \{0, 4\}.$$

So we assume that  $m, n \geq 2$  and  $\max\{m, n\} \geq 3$ .

**Lemma 3.1.** *For  $k \geq 1$ , there is no edge-friendly labeling  $f$  of  $K_{2,4k+2}$  such that  $v_f(1) = 0$ . But there is an edge-friendly labeling  $\mu$  of  $K_{2,4k+2}$  such that  $v_\mu(1) = 2$ .*

**Proof.** Suppose there is an edge-friendly labeling  $f$  of  $K_{2,4k+2}$  such that  $v_f(1) = 0$ . Let  $H$  be the subgraph of  $K_{2,4k+2}$  induced by all 1-edges. Since  $f$  is edge-friendly, the size of  $H$  is  $4k + 2$ . Let  $X$  and  $Y$  be the bipartition of  $K_{2,4k+2}$ . Also let  $U \subseteq X$  and  $W \subseteq Y$  be the bipartition of  $H$ . Since  $v_f(1) = 0$ , the degree of each vertex in  $H$  must be even and positive. Hence  $\deg_H(w) = 2$  for each  $w \in W$ . Thus  $U = X$ . By means of the size of  $H$ , we have  $|W| = 2k + 1$ . Hence  $H \cong K_{2,2k+1}$  which contradicts  $\deg_H(u)$  being even for  $u \in U$ .

Let the block matrix  $A_{2,4k+2} = \begin{pmatrix} J_{2,2k+1} & O_{2,2k+1} \end{pmatrix}$ .  $A_{2,4k+2}$  is the required friendly matrix which induces an edge-friendly labeling  $\mu$  for  $K_{2,4k+2}$  with  $v_\mu(1) = v_{A_{2,4k+2}}(1) = 2$ . □

We define some useful friendly matrices first. Let

$$A_{2h,4} = \begin{pmatrix} J_{2h,2} & O_{2h,2} \end{pmatrix} \text{ for } h \geq 1, \quad A_{3,4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$D_h = \begin{pmatrix} J_{h,6} \\ O_{h,6} \end{pmatrix} \text{ for } h \geq 1, \quad A_{6,6} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Lemma 3.2.** For  $k \geq 1$ , there is a friendly matrix  $A$  of  $K_{m,4k}$  such that  $v_A(1) = 0$ .

**Proof.** For  $m = 2s \geq 2$ , let

$$A = A_{2s,4k} = \begin{pmatrix} A_{2s,4} & A_{2s,4} & \cdots & A_{2s,4} \end{pmatrix}.$$

For  $m = 2s + 1 \geq 3$ , let

$$(3.1) \quad A = A_{2s+1,4k} = \begin{pmatrix} A_{2s-2,4} & A_{2s-2,4} & \cdots & A_{2s-2,4} \\ A_{3,4} & A_{3,4} & \cdots & A_{3,4} \end{pmatrix}.$$

Note that, when  $s = 1$ , the first row of block matrices does not appear. It is easy to check that each matrix above is friendly  $v_A(1) = 0$ . □

**Example 3.1.** Let  $A_{4,8} = \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array} \right)$  and  $A_{7,8} = \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array} \right)$ .

They define edge-friendly labelings for  $K_{4,8}$  and  $K_{7,8}$  with  $v_{A_{4,8}}(1) = 0$  and  $v_{A_{7,8}}(1) = 0$ , respectively.

**Lemma 3.3.** For  $k \geq 0$  and odd  $m$ , there is no edge-friendly labeling  $f$  of  $K_{m,4k+2}$  such that  $v_f(1) = 0$ . But there is an edge-friendly labeling  $\mu$  of  $K_{m,4k+2}$  such that  $v_\mu(1) = 2$ .

**Proof.** Suppose there is an edge-friendly labeling  $f$  of  $K_{m,4k+2}$  such that  $v_f(1) = 0$ . Let  $H$  be the subgraph of  $K_{m,4k+2}$  induced by all 1-edges. Then  $q(H) = m(2k + 1)$ , which is odd. Let  $U$  and  $W$  be the bipartition of  $H$ . Since  $v_f(1) = 0$ , the degree of each vertex in  $H$  must be even and positive. By Lemma 2.2 we know that it is impossible.

