Remarks on flat and differential K-theory

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REMARKS ON FLAT AND DIFFERENTIAL $K$-THEORY

MAN-HO HO

Abstract. In this note we prove some results in flat and differential $K$-theory. The first one is a proof of the compatibility of the differential topological index and the flat topological index by a direct computation. The second one is the explicit isomorphisms between Bunke-Schick differential $K$-theory and Freed-Lott differential $K$-theory.

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1. Introduction

In this note we prove some results in flat and differential $K$-theory. While some of these results are known to the experts, the proofs given here have not appeared in the literature. We first prove the compatibility of the flat topological index $\text{ind}_L^t$ and the differential topological index $\text{ind}_\text{FL}^t$ by a direct computation, i.e., the following diagram commutes ([17, Proposition 8.10])

\[
\begin{array}{ccc}
K_L^{-1}(X; \mathbb{R}/\mathbb{Z}) & \overset{i}{\longrightarrow} & \hat{K}_{\text{FL}}(X) \\
\downarrow \text{ind}_L^t & & \downarrow \text{ind}_\text{FL}^t \\
K_L^{-1}(B; \mathbb{R}/\mathbb{Z}) & \overset{i}{\longrightarrow} & \hat{K}_{\text{FL}}(B)
\end{array}
\]
where $i$ is the canonical inclusion, $K^{-1}_L(X; \mathbb{R}/\mathbb{Z})$ is the geometric model of $K$-theory with $\mathbb{R}/\mathbb{Z}$ coefficients and $\hat{K}_{\text{FL}}(X)$ is Freed-Lott differential $K$-theory. The commutativity of (1) is a consequence of the compatibility of the differential analytic index $\text{ind}_{\text{FL}}^a$ and the flat analytic index $\text{ind}_{\text{L}}^a$ together with the differential family index theorem [7, Theorem 7.35]. The differential topological index $\text{ind}_{\text{FL}}^t$ is defined to be the composition of an embedding pushforward and a projection pushforward. When defining the embedding pushforward, currential $K$-theory [7, §2.28] is used instead of differential $K$-theory due to the Bismut-Zhang current [2, Definition 1.3]. It is not clear whether currential $K$-theory should be regarded as a differential cohomology or a “differential homology” (see [6, §4.5] for a detailed discussion), so it may be clearer by looking at the direct computation.

Second we construct the unique natural isomorphisms between Bunke-Schick differential $K$-theory [4] and Freed-Lott differential $K$-theory by writing down the explicit formulas, which are inspired by [4, Corollary 5.5]. The uniqueness follows from [5, Theorem 3.10]. Together with [10, Theorem 4.34] and [8, Theorem 1] all the explicit isomorphisms between all the existing differential $K$-groups [9, 4, 7, 12] are known.

The paper is organized as follows: Section 2 contains all the necessary background material, including the Freed-Lott differential $K$-theory, the differential topological index, the pairing between flat $K$-theory and $K$-homology, and Bunke-Schick differential $K$-theory. In Section 3 we prove the main results.

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2. Background material


The Freed–Lott differential $K$-group $\hat{K}_{\text{FL}}(X)$ is the abelian group generated by quadruples $\mathcal{E} = (E, h, \nabla, \phi)$, where $(E, h, \nabla) \to X$ is a complex vector bundle with a Hermitian metric $h$ and a unitary connection $\nabla$, and $\phi \in \Omega^{\text{odd}}(X) / \text{Im}(d)$. The only relation is $\mathcal{E}_1 = \mathcal{E}_2$ if and only if there exists a generator $(F, h^F, \nabla^F, \phi^F)$ of $\hat{K}_{\text{FL}}(X)$ such that $E_1 \oplus F \cong E_2 \oplus F$ and $\phi_1 - \phi_2 = \text{CS}(\nabla^{E_2} \oplus \nabla^F, \nabla^{E_1} \oplus \nabla^F)$. 

