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Effects on the normalized Laplacian spectral radius of non-bipartite graphs under perturbation and their applications*

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Abstract

The normalized Laplacian eigenvalues of a network play an important role in its structural and dynamical aspects associated with the network. In this paper, we consider how the normalized Laplacian spectral radius of a non-bipartite graph behaves by several graph operations. As an example of the application, the smallest normalized Laplacian spectral radius of non-bipartite unicyclic graphs with fixed order is determined.

Keywords: Normalized Laplacian spectral radius; non-bipartite graph; unicyclic graph

AMS classification: 05C50; 15A18.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E. Let $d_G(v_i)$ (or simply $d(v_i)$) denote the degree of the vertex $v_i \in V$ $(i = 1, 2, \ldots, n)$, and $D = D(G) = diag(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. The *Laplacian matrix* of G is defined by $L(G) = D(G) - A(G)$ and the *normalized Laplacian matrix* of G is defined by $\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2} = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$ (with the convention that if the degree of v is 0 then $d(v)^{-1/2} = 0$), where $A(G)$ and I denote the adjacency matrix of G and the identity matrix, respectively. It is easy to see that $\mathcal{L}(G)$ is a symmetric positive semidefinite matrix and $D(G)^{1/2}\boldsymbol{j}_n$ is an eigenvector of $\mathcal{L}(G)$ with eigenvalue 0,

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where $\boldsymbol{j}_n$ is the vector in $\mathbb{R}^n$ whose entries are 1. Denote the eigenvalues of $\mathcal{L}(G)$ by

$$0 = \lambda_n(G) \leq \lambda_{n-1}(G) \leq \cdots \leq \lambda_2(G) \leq \lambda_1(G),$$

which are always enumerated in non-increasing order and repeated according to their multiplicity.

Since it is related to Markov chains and random walks [3] and the application in chemistry [18], the matrices $D^{-1}A$ and $D(G)^{-1/2}A(G)D(G)^{-1/2}$ are known as the transition matrix and the Randić matrix, respectively. It is easy to see that the transition matrix and the Randić matrix are similar with the same eigenvalues $1-\lambda_n(G), 1-\lambda_{n-1}(G), \cdots, 1-\lambda_2(G), 1-\lambda_1(G)$. The largest eigenvalue $\lambda_1(G)$ of $\mathcal{L}(G)$ is called the *normalized Laplacian spectral radius* of G, denoted by $\lambda(G)$. Chung [6] proved that for a connected graph G with $n \geq 2$ vertices, $\frac{n}{n-1} \leq \lambda(G) \leq 2$, the left equality holds if and only if G is a complete graph, and the right equality holds if and only if G is a bipartite graph.

The normalized Laplacian is a rather new but important tool popularized by Chung in the mid 1990s. As pointed out by Fan Chung [6], the eigenvalues of the normalized Laplacians are in a normalized form, and the spectra of the normalized Laplacians relate well to other graph invariants for general graphs in a way that the other two definitions (such as the eigenvalues of adjacency matrix) fail to do. The eigenvalues of the normalized Laplacian matrix of a network play an important role in its structural and dynamical aspects associated with the network [12]. The advantages of this definition are perhaps due to the fact that it is consistent with the eigenvalues in spectral geometry and in stochastic processes. For more details, see [1,4,6,7]. For a fixed list $\{v_1, \ldots, v_n\}$ of vertices of G, let $X = (x_1, \ldots, x_n)^T$ be a real vector. It can be viewed as a labeling of G in which vertex v_i is labeled by x_i (or $X(v_i)$). Such labeling is sometimes called a *valuation* [17] of G. If X is a unit eigenvector of G corresponding to $\lambda(G)$, then we have

$$\lambda(G) = \max_{\substack{Y\in\mathbb{R}^n \\ \|Y\|=1}} Y^T\mathcal{L}(G)Y = X^T\mathcal{L}(G)X = \sum_{\substack{v_iv_j\in E \\ 1\leq i<j\leq n}} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}}\right)^2. \tag{1.1}$$

Let P_n and C_n denote the path and the cycle with n vertices, respectively. It is a well known fact [6] that if g is odd, then we have

$$\lambda(C_g) = \lambda_g(P_{g+1}) = 1 - \cos\frac{(g-1)\pi}{g}. \tag{1.2}$$

A unicyclic graph is a connected graph in which the number of vertices equals the number of edges. So, if G is a unicyclic graph with girth g, then G consists of the unique cycle (say C_g) of length g and a certain number of trees attached at vertices of C_g having in total $n-g$ edges. The adjacency spectral radius of unicyclic graphs is well studied (see [11] and the references therein).