For the second part of the lemma. Let

$$Z_{m,2} = \begin{pmatrix} J_{2h,2} \\ 1 & 0 \\ O_{2h,2} \end{pmatrix} \text{ if } m = 4h + 1, \quad Z_{m,2} = \begin{pmatrix} J_{2h+1,2} \\ 1 & 0 \\ O_{2h+1,2} \end{pmatrix} \text{ if } m = 4h + 3.$$

It is easy to see that the matrices above are friendly. Then

$$A_{m,4k+2} = \begin{pmatrix} A_{m,4k} & Z_{m,2} \end{pmatrix}$$

is the required friendly matrix which induces an edge-friendly labeling  $\mu$  of  $K_{m,4k+2}$  such that  $v_\mu(1) = 2$ , where  $A_{m,4k}$  is defined in (3.1).  $\square$

**Example 3.2.** Let  $A_\mu(K_{3,10}) = \left( \begin{array}{cccc|cccc|cc} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$  and

$A_\mu(K_{5,10}) = \left( \begin{array}{cccc|cccc|cc} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$ . They define edge-friendly labelings  $\mu$  for  $K_{3,10}$  and  $K_{5,10}$  with  $v_\mu(1) = 2$ , respectively.

**Lemma 3.4.** For  $h, k \geq 1$ , there is a friendly matrix  $A$  of  $K_{4h+2,4k+2}$  such that  $v_A(1) = 0$ .

**Proof.** Let

$$A = A_{4h+2,4k+2} = \left( \begin{array}{cccc|c} A_{4h-4,4} & A_{4h-4,4} & \cdots & A_{4h-4,4} & D_{2h-2} \\ A_{3,4} & A_{3,4} & \cdots & A_{3,4} & \\ A_{3,4} & A_{3,4} & \cdots & A_{3,4} & A_{6,6} \end{array} \right)$$

It is easy to check that  $A$  is friendly and  $v_A(1) = 0$ .  $\square$

**Example 3.3.** Let  $A_\mu(K_{10,10}) = \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$ . This matrix defines an edge-friendly labeling  $\mu$  for  $K_{10,10}$  with  $v_\mu(1) = 0$ .

**Lemma 3.5.** For odd  $m$  and  $n$ , there is a friendly matrix  $A$  of  $K_{m,n}$  such that  $v_A(1) = 0$ .

**Proof.** Suppose one of  $m$  and  $n$  is of the form  $4k + 3$  for some  $k \geq 0$ . Without loss of generality, we assume that  $n = 4k + 3$ .

We consider  $k = 0$  first. Let

$$A_{3,3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_{4,3} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A_{5,3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly,  $v_{A_{3,3}}(1) = 0$ ,  $v_{A_{4,3}}(1) = 0$  and  $v_{A_{5,3}}(1) = 0$ .

Suppose there is a friendly matrix  $A_{2s-3,3}$  of size  $(2s-3) \times 3$  such that  $v_{A_{2s-3,3}}(1) = 0$ , where  $s \geq 3$ . Let  $A_{2s+1,3} = \begin{pmatrix} A_{4,3} \\ A_{2s-3,3} \end{pmatrix}$ . Clearly  $A_{2s+1,3}$  is a friendly matrix of  $K_{2s+1,3}$  such that  $v_{A_{2s+1,3}}(1) = 0$ . By mathematical induction, we know that there is friendly matrix  $A$  of  $K_{2s+1,3}$  such that  $v_A(1) = 0$  for all  $s \geq 1$ .

For the general case, i.e.,  $m = 2s + 1$  and  $n = 4k + 3$ , let

$$A_{2s+1,4k+3} = \begin{pmatrix} A_{2s+1,4k} & A_{2s+1,3} \end{pmatrix},$$

where  $A_{2s+1,4k}$  is defined in (3.1) and  $A_{2s+1,3}$  is defined above. Then  $A_{2s+1,4k+3}$  is a required friendly matrix.

The remaining case is when  $m = 4h + 1$  and  $n = 4k + 1$  for some  $h, k \geq 1$ . When  $n = 5$ . Let

$$A_{4,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A_{5,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly,  $A_{5,5}$  is a friendly matrix of  $K_{5,5}$  such that  $v_{A_{5,5}}(1) = 0$ . Suppose there is a friendly matrix  $A_{4h-3,5}$  of  $K_{4h-3,5}$  such that  $v_{A_{4h-3,5}}(1) = 0$ , where  $h \geq 2$ . Let  $A_{4h+1,5} = \begin{pmatrix} A_{4,5} \\ A_{4h-3,5} \end{pmatrix}$ . This is a required friendly matrix. By mathematical induction, we know that there is a friendly matrix  $A$  of  $K_{4h+1,5}$  such that  $v_A(1) = 0$  for all  $h \geq 1$ .