There is an exact sequence \cite[(2.20)]{7}

\[ 0 \longrightarrow K_L^{-1}(X; \mathbb{R}/\mathbb{Z}) \overset{i}{\longrightarrow} \tilde{K}_{FL}(X) \overset{\text{ch}_{\tilde{K}}}{\longrightarrow} \Omega_{\text{BU}}^{\text{even}}(X) \longrightarrow 0 \]  

(2)

where $K_L^{-1}(X; \mathbb{R}/\mathbb{Z})$ is the geometric model of $\mathbb{R}/\mathbb{Z}$ $K$-theory \cite{11}, $i$ is the canonical inclusion map,

$\Omega_{\text{BU}}^{\text{even}}(X) = \{ \omega \in \Omega_{d=0}^{\text{even}}(X)[[\omega]] \in \text{Im}(r \circ \text{ch} : K^0(X) \to H^{\text{even}}(X; \mathbb{R})) \}$,

and $\text{ch}_{\tilde{K}_{FL}}(E, h, \nabla, \phi) := \text{ch}(\nabla) + d\phi$. Elements in $K_L^{-1}(X; \mathbb{R}/\mathbb{Z})$ are required to have virtual rank zero. The canonical inclusion map $i$ in \cite{2} is defined by $i(E, h, \nabla, \phi) = (E, h, \nabla, \phi)$.

Let $X \to B$ and $Y \to B$ be fiber bundles of smooth manifolds with $X$ compact. Let $g^{TV}X$ and $g^{TV}Y$ be metrics on the vertical bundles $T^V X \to X$ and $T^V Y \to Y$ respectively, and assume there are horizontal distributions $T^H X$ and $T^H Y$. Let $\mathcal{E} = (E, h, \nabla^E, \phi) \in \tilde{K}_{FL}(X)$ and $\iota : X \hookrightarrow Y$ be an embedding of manifolds. We assume the codimension of $X$ in $Y$ is even, and the normal bundle $\nu \to X$ of $X$ in $Y$ carries a spin$^c$ structure. As in \cite[§5]{7} we assume for each $b \in B$, the map $\iota_b : X_b \to Y_b$ is an isometric embedding and is compatible with projections to $B$. Denote by $S(\nu) \to X$ the spinor bundle associated to the spin$^c$-structure of $\nu \to X$. We can locally choose a spin structure for $\nu \to X$ with spinor bundle $S^{\text{spin}}(\nu)$. Then there exists a locally defined Hermitian line bundle $L^2(\nu)$ such that $S(\nu) \cong S^{\text{spin}}(\nu) \otimes L^2(\nu)$.

Note that the tensor product on the right is globally defined, and so is the Hermitian line bundle $L(\nu) \to X$ defined by $L(\nu) := (L^2(\nu))^2$. Let $\nabla^\nu$ be a metric compatible connection on $\nu \to X$. It has a unique lift to a connection on $S^{\text{spin}}(\nu)$, still denoted by $\nabla^\nu$. Choose a unitary connection $\nabla^{L(\nu)}$ on $L(\nu) \to X$, which induces a connection on $L^2(\nu)$. The tensor product of $\nabla^\nu$ and the induced connection on $L^2(\nu)$ is a connection on $S(\nu) \to X$, denoted by $\tilde{\nabla}^\nu$. Define

$\text{Todd}(\tilde{\nabla}^\nu) := \tilde{A}(\nabla^\nu) \wedge e^{\frac{i}{2}c_1(\nabla^{L(\nu)})}$. 

The embedding pushforward $\tilde{\iota}_* : \tilde{K}_{FL}(X) \to \delta\tilde{K}_{FL}(Y)$ \cite[Definition 4.14]{7} is defined to be

$$\tilde{\iota}_*(\mathcal{E}) = \left( F, h^F, \nabla^F, \phi^E_{\text{Todd}(\nabla^\nu)} \wedge \delta_X - \gamma \right),$$

where $\delta\tilde{K}_{FL}(Y)$ is the currential $K$-group, $\delta_X$ is the current of integration over $X$ and $\gamma$ is the Bismut-Zhang current. $(F, h^F, \nabla^F)$ is a Hermitian bundle with a Hermitian metric and a unitary connection chosen as in \cite[Lemma 4.4]{7}. Note that $\gamma$ satisfies the following transgression formula \cite[Theorem 1.4]{2}

$$d\gamma = \text{ch}(\nabla^F) - \frac{\text{ch}(\nabla^E)}{\text{Todd}(\nabla^\nu)} \wedge \delta_X.$$
As noted in [7, p.926] the horizontal distributions of the fiber bundles \( X \to B \) and \( Y \to B \) need not be compatible. An odd form \( \widetilde{C} \in \Omega^{\text{odd}}(X) \) is defined to correct this non-compatibility, and it satisfies the following transgression formula \([7, (5.6)]\)

\[
d\widetilde{C} = \nu^* \text{Todd}(\widehat{\nabla}^T V_Y) - \text{Todd}(\widehat{\nabla}^T V_X) \wedge \text{Todd}(\widehat{\nabla}^\nu).
\]