For $v \in V(G)$, let $\mathcal{L}_v(G))$ be the principal submatrix of $\mathcal{L}(G)$ formed by deleting the row and column corresponding to the vertex v. Throughout this paper, we shall denote by $\Phi(B) = \Phi(B;x) = \det(xI-B)$ the *characteristic polynomial* of the square matrix B. In particular, if $B = \mathcal{L}(G)$, we write $\Phi(\mathcal{L}(G))$ by $\Phi(G;x)$ or simply by $\Phi(G)$ and call $\Phi(G)$ the *normalized Laplacian characteristic polynomial* of G. If $G = v$, it is both convenient and consistent to define $\Phi(\mathcal{L}_v(G)) = 1$.

In [8], Guo investigated how the Laplacian spectral radius changes when one graph is transferred to another graph obtained from the original graph by adding some edges, or subdivision, or removing some edges from one vertex to another. In [2], Banerjee and Jostwe investigated the behavior of the normalized Laplacian spectrum under local and global operations like motif doubling, graph joining or splitting. In this paper, we consider how the normalized Laplacian spectral radius of a non-bipartite graph behaves by several graph operations, such as deleting an edge, coalescence, separating an edge, subdivision and so on. As an example of the application, the smallest normalized Laplacian spectral radius among non-bipartite unicyclic graphs with fixed order is determined.

2 The effect on the normalized Laplacian spectral radius of a graph under the operations: deleting an edge; coalescence

The following inequalities are known as *Cauchy's* inequalities and the whole theorem is also known as interlacing theorem.

Lemma 2.1 (**[14]**) *Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and B be a principal submatrix of order m; let B have eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ $(i = 1, 2, \cdots, m)$ hold.*

In [5,15], the authors considered how the normalized Laplacian eigenvalues of a graph behave by deleting an edge or some vertices.

Lemma 2.2 *Let G be a simple graph and $H = G - e$ be the graph obtained from G by removing an edge e. Then $\lambda_{i+1}(G) \leq \lambda_i(H) \leq \lambda_{i-1}(G)$ for $i = 1, \ldots, n$, where $\lambda_0(G) = 2$ and $\lambda_{n+1}(G) = 0$.*

For the normalized Laplacian spectral radius of a non-bipartite graph by deleting a pendant vertex, we have the following result.

Lemma 2.3 (**[10]**) *Suppose that u is a vertex of a non-bipartite graph G. Let G_v be the graph obtained from G by attaching a pendant edge uv at u of G. Let X be a unit eigenvector of G corresponding to $\lambda(G)$. Then we have $\lambda(G_v) \geq \lambda(G)$, the inequality is strict if $X(u) \neq 0$.*

Let G, r and H, s be two disjoint rooted graphs with roots r and s, respectively. The coalescence of two rooted graphs G, r and H, s, denoted by $G \cdot H$, is the graph formed by identifying the two roots r and s.

Lemma 2.4 (**[9]**) *If G and H are two rooted graphs with roots r and s, respectively, then the normalized Laplacian characteristic polynomial of the coalescence $G \cdot H$ satisfies*

$$\Phi(G \cdot H) = \frac{d_G(r)\Phi(G)\Phi(\mathcal{L}_s(H)) + d_H(s)\Phi(H)\Phi(\mathcal{L}_r(G))}{d_G(r) + d_H(s)}.$$

As the generalization of Lemma 2.3, we have the following result.

Theorem 2.5 *Let G, r and H, s be two disjoint rooted connected graphs with roots r and s, respectively, and $G \cdot H$ be the coalescence of G and H. If $\lambda(G) \geq \lambda(H)$, then we have $\lambda(G) \geq \lambda(G \cdot H) \geq \lambda(H)$*

Proof. From Lemma 2.1, we have

$$\lambda(G \cdot H) \geq \lambda(\mathcal{L}_r(G)) \geq \lambda_2(G)$$

and

$$\lambda(G \cdot H) \geq \lambda(\mathcal{L}_r(H)) \geq \lambda_2(H).$$

Thus from Lemma 2.4, the result follows. □

In the following, we will consider how the normalized Laplacian spectral radius of a graph behaves by deleting an edge.