For the general case, similar to the construction above, we let

$$A_{4h+1,4k+1} = \begin{pmatrix} A_{4h+1,4k-4} & A_{4h+1,5} \end{pmatrix},$$

where  $k \geq 2$ . Hence this matrix is a required friendly matrix. □

**Corollary 3.6.** *Let  $\mu$  be an edge-friendly labeling for  $K_{m,n}$  attaining the minimum number of 1-edges.*

*Then  $v_\mu(1) = 0$  except the following cases:*

*(a)  $n = 4k + 2$  and  $m = 2$  or  $m$  is odd, for some  $k \geq 0$ ;*

*(b)  $m = 4h + 2$  and  $n = 2$  or  $n$  is odd, for some  $h \geq 0$ .*

*Moreover, for these exceptional cases,  $v_\mu(1) = 2$ .*

Note that, the above exceptional cases are the same by symmetry. In the following, we shall keep the definitions of  $A_{i,j}$ 's defined in this section.

#### 4. Maximum Value of $v_f(1)$

In this section, we shall define a friendly matrix  $M_{m,n}$  which induces an edge-friendly labeling  $\varphi$  for  $K_{m,n}$  such that  $v_\varphi(1) = v_{M_{m,n}}(1)$  is maximum.

From Lemma 2.1 we have

**Corollary 4.1.** *Suppose  $m$  and  $n$  are of opposite parity. There is no edge-friendly labeling  $f$  of  $K_{m,n}$  such that  $v_f(1) = m + n$ .*

Let

$$\begin{aligned}
 M_{2,4k+2} &= \begin{pmatrix} J_{1,2k+1} & O_{1,2k+1} \\ O_{1,2k+1} & J_{1,2k+1} \end{pmatrix}, k \geq 0; & M_{2,4k} &= \begin{pmatrix} J_{1,2k+1} & O_{1,2k-1} \\ O_{1,2k+1} & J_{1,2k-1} \end{pmatrix}, k \geq 1; \\
 N_{2s,4k+2} &= \begin{pmatrix} J_{2s,2k+1} & O_{2s,2k+1} \end{pmatrix}, k \geq 0, s \geq 1; & N_{4h,4k} &= \begin{pmatrix} J_{2h,2k+1} & O_{2h,2k-1} \\ O_{2h,2k+1} & J_{2h,2k-1} \end{pmatrix}, h, k \geq 1. \\
 N_{4h,2t+1} &= \begin{pmatrix} A_{2t-1,4h}^T & N_{4h,2} \end{pmatrix}, h \geq 1, t \geq 2; \\
 M_{4,4} &= \begin{pmatrix} J_{2,2} & I_2 \\ I_2 & O_{2,2} \end{pmatrix}; & M_{4,4k} &= \begin{pmatrix} M_{4,4} & N_{4k-4,4}^T \end{pmatrix}, k \geq 2,
 \end{aligned}$$

where  $I_2$  is the identity matrix of order 2. Note that  $v_{M_{2,4k+2}}(1) = 4k + 4$ ,  $v_{M_{2,4k}}(1) = 4k + 2$ ,  $v_{M_{4,4k}}(1) = 4k + 4$ ,  $v_{N_{2s,4k+2}}(1) = 2s$ ,  $v_{N_{4h,4k}}(1) = 4h$ ,  $v_{N_{4h,2t+1}}(1) = 4h$ . More precisely, each row sum of  $N_{2s,4k+2}$  (resp.  $N_{4h,4k}$ ,  $N_{4h,2t+1}$ ) is 1 and each column sum of  $N_{2s,4k+2}$  (resp.  $N_{4h,4k}$ ,  $N_{4h,2t+1}$ ) is 0.

**Lemma 4.2.** *Suppose  $m$  and  $n$  are even. There is a friendly matrix  $M$  of  $K_{m,n}$  such that  $v_M(1) = m + n$ .*

**Proof.** Suppose  $n = 4k + 2$  for some  $k \geq 0$ . Let  $m = 2s$  for some  $s \geq 1$ . When  $s = 1$ , the required matrix is  $M_{2,4k+2}$ . When  $s \geq 2$ . Then  $M_{2s,4k+2} = \begin{pmatrix} M_{2,4k+2} \\ N_{2s-2,4k+2} \end{pmatrix}$  is a required matrix.