The modified embedding pushforward \( \hat{\iota}^*_\text{mod} : \hat{K}_{\text{FL}}(X) \to \text{WF} \hat{K}_{\text{FL}}(Y) \) \([7, \text{Definition } 5.8]\) is defined to be

\[
\hat{\iota}^*_\text{mod}(E) := \hat{\iota}^*(E) - j \left( \nu^* \text{Todd}(\widehat{\nabla}^T V_Y) \wedge \text{ch}_{\hat{K}_{\text{FL}}}(E) \wedge \delta_X \right).
\]

See \([7, \S 3.1]\) for the definition of \( \text{WF} \hat{K}_{\text{FL}}(X) \).

The differential topological index \( \text{ind}^t_{\hat{K}_{\text{FL}}} : \hat{K}_{\text{FL}}(X) \to \hat{K}_{\text{FL}}(B) \) \([7, \text{Definition } 5.34]\) is defined by taking \( Y = S^N \times B \) for some even \( N \) and composing the embedding pushforward with the submersion pushforward \( \hat{\pi}^\text{prod} \) defined in \([7, \text{Lemma } 5.13]\), i.e., \( \text{ind}^t_{\hat{K}_{\text{FL}}} := \hat{\pi}^\text{prod} \circ \hat{\iota}^*_\text{mod} \).

### 2.2. Pairing between flat \( K \)-theory and topological \( K \)-homology.

Let \( X \) be an odd-dimensional closed spin\(^c\) manifold. Let \( \mathcal{E} = (E, h^E, \nabla^E, \phi) \in \delta \hat{K}_{\text{FL}}(X) \), and \( D^{X,E} \) be the twisted Dirac operator on \( S(X) \otimes E \to X \). A modified reduced eta-invariant \( \bar{\eta}(X, \mathcal{E}) \in \mathbb{R}/\mathbb{Z} \) \([7, \text{Definition } 2.33]\) is defined by

\[
\bar{\eta}(X, \mathcal{E}) := \eta(D^{X,E}) + \int_X \text{Todd}(\widehat{\nabla}^{TX}) \wedge \phi \mod \mathbb{Z}.
\]

\( \bar{\eta} : \delta \hat{K}_{\text{FL}}(X) \to \mathbb{R}/\mathbb{Z} \) is a well defined homomorphism \([7, \text{Prop } 2.25]\). If \( \mathcal{E} \) is a generator of \( K^{-1}_L(X; \mathbb{R}/\mathbb{Z}) \), by \([7, (2.37)]\) we have

\[
\bar{\eta}(X, i(\mathcal{E})) = \langle [X], \mathcal{E} \rangle,
\]

where \([X] \in K^{-1}_L(X)\) is the fundamental \( K \)-homology class. Here \( \langle [X], \mathcal{E} \rangle \) is the perfect pairing between flat \( K \)-theory and topological \( K \)-homology \([11, \text{Prop } 3]\)

\[
K^{-1}_L(X; \mathbb{R}/\mathbb{Z}) \times K^{-1}_L(X) \to \mathbb{R}/\mathbb{Z}.
\]

### 2.3. Bunke-Schick differential \( K \)-theory.

In this subsection we briefly recall Bunke-Schick differential \( K \)-theory \( \hat{K}_{\text{BS}} \), and refer to \([4]\) for the details.