Theorem 2.6 *Let uv be an edge of the connected graph G, and X be a unit eigenvector of G corresponding to $\lambda(G)$. If $X(u)X(v) \geq 0$, then*

$$\lambda(G-uv) \geq \lambda(G),$$

the inequality is strict if $X(u) \neq 0$ or $X(v) \neq 0$.

Proof. Let Y be a valuation of $G-uv$ defined by

$$\begin{cases} Y(u) = \sqrt{\frac{d_G(u)-1}{d_G(u)}}X(u); \\ Y(v) = \sqrt{\frac{d_G(v)-1}{d_G(v)}}X(v); \\ Y(w) = X(w), \ w \neq u, v. \end{cases}$$

From Eq. (1.1), we have

$$\begin{aligned} \lambda(G-uv) - \lambda(G) &\geq \frac{Y^T\mathcal{L}(G-uv)Y}{Y^TY} - \lambda(G) \\ &= \frac{\lambda(G) - (\frac{X(u)}{\sqrt{d_G(u)}} - \frac{X(v)}{\sqrt{d_G(v)}})^2}{1 - \frac{X^2(u)}{d_G(u)} - \frac{X^2(v)}{d_G(v)}} - \lambda(G) \\ &= \frac{(\lambda(G)-1)(\frac{X^2(u)}{d_G(u)} + \frac{X^2(v)}{d_G(v)}) + \frac{2X(u)X(v)}{\sqrt{d_G(u)d_G(v)}}}{1 - \frac{X^2(u)}{d_G(u)} - \frac{X^2(v)}{d_G(v)}} \\ &\geq 0. \end{aligned}$$

Therefore, we have $\lambda(G-uv) \geq \lambda(G)$, and the inequality is strict if $X(u) \neq 0$ or $X(v) \neq 0$. □

3 The effect on the normalized Laplacian spectral radius by separating an edge

Lemma 3.1 *Let X be an eigenvector of $\mathcal{L}(G)$ corresponding to $\lambda(G)$. Then for any $v \in V(G)$, we have*

$$\sum_{uv \in E(G)} \frac{X(u)}{\sqrt{d(u)}} = \sqrt{d(v)}(1-\lambda(G))X(v).$$

Proof. The result follows from $D(G)^{-1/2}(D(G) - A(G))D(G)^{-1/2}X = \lambda(G)X$. □

Lemma 3.2 ([9]) *Let v be a pendant vertex of graph G, and u be the vertex adjacent to v. Then*

$$\Phi(G) = \frac{(d(u)-1)(x-1)}{d(u)}\Phi(G-v) + \frac{x^2-2x}{d(u)}\Phi(\mathcal{L}_{uv}(G)).$$

Lemma 3.3 ([9]) *Let $e = uv$ be a cut edge of G. The normalized Laplacian characteristic polynomial of G satisfies*

$$\Phi(G) = \frac{(d(u)-1)(d(v)-1)}{d(u)d(v)}\Phi(G-e) + \frac{(d(v)-1)(x-1)}{d(u)d(v)}\Phi(\mathcal{L}_u(G-e))$$
$$+ \frac{(d(u)-1)(x-1)}{d(u)d(v)}\Phi(\mathcal{L}_v(G-e)) + \frac{x(x-2)}{d(u)d(v)}\Phi(\mathcal{L}_{uv}(G)).$$

Let $S_1 = (a_1, a_2, \cdots, a_n)$ and $S_2 = (b_1, b_2, \cdots, b_n)$ be two n-tuples satisfying that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. Set $a_i = +\infty$ when $i \leq 0$ and $a_i = -\infty$ when $i \geq n+1$, and let k be a nonnegative integer. S_2 is said to k-interlace S_1 if $a_{i-k} \geq b_i \geq a_{i+k}$ for each $i = 1, 2, \cdots, n$. Let G be a graph and u, v be two vertices of G. Suppose $\{v_1, v_2, \cdots, v_s\} \subseteq N_G(v)\backslash\{N_G(u) \cup \{u\}\}$ where $1 \leq s \leq d_G(v)$, and $H = G - vv_1 - \cdots - vv_s + uv_1 + \cdots + uv_s$. Then we say that the graph H is obtained from G by $\{v_1, v_2, \cdots, v_s\}$-neighbors moving from v to u.