Suppose  $n = 4k$  for some  $k \geq 0$ . Suppose  $m = 4h + 2$  for some  $h \geq 0$ . When  $h = 0$ , the required matrix is  $M_{2,4k}$ . When  $h \geq 1$ . Then  $M_{4h+2,4k} = \begin{pmatrix} M_{2,4k} \\ N_{4h,4k} \end{pmatrix}$  is a required matrix. Suppose  $m = 4h$  for

some  $h \geq 1$ . When  $h = 1$ . The required matrix is  $M_{4,4k}$ . When  $h \geq 2$ . Then  $M_{4h,4k} = \begin{pmatrix} M_{4,4k} \\ N_{4h-4,4k} \end{pmatrix}$  is a required matrix. □

**Example 4.1.**  $M_{4,6} = \begin{pmatrix} M_{2,6} \\ N_{2,6} \end{pmatrix} = \left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array} \right),$

$$M_{6,8} = \begin{pmatrix} M_{2,8} \\ N_{4,8} \end{pmatrix} = \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right), M_{8,8} = \begin{pmatrix} M_{4,8} \\ N_{4,8} \end{pmatrix} = \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right).$$

**Lemma 4.3.** *Suppose  $m$  and  $n$  are odd. There is a friendly matrix  $M$  of  $K_{m,n}$  such that  $v_M(1) = m + n$ .*



**Proof.** Suppose one of  $m$  and  $n$  is congruence 3 modulo 4. Without loss of generality, we assume that

$$m = 4h + 3 \text{ for some } h \geq 0. \text{ Consider } m = 3 \text{ first. Let } M_{3,3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, M_{3,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } R_{3,4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \text{ Clearly, } v_{M_{3,3}}(1) = 6, v_{M_{3,5}}(1) = 8 \text{ and } v_{R_{3,4}}(1) = 4. \text{ Moreover, the row}$$

sums of  $R_{3,4}$  are 0 and columns sums are 1. Putting a suitable numbers of  $R_{3,4}$  on the left hand side of  $M_{3,3}$  and  $M_{3,5}$  to form a large matrix will get a friendly matrix  $M_{3,2t+1}$  of  $K_{3,2t+1}$  such that  $v_{M_{3,2t+1}}(1) = 2t + 4$ , for some  $t \geq 1$ .

For the general case, let  $M_{4h+3,2t+1} = \begin{pmatrix} N_{4h,2t+1} \\ M_{3,2t+1} \end{pmatrix}$ . Then  $v_{M_{4h+3,2t+1}}(1) = v_{M_{3,2t+1}}(1) + 4h = 4h + 2t + 4$ .

Suppose both of  $m$  and  $n$  are congruence 1 modulo 4. Let  $m = 4h + 1$  and  $n = 4k + 1$  with  $h, k \geq 1$ . Suppose one of  $m$  and  $n$  equal to 5. Without loss of generality, we assume  $m = 5$ . Let

$$M_{5,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, M_{5,4k+1} = \begin{pmatrix} N_{4k-4,5}^T & M_{5,5} \end{pmatrix} \text{ if } k \geq 2.$$

Clearly, these two matrices are required matrices.

The remaining case is when  $h, k \geq 2$ . Let

$$M_{4h+1,4k+1} = \begin{pmatrix} A_{4h-4,4k-4} & N_{4h-4,5} \\ N_{4k-4,5}^T & M_{5,5} \end{pmatrix}.$$

We may check that  $v_{M_{4h+1,4k+1}}(1) = 4h + 4k + 2$ . □

Note that  $e_{M_{4h+3,4k+3}}(1) = e_{M_{4h+3,4k+3}}(0) + 1$ ;  $e_{M_{4h+1,4k+1}}(1) = e_{M_{4h+1,4k+1}}(0) + 1$  and  $e_{M_{4h+3,4k+1}}(1) = e_{M_{4h+3,4k+1}}(0) - 1$ .