A generator of \( \hat{K}_{\text{BS}}(B) \) is of the form \((\mathcal{E}, \phi)\), where \( \mathcal{E} \) is an even-dimensional geometric family \([4, \text{Definition } 2.2]\) over a compact manifold \( B \) and \( \phi \in \Omega^{\text{odd}}(B) / \text{Im}(d) \). Roughly speaking a geometric family over \( B \) is the geometric data needed to construct the index bundle. There is a well defined notion of isomorphic and sum of generators \([4, \text{Definition } 2.5, 2.6]\). Two geometric families \((\mathcal{E}_0, \phi_0)\) and \((\mathcal{E}_1, \phi_1)\) are equivalent if there exists a geometric family
(E′, φ′) such that (E′₀, ρ₀) + (E′, φ′) is paired with (E₁, ρ₁) + (E′, φ′) \[4, Definition 2.10, Lemma 2.13\]. Two generators (E₀, φ₀) and (E₁, φ₁) are paired if

ρ₁ − ρ₀ = ηₘₐₜₜ \((E₀ ⊔ (E₁)\text{op})\)

where \((E ⊔ (E′)\text{op})\) is a certain tamed geometric family \[4, Definition 2.7\], and \(ηₘₐₜₜ\) is the Bunke eta form \[3\].

As noted in \[4, 2.14\] and \[3, 4.2.1\], a complex vector bundle \(E → B\) with a Hermitian metric \(h\) and a unitary connection \(∇\) can be naturally considered as a zero-dimensional geometric family over \(B\), denoted by \(E\).

3. Main results

3.1. Compatibility of the topological indices. Note that every element \(E − F ∈ ˆK FL(X)\) can be written in the form

\(\tilde{E} − [n]\).

Here \(\tilde{E} = (E ⊕ G, hE ⊕ hG, ∇E ⊕ ∇G, φE + φG)\), where \((G, hG, ∇G, φG)\) is a generator of \(\tilde{K}_{FL}(X)\) such that

\((F ⊕ G, hF ⊕ hG, ∇F ⊕ ∇G, φF + φG) = (\mathbb{C}^n, h, d, 0) =: [n]\).

The existence of the connection \(∇G\) such that \(CS(∇E ⊕ ∇G, d) = 0\), where \(d\) is the trivial connection on the trivial bundle \(X × \mathbb{C}^n → X\), follows from \[12, Theorem 1.8\]. Here \(φG := −φF\). Henceforth we assume an element of \(\tilde{K}_{FL}(X; \mathbb{R}/\mathbb{Z})\) is of the form \(E − [n]\). These arguments also apply to elements in \(K_{L}^{-1}(X; \mathbb{R}/\mathbb{Z})\).

Proposition 1. Let \(π : X → B\) be a fiber bundle with \(X\) compact and such that the fibers are of even dimension. The following diagram commutes.

\[
\begin{array}{ccc}
K_{L}^{-1}(X; \mathbb{R}/\mathbb{Z}) & \xrightarrow{i} & \tilde{K}_{FL}(X) \\
\text{ind} & \downarrow & \text{ind}_{FL} \\
K_{L}^{-1}(B; \mathbb{R}/\mathbb{Z}) & \xrightarrow{i} & \tilde{K}_{FL}(B)
\end{array}
\]

Proof. Let \(\mathcal{E}' − [n'] ∈ K_{L}^{-1}(X)\) and write \(\mathcal{E} − [n] = i(\mathcal{E}' − [n'])\), where \(i\) is given in \[2\]. Consider the difference

\[
h := \text{ind}_{FL}(\mathcal{E} − [n]) − i(\text{ind}^{t}(\mathcal{E}' − [n'])).
\]

1It differs by a sign in \[4\].
We prove that \( h = 0 \). By [7, Lemma 5.36] and the fact that \( \text{ch}_{K_{FL}} \circ i = 0 \) (see (2)), we have

\[
\text{ch}_{K_{FL}}(\text{ind}_{FL}(E - [n])) - \text{ch}_{K_{FL}}(i(\text{ind}^i(E' - [n']))) = \int_{X/B} \text{Todd}(\nabla^{VX}) \wedge (\text{ch}(E) - \text{rank}(E) + d\phi^E) = 0.
\]

By [2], there exists an element \( a \in K^{-1}(B; \mathbb{R}/\mathbb{Z}) \) such that \( i(a) = h \). To prove \( a = 0 \in K^{-1}(B; \mathbb{R}/\mathbb{Z}) \), it follows from (5) that it is sufficient to show that for all \( \alpha \in \check{K}^{-1}_1(B; \mathbb{Z}) \),

\[
\langle \alpha, a \rangle = 0 \in \mathbb{R}/\mathbb{Z}.
\]