Lemma 3.4 ([19]) *Let G be a graph and u, v be two vertices of G. Suppose $\{v_1, v_2, \cdots, v_s\} \subseteq N_G(v)\backslash N_G(u) \bigcup\{u\}$, and H is the graph obtained from G by $\{v_1, v_2, \cdots, v_s\}$-neighbors moving from v to u. Then the spectra of $A(H)$ and $A(G)$ 1-interlace with each other. So do the spectra of $L(H)$ and $L(G)$, and the spectra of $\mathcal{L}(H)$ and $\mathcal{L}(G)$.*

Separating an edge: Let $e = uv$ be an edge of a graph G. By G' we denote the graph obtained from G by contracting the edge e into a new vertex u_e, which becomes adjacent to all the former neighbours of u and of v, and adding a new pendent edge u_ev_e, where v_e is a new pendent vertex. We say that G' is obtained from G by *separating* an edge uv (see Fig. 1).

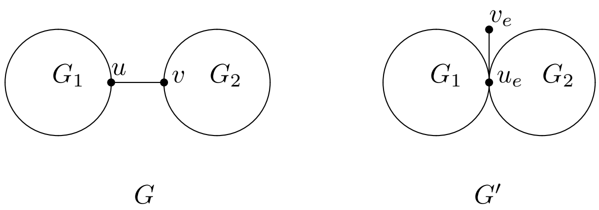

Fig. 1. Separating a cut edge uv

In the following, we consider how the normalized Laplacian spectral radius of a graph behaves by separating a cut edge.

Theorem 3.5 *Let $e = uv$ be a cut edge of the connected graph G and suppose that $G - uv = G_1 \cup G_2$, $|V(G_i)| = n_i \geq 2$, $(i = 1, 2)$, $u \in V(G_1), v \in V(G_2)$, $\lambda(G_1) \geq \lambda(G_2)$. Let G' be the graph obtained from G by separating the edge uv (see Fig. 1). Then we have*

(1) If both G_1 and G_2 are bipartite graphs, then $\lambda(G') = \lambda(G) = 2$;

(2) If G_1 is a bipartite graph and G_2 is a nonbipartite graph, then $\lambda(G') \leq \lambda(G)$;

(3) If both G_1 and G_2 are nonbipartite graphs and $\lambda(G_1) = \lambda(G_2 + vu)$, then $\lambda(G') = \lambda(G) = \lambda(G_1)$;

(4) If both G_1 and G_2 are nonbipartite graphs and $\lambda(G_1) < \lambda(G_2 + vu)$, then $\lambda(G') \geq \lambda(G)$;

(5) If both G_1 and G_2 are nonbipartite graphs and $\lambda(G_1) > \lambda(G_2 + vu)$, then $\lambda(G') \leq \lambda(G)$.

Proof. Since both G_1 and G_2 are bipartite graphs, we have both G and G' are bipartite graphs. Then $\lambda(G') = \lambda(G) = 2$. (1) holds.

We now prove that (2) holds. From Lemma 3.2, we have

$$\Phi(G') = \frac{(d_{G_1}(u) + d_{G_2}(v))(x-1)}{d_{G_1}(u) + d_{G_2}(v) + 1}\Phi(G' - v_e) + \frac{x(x-2)}{d_{G_1}(u) + d_{G_2}(v) + 1}\Phi(\mathcal{L}_u(G_1))\Phi(\mathcal{L}_v(G_2)). \tag{3.3}$$

From Lemma 2.4, we have

$$\Phi(G' - v_e) = \frac{d_{G_1}(u)\Phi(G_1)\Phi(\mathcal{L}_v(G_2)) + d_{G_2}(v)\Phi(G_2)\Phi(\mathcal{L}_u(G_1))}{d_{G_1}(u) + d_{G_2}(v)}. \tag{3.4}$$

From Lemma 3.3, we have

$$\begin{aligned}
\Phi(G) = {} & \frac{d_{G_1}(u)d_{G_2}(v)}{(d_{G_1}(u)+1)(d_{G_2}(v)+1)}\Phi(G_1)\Phi(G_2) + \frac{d_{G_2}(v)(x-1)}{(d_{G_1}(u)+1)(d_{G_2}(v)+1)}\Phi(\mathcal{L}_u(G_1))\Phi(G_2) \\
& + \frac{d_{G_1}(u)(x-1)}{(d_{G_1}(u)+1)(d_{G_2}(v)+1)}\Phi(G_1)\Phi(\mathcal{L}_v(G_2)) \\
& + \frac{x(x-2)}{(d_{G_1}(u)+1)(d_{G_2}(v)+1)}\Phi(\mathcal{L}_u(G_1))\Phi(\mathcal{L}_v(G_2)).
\end{aligned} \tag{3.5}$$