**Example 4.2.**  $M_{7,7} = \begin{pmatrix} N_{4,7} \\ M_{3,7} \end{pmatrix}$ ,  $N_{4,7} = \begin{pmatrix} A_{5,4}^T & N_{4,2} \end{pmatrix}$ ,  $A_{5,4} = \begin{pmatrix} A_{2,4} \\ A_{3,4} \end{pmatrix}$ .

$$\text{Hence } A_{5,4}^T = \left( \begin{array}{cc|cc} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), N_{4,7} = \left( \begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right),$$

$$M_{3,7} = (R_{3,4} \quad M_{3,3}) = \left( \begin{array}{cccc|ccc} 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array} \right), M_{7,7} = \left( \begin{array}{cccccc|ccc} 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array} \right).$$

Similarly,  $M_{7,9} = \begin{pmatrix} N_{4,9} \\ M_{3,9} \end{pmatrix}$ ,

$$N_{4,9} = \begin{pmatrix} A_{7,4}^T & N_{4,2} \end{pmatrix} = \begin{pmatrix} A_{4,4}^T & A_{3,4}^T & N_{4,2} \end{pmatrix} = \left( \begin{array}{cccc|ccc} 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \middle| \begin{array}{c} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right) \text{ and}$$

$$M_{3,9} = \begin{pmatrix} R_{3,4} & M_{3,5} \end{pmatrix} = \left( \begin{array}{cccc|ccccc} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array} \right). \text{ So } M_{7,9} = \left( \begin{array}{cccccccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array} \right).$$

$$M_{9,13} = \begin{pmatrix} A_{4,8} & N_{4,5} \\ N_{8,5}^T & M_{5,5} \end{pmatrix}. \text{ So we have } M_{9,13} = \left( \begin{array}{cccccc|cccc} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right).$$

**Lemma 4.4.** *There is a friendly matrix  $M$  of  $K_{m,n}$  such that  $v_M(1) = m + n - 1$ , where  $m \not\equiv n \pmod{2}$ .*

**Proof.** Without loss of generality, we may assume  $m$  is even.

$$\text{Let } S_{3,3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; T_{3,3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; S_{3,5} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}; T_{3,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix};$$

$$S_{3,4k+1} = \begin{pmatrix} A_{3,4k-4} & S_{3,5} \end{pmatrix} \text{ and } T_{3,4k+1} = \begin{pmatrix} A_{3,4k-4} & T_{3,5} \end{pmatrix} \text{ if } k \geq 2;$$

$S_{3,4k+3} = \begin{pmatrix} A_{3,4k} & S_{3,3} \end{pmatrix}$  and  $T_{3,4k+3} = \begin{pmatrix} A_{3,4k} & T_{3,3} \end{pmatrix}$  if  $k \geq 1$ . Then  $v_{S_{3,4k+1}}(1) = v_{S_{3,4k+3}}(1) = 4$  in which all row sums are 1 and exactly one column sum is 1.  $v_{T_{3,4k+1}}(1) = v_{T_{3,4k+3}}(1) = 2$  in which all column sums are 0. Note that  $e_{S_{3,4k+1}}(1) = e_{S_{3,4k+1}}(0) - 1$ ;  $e_{T_{3,4k+1}}(1) = e_{T_{3,4k+1}}(0) + 1$ ;  $e_{S_{3,4k+3}}(1) = e_{S_{3,4k+3}}(0) + 1$ ; and  $e_{T_{3,4k+3}}(1) = e_{T_{3,4k+3}}(0) - 1$ .

1.  $m = 4h$  and  $n = 4k + 1$ .

Let  $M_{4h,4k+1} = \begin{pmatrix} M_{4h-3,4k+1} \\ S_{3,4k+1} \end{pmatrix}$ . Since  $e_{M_{4h-3,4k+1}}(1) = e_{M_{4h-3,4k+1}}(0) + 1$ ,  $M_{4h,4k+1}$  is friendly.

Moreover,  $v_{M_{4h,4k+1}}(1) = (4h - 3) + (4k + 1) + 3 - 1 = 4h + 4k = m + n - 1$ .

2.  $m = 4h$  and  $n = 4k + 3$ .

Let  $M_{4h-3,4k+3} = M_{4k+3,4h-3}^T$  and  $M_{4h,4k+3} = \begin{pmatrix} M_{4h-3,4k+3} \\ S_{3,4k+3} \end{pmatrix}$ . Since  $e_{M_{4h-3,4k+3}}(1) = e_{M_{4h-3,4k+3}}(0) - 1$ ,  $M_{4h,4k+3}$  is friendly. Moreover,  $v_{M_{4h,4k+3}}(1) = (4h - 3) + (4k + 3) + 3 - 1 = 4h + 4k + 2 = m + n - 1$ .

3.  $m = 2$  and  $n$  is odd. The required labeling matrix is  $\begin{pmatrix} J_{1,n} \\ O_{1,n} \end{pmatrix}$ .