Using the geometric picture of \( K \)-homology [1], we may, without loss of generality, let \( \alpha = f_*[M] \) for some smooth map \( f : M \to B \), where \( M \) is a closed odd-dimensional spin\(^c\) manifold, and \([M]\) is the fundamental \( K \)homology in \( K_1(M) \). Since \( \langle \alpha, a \rangle = \langle [M], f^*a \rangle \), we pull everything back to \( M \) and we may assume \( B \) is an arbitrary closed odd-dimensional spin\(^c\) manifold. Thus proving (6) is equivalent to proving

\[
\langle [B], a \rangle = 0 \in \mathbb{R}/\mathbb{Z}.
\]

Since

\[
\langle [B], a \rangle = \bar{\eta}(B, \text{ind}_{FL}(E - [n])) - \bar{\eta}(B, i(\text{ind}^i(E' - [n']))) \mod \mathbb{Z},
\]

proving (7) is equivalent to proving

\[
\bar{\eta}(B, \text{ind}_{FL}(E - [n])) = \bar{\eta}(B, i(\text{ind}^i(E' - [n']))) \mod \mathbb{Z}.
\]

In the following, we write \( a \equiv b \mod \mathbb{Z} \). By [7, (6.7)], we have

\[
\bar{\eta}(B, \text{ind}_{FL}(E - [n])) \equiv \bar{\eta}(D^{X,E-n}) + \int_X \frac{t^* \text{Todd}(\nabla^{(S^N \times B)})}{\text{Todd}(\nabla^F)} \wedge \phi^E
\]

\[
- \int_X \frac{\pi^* \text{Todd}(\nabla^B)}{\text{Todd}(\nabla^F)} \wedge \check{C} \wedge \text{ch}_{K_{FL}}(E - [n]) \equiv \bar{\eta}(D^{X,E-n}) + \int_X \frac{t^* \text{Todd}(\nabla^{(S^N \times B)})}{\text{Todd}(\nabla^F)} \wedge \phi^E
\]

as \( \text{ch}_{K_{FL}}(E - [n]) = \text{ch}_{K_{FL}}(i(E' - [n'])) = 0 \). On the other hand, by [11] (49), we have

\[
\bar{\eta}(B, i(\text{ind}^i(E' - [n']))) \equiv \langle [B], \text{ind}^i(E' - [n']) \rangle
\]

\[
= \langle \pi^*[B], E - [n] \rangle = \langle [X], E - [n] \rangle = \bar{\eta}(X, E - [n]) \equiv \bar{\eta}(D^{X,E-n}) + \int_X \text{Todd}(\nabla^{TX}) \wedge \phi^E.
\]
From (9) and (10) we have
\[ \bar{\eta}(B, \text{ind}_{\text{FL}}(E-[n])) - \bar{\eta}(B, i(\text{ind}(E-[n]))) \]
\[ \equiv \int_X \left( \frac{i^* \text{Todd}(\nabla_{T(S^N \times B)})}{\text{Todd}(\nabla)} - \text{Todd}(\nabla_{TX}) \right) \wedge \phi_E \]  
\[ \equiv \int_X \left( \frac{i^* \text{Todd}(\nabla_{T(S^N \times B)}) - \text{Todd}(\nabla_{TX}) \wedge \text{Todd}(\nabla)}{\text{Todd}(\nabla)} \right) \wedge \phi_E. \]  
\[ (11) \]

Since \( \text{ch}_{\hat{K}_{\text{FL}}}(E-[n]) = 0 \), it follows from (3) that
\[ \tilde{\iota}^\text{mod}(E-[n]) = \tilde{\iota}(E-[n]). \]  
\[ (12) \]

Recall that the purpose of the modified embedding pushforward \( \tilde{\iota}^\text{mod} \) is to correct the non-compatibility of the horizontal distributions \( T^H(S^N \times B) \) and \( T^H X \). By (12) we may assume that the horizontal distributions \( T^H(S^N \times B) \) and \( T^H X \) are compatible by changing the one for \( X \) to be the restriction of the one for \( S^N \times B \). Thus
\[ i^* \text{Todd}(\nabla_{T(S^N \times B)}) = \text{Todd}(\nabla_{TX}) \wedge \text{Todd}(\nabla), \]
which implies that (11) is zero, and therefore \( h = 0 \). \( \square \)

3.2. Explicit isomorphisms between \( \hat{K}_{\text{BS}} \) and \( \hat{K}_{\text{FL}} \). In this subsection we construct the explicit isomorphisms between Bunke-Schick differential \( K \)-group and the Freed-Lott differential \( K \)-group.