From Eqs. (3.3)-(3.5), we have

$$\begin{aligned}
& (d_{G_1}(u) + d_{G_2}(v) + 1)\Phi(G') - (d_{G_1}(u)+1)(d_{G_2}(v)+1)\Phi(G) \\
& = (d_{G_1}(u) + d_{G_2}(v))(x-1)\Phi(G' - v) + x(x-2)\Phi(\mathcal{L}_u(G_1))\Phi(\mathcal{L}_v(G_2)) \\
& - d_{G_1}(u)d_{G_2}(v)\Phi(G_1)\Phi(G_2) - d_{G_2}(v)(x-1)\Phi(\mathcal{L}_u(G_1))\Phi(G_2) \\
& - d_{G_1}(u)(x-1)\Phi(G_1)\Phi(\mathcal{L}_v(G_2)) - x(x-2)\Phi(\mathcal{L}_u(G_1))\Phi(\mathcal{L}_v(G_2)) \\
& = d_{G_1}(u)(x-1)\Phi(G_1)\Phi(\mathcal{L}_v(G_2)) + d_{G_2}(v)(x-1)\Phi(\mathcal{L}_u(G_1))\Phi(G_2) \\
& - d_{G_1}(u)d_{G_2}(v)\Phi(G_1)\Phi(G_2) - d_{G_2}(v)(x-1)\Phi(\mathcal{L}_u(G_1))\Phi(G_2) \\
& - d_{G_1}(u)(x-1)\Phi(G_1)\Phi(\mathcal{L}_v(G_2)) \\
& = -d_{G_1}(u)d_{G_2}(v)\Phi(G_1)\Phi(G_2).
\end{aligned} \tag{3.6}$$

Since G_1 is a bipartite graph and G_2 is a nonbipartite graph, we have $\lambda(G_1) = 2 > \lambda(G')$. Thus from Lemma 2.2, we have $\lambda(G) \geq \max\{\lambda_2(G_1), \lambda(G_2)\}$. If $\lambda(G') \leq \max\{\lambda_2(G_1), \lambda(G_2)\}$, then we have $\lambda(G) \geq \lambda(G')$; If $\lambda(G') > \max\{\lambda_2(G_1), \lambda(G_2)\}$, then from Eq. (3.6), we have

$$\begin{aligned}
& (d_{G_1}(u) + d_{G_2}(v) + 1)\Phi(G'; \lambda(G')) - (d_{G_1}(u)+1)(d_{G_2}(v)+1)\Phi(G; \lambda(G')) \\
& = -d_{G_1}(u)d_{G_2}(v)\Phi(G_1; \lambda(G'))\Phi(G_2; \lambda(G')) > 0.
\end{aligned}$$

Then, we have $\lambda(G) > \lambda(G')$. This completes the proof of (2).

In the following, we prove that (3) holds. Note that G can be considered as the coalescence of G_1 and $G_2 + vu$. Then from Lemma 2.4, we have

$$\Phi(G) = \frac{\Phi(\mathcal{L}_u(G_1))\Phi(G_2 + vu) + d_{G_1}(u)\Phi(G_1)\Phi(\mathcal{L}_u(G_2 + vu))}{d_{G_1}(u) + 1}. \tag{3.7}$$

and

$$\Phi(G') = \frac{d_{G_1}(u)(x-1)\Phi(G_1)\Phi(\mathcal{L}_v(G_2)) + (d_{G_2}(v)+1)\Phi(G_2 + vu)\Phi(\mathcal{L}_u(G_1))}{d_{G_1}(u) + d_{G_2}(v) + 1}. \tag{3.8}$$

From Lemmas 2.1 and 2.3, we have

$$\lambda(G_1) \geq \lambda(\mathcal{L}_u(G_1)), \lambda(G_2 + vu) \geq \lambda(\mathcal{L}_u(G_2 + vu)), \lambda(G_2 + vu) \geq \lambda(G_2) \geq \lambda(\mathcal{L}_v(G_2)).$$

Thus from Eqs. (3.7) and (3.8) and the assumption of (3), we have

$$\lambda(G) = \lambda(G') = \lambda(G_1) = \lambda(G_2 + vu).$$

This completes the proof of (3).