4.  $m = 4h + 2$  and  $n = 4k + 1$ , where  $h \geq 1$ .

Let  $M_{4h+2,4k+1} = \begin{pmatrix} M_{4h-1,4k+1} \\ T_{3,4k+1} \end{pmatrix}$ . Since  $e_{M_{4h-1,4k+1}}(1) = e_{M_{4h-1,4k+1}}(0) - 1$ ,  $M_{4h+2,4k+1}$  is friendly.

Moreover,  $v_{M_{4h+2,4k+1}}(1) = (4h - 1) + (4k + 1) + 2 = 4h + 4k + 2 = m + n - 1$ .

5.  $m = 4h + 2$  and  $n = 4k + 3$ , where  $h \geq 1$ .

Let  $M_{4h+2,4k+3} = \begin{pmatrix} M_{4h-1,4k+3} \\ T_{3,4k+3} \end{pmatrix}$ . Since  $e_{M_{4h-1,4k+3}}(1) = e_{M_{4h-1,4k+3}}(0) + 1$ ,  $M_{4h+2,4k+3}$  is friendly.

Moreover,  $v_{M_{4h+2,4k+3}}(1) = (4h - 1) + (4k + 3) + 2 = 4h + 4k + 4 = m + n - 1$ .

This completes the proof. □

**Example 4.3.**  $M_{8,5} = \begin{pmatrix} M_{5,5} \\ S_{3,5} \end{pmatrix} = \left( \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right),$

$$M_{8,7} = \begin{pmatrix} M_{5,7} \\ S_{3,7} \end{pmatrix} = \left( \begin{array}{ccc|ccc} A_{3,4} & & & M_{3,5}^T & & \\ \hline N_{4,2}^T & & & & & \\ A_{3,4} & & & S_{3,3} & & \end{array} \right) = \left( \begin{array}{cccc|ccc} 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right),$$

$$M_{6,5} = \begin{pmatrix} M_{3,5} \\ T_{3,5} \end{pmatrix} = \left( \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right),$$

$$M_{6,7} = \begin{pmatrix} M_{3,7} \\ T_{3,7} \end{pmatrix} = \begin{pmatrix} R_{3,4} & M_{3,3} \\ A_{3,4} & T_{3,3} \end{pmatrix} = \left( \begin{array}{cccc|ccc} 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right).$$

The next problem is to determine the full edge-friendly index set of  $K_{m,n}$ . We conjecture that

$$\text{FEFI}(K_{m,n}) = \begin{cases} \{4j - (m + n) \mid 1 \leq j \leq \lfloor (m + n)/2 \rfloor\}, & \text{if } n \equiv 2 \pmod{4} \text{ and } m = 2 \text{ or } m \text{ is odd;} \\ \{4j - (m + n) \mid 1 \leq j \leq \lfloor (m + n)/2 \rfloor\}, & \text{if } m \equiv 2 \pmod{4} \text{ and } n = 2 \text{ or } n \text{ is odd;} \\ \{4j - (m + n) \mid 0 \leq j \leq \lfloor (m + n)/2 \rfloor\}, & \text{otherwise.} \end{cases}$$

It will take more iterations on finding each possible value of edge-friendly index. Since this paper seems too long, we will study this problem at the next paper. Following we provide two examples as the ending of this paper.

**Example 4.4.** Matrices  $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$  induce the edge-friendly labelings  $\mu$  and  $\varphi$  for  $K_{2,3}$  such that  $v_\mu(1) = 2$  and  $v_\varphi(1) = 4$ , respectively. Hence  $FEFI(K_{2,3}) = \{-1, 3\}$ .

**Example 4.5.** Consider  $K_{2,4k+2}$  for  $k \geq 1$ . We start from  $A_0 = A_{2,4k+2}$ . Note that  $v_{A_0}(1) = 2$ . Obtain a new matrix  $A_1$  by shifting the second row of  $A_0$  to right by one entry cyclically. Then  $v_{A_1}(1) = 4$ . Repeat shifting the second row to right of the newly matrix. We can see that the number of 1-edges increase by 2 after each shifting. For example, when  $k = 1$ :

$$A_0 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow A_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \rightarrow A_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \dots$$

So, we get that

$$FEFI(K_{2,4k+2}) = \{4j - 4k - 4 \mid 1 \leq j \leq 2k + 2\}.$$

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