**Proposition 2.** Let \( B \) be a compact manifold. Define two maps \( f : \hat{K}_{\text{FL}}(B) \rightarrow \hat{K}_{\text{BS}}(B) \) and \( g : \hat{K}_{\text{BS}}(B) \rightarrow \hat{K}_{\text{FL}}(B) \) by
\[ f(E, h, \nabla, \phi) = [E, \phi], \]
\[ g([E, \phi]) = (\text{ind}^a(\phi), h^\text{ind}^a(\phi), \nabla^\text{ind}^a(\phi), \phi), \]
where, in the definition of \( f \), \( E \) is the zero-dimensional geometric family associated to \( (E, h, \nabla) \). Then \( f \) and \( g \) are well defined ring isomorphisms and are inverses to each other.

**Proof.** Note that it suffices to prove the statement under the assumption that \( \text{ind}^a(\phi) \rightarrow B \) is actually given by a kernel bundle \( \ker(D^E) \rightarrow B \) in the definition of \( g \). Indeed, by a standard perturbation argument every class in \( \hat{K}_{\text{BS}} \) has a representative where this is satisfied.

First of all we prove that \( f \) is well defined. Suppose
\[ (E, h^E, \nabla^E, \phi^E) = (F, h^F, \nabla^F, \phi^F) \in \hat{K}_{\text{FL}}(B). \]
Then there exists a generator \( (G, h^G, \nabla^G, \phi^G) \) of \( \hat{K}_{\text{FL}}(B) \) such that
\[ E \oplus G \cong F \oplus G, \]
\[ \phi^F - \phi^E = CS(\nabla^E \oplus \nabla^G, \nabla^F \oplus \nabla^G). \]  
\[ (13) \]
Denote by \( \mathbb{F} \) and \( \mathbb{G} \) the zero-dimensional geometric families associated to \( (F, h^F, \nabla^F) \) and \( (G, h^G, \nabla^G) \), respectively. We prove that \( [E, \phi^E] = [F, \phi^F] \in \)
\( \hat{K}_{BS}(B) \). Indeed, we prove that \((\mathcal{E} + \mathcal{G}, \phi^E + \phi^G)\) is paired with \((\mathcal{F} + \mathcal{G}, \phi^F + \phi^G)\). We need to show \(\mathcal{E} \cup_B \mathcal{G} \cong \mathcal{F} \cup_B \mathcal{G}\) and

\[
(\phi^F + \phi^G) - (\phi^E + \phi^G) = \eta^B(\mathcal{E} \cup_B \mathcal{G} \oplus \mathcal{F} \cup_B \mathcal{G})
\]

if such a taming exists. In the case of zero-dimensional geometric family, \(\mathcal{E} \cup_B \mathcal{G} \cong \mathcal{E} \oplus \mathcal{G}\) as vector bundles over \(B\). Thus the first equality \((13)\) implies \(\mathcal{E} \cup_B \mathcal{G} \cong \mathcal{F} \cup_B \mathcal{G}\). Since the underlying proper submersion of the trivial geometric family is the identity map, the corresponding kernel bundle is just \(E \to B\) by the remark of \(\mathcal{E}\) Definition 4.7. Thus the taming in \((14)\) exists and the definition of \(\eta^B\) shows that

\[
\eta^B(\mathcal{E} \cup_B \mathcal{G} \oplus \mathcal{F} \cup_B \mathcal{G}) = \text{CS}(\nabla^E \oplus \nabla^G, \nabla^F \oplus \nabla^G).
\]

From \((13)\) and \((14)\) we see that \((\mathcal{E} + \mathcal{G}, \phi^E + \phi^G)\) is paired with \((\mathcal{F} + \mathcal{G}, \phi^F + \phi^G)\). Thus \(f\) is well defined.

For the map \(g\), note that under our assumption we have \([\mathcal{E}, 0] = [\mathcal{K}, \tilde{\eta}(\mathcal{E})]\) by \([1]\) Corollary 5.5, where \(\mathcal{K}\) is the trivial geometric family associated to \((\ker(D^E), h_{\ker(D^E)}, \nabla_{\ker(D^E)})\) and \(\tilde{\eta}(\mathcal{E})\) is the associated Bismut-Cheeger eta form. Since \([\mathcal{E}, \phi] = [\mathcal{K}, \tilde{\eta}(\mathcal{E}) + \phi], g\) can be written as

\[
g([\mathcal{E}, \phi]) = g([\mathcal{K}, \tilde{\eta}(\mathcal{E}) + \phi]) = (\ker(D^E), h_{\ker(D^E)}, \nabla_{\ker(D^E)}, \tilde{\eta}(\mathcal{E}) + \phi).
\]

We prove that \(g\) is well defined. Suppose \([\mathcal{E}_1, \phi^1] = [\mathcal{E}_2, \phi^2] \in \hat{K}_{BS}(B)\). Since \([\mathcal{E}_i, \phi^i] = [\mathcal{K}_i, \tilde{\eta}(\mathcal{E}_i) + \phi^i]\) for \(i = 1, 2\), to prove \(g([\mathcal{E}_1, \phi^1]) = g([\mathcal{E}_2, \phi^2])\) it suffices to show

\[
(\ker(D^{E_1}), h_{\ker(D^{E_1})}, \nabla_{\ker(D^{E_1})}, \tilde{\eta}^{(1)}) = (\ker(D^{E_2}), h_{\ker(D^{E_2})}, \nabla_{\ker(D^{E_2})}, \tilde{\eta}^{(2)}).
\]

Since \([\mathcal{E}_1, \phi^1] = [\mathcal{E}_2, \phi^2]\), there exists a taming \((\mathcal{E}_1 \cup_B (\mathcal{E}_2)^{\text{op}})\), and therefore \(\ker(D^{E_1}) = \ker(D^{E_2}) \in K(B)\). Thus it suffices to show

\[
\text{CS}(\nabla_{\ker(D^{E_2})}, \nabla_{\ker(D^{E_1})}) = \tilde{\eta}(\mathcal{E}_1) - \tilde{\eta}(\mathcal{E}_2) - \phi^2 < \eta^B((\mathcal{K}_2 \cup_B (\mathcal{K}_1)^{\text{op}})\).
\]

by the exactness of \([7]\) (2.21)]. Since

\[
[\mathcal{K}_1, \tilde{\eta}(\mathcal{E}_1) + \phi^1] = [\mathcal{E}_1, \phi^1] = [\mathcal{E}_2, \phi^2] = [\mathcal{K}_2, \tilde{\eta}(\mathcal{E}_2) + \phi^2],
\]

it follows that there exists a taming \((\mathcal{K}_2 \cup_B (\mathcal{K}_1)^{\text{op}})\) such that

\[
\tilde{\eta}(\mathcal{E}_1) - \tilde{\eta}(\mathcal{E}_2) + \phi^1 - \phi^2 = \eta^B((\mathcal{K}_2 \cup_B (\mathcal{K}_1)^{\text{op}})\).
\]

By the same reason as in \((15)\) we have

\[
\eta^B((\mathcal{K}_2 \cup_B (\mathcal{K}_1)^{\text{op}})\) = \text{CS}(\nabla_{\ker(D^{E_2})}, \nabla_{\ker(D^{E_1})}).
\]

\([17]\) follows by comparing \((18)\) and \((19)\). Thus \(g\) is well defined.

We prove that \(f\) and \(g\) are inverses to each other. Let \((E, h, \nabla, \phi)\) be a generator of \(\hat{K}_{FL}(B)\). Then

\[
(g \circ f)(E, h, \nabla, \phi) = g([\mathcal{E}, \phi]) = (E, h, \nabla, \phi).
\]
On the other hand, for a generator \((\mathcal{E}, \phi)\) of \(\hat{K}_{BS}(B)\),

\[
(f \circ g)([\mathcal{E}, \phi]) = f(\ker(D^E), h^{\ker(D^E)}, \overline{\nabla}^{\ker(D^E)}, \overline{\eta}(\mathcal{E}) + \phi) = [K, \overline{\eta}(\mathcal{E}) + \phi] = [\mathcal{E}, \phi]
\]

by [4, Corollary 5.5] again.

Since \(f\) is a ring homomorphism, the same is true for \(g\). Thus \(f\) and \(g\) are ring isomorphisms and are inverses to each other. 

\[\square\]

References


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