Fourthly, we prove that (4) holds. From Lemma 3.4 and the assumption, we have

$$\lambda(G) \geq \max\{\lambda(G_1), \lambda_2(G_2 + vu)\} \geq \lambda(G_1) \geq \lambda(G_2).$$

Then from Eq. (3.6), we have

$$\begin{aligned}&(d_{G_1}(u) + d_{G_2}(v) + 1)\Phi(G'; \lambda(G)) - (d_{G_1}(u) + 1)(d_{G_2}(v) + 1)\Phi(G; \lambda(G))\\ &= -d_{G_1}(u)d_{G_2}(v)\Phi(G_1; \lambda(G))\Phi(G_2; \lambda(G)) \leq 0.\end{aligned}$$

Thus, we have $\lambda(G) \leq \lambda(G')$. This completes the proof of (4).

Finally, we prove that (5) holds. From Lemmas 2.3 and 3.4, we have

$$\lambda(G') \geq \max\{\lambda_2(G_1), \lambda(G_2 + vu)\} \geq \max\{\lambda_2(G_1), \lambda(G_2)\}. \tag{3.9}$$

From Lemma 2.1, we have

$$\lambda(G') \geq \max\{\lambda_1(\mathcal{L}_u(G_1)), \lambda_1(\mathcal{L}_v(G_2))\}.$$

Thus from Eq. (3.8), we have for $x \geq \lambda(G_1)$, $\Phi(G'; x) \geq 0$. Then we have $\lambda(G_1) \geq \lambda(G')$. From Eqs. (3.6) and (3.9), we have

$$\begin{aligned}&(d_{G_1}(u) + d_{G_2}(v) + 1)\Phi(G'; \lambda(G')) - (d_{G_1}(u) + 1)(d_{G_2}(v) + 1)\Phi(G; \lambda(G'))\\ &= -d_{G_1}(u)d_{G_2}(v)\Phi(G_1; \lambda(G'))\Phi(G_2; \lambda(G')) \geq 0.\end{aligned}$$

Then, we have $\lambda(G) \geq \lambda(G')$. This completes the proof of (5). $\square$

From Theorem 3.5, we immediately have the following result:

Corollary 3.6 *Let C_g : $v_1v_2 \cdots v_gv_1$ be a cycle with length g, and $C_{g;\ n_1n_2\cdots n_g}$ be the unicyclic graph obtained from C_g by attaching n_i pendent edges at v_i $(1 \leq i \leq g)$, respectively, where $n_1 + n_2 + \cdots + n_g = n - g$. Then for some choice of the parameters $n_1, \cdots, n_g$, the graph $C_{g;\ n_1n_2\cdots n_g}$ minimizes the normalized Laplacian spectral radius over the class of unicyclic graphs with girth g.*

Let G and G' be the graphs defined in Theorem 3.5. From Lemma 2.3, we have $\lambda(G') \geq \lambda(G' - v_e)$. The following result displays the relation between $\lambda(G)$ and $\lambda(G' - v_e)$.

Theorem 3.7 *Let G and G' be the graphs defined in Theorem 3.5. Then we have $\lambda(G) \geq \lambda(G' - v_e)$.*

Proof. Without loss of generality, we suppose that $\lambda(G_1) \geq \lambda(G_2)$ and $G'' = G' - v_e$.

From Eqs. (3.4) and (3.5), we have

$$\begin{aligned}&(d_{G_1}(u)+d_{G_2}(v))(x-1)\Phi(G'')-(d_{G_1}(u)+1)(d_{G_2}(v)+1)\Phi(G)\\&=-d_{G_1}(u)d_{G_2}(v)\Phi(G_1)\Phi(G_2)-x(x-2)\Phi(\mathcal{L}_u(G_1))\Phi(\mathcal{L}_v(G_2)).\end{aligned}\tag{3.10}$$

From Lemma 2.2, we have $\lambda(G)\geq\max\{\lambda_2(G_1),\lambda(G_2)\}$. If $\lambda(G'')\leq\max\{\lambda_2(G_1),\lambda(G_2)\}$, the result follows. Suppose that $\lambda(G'')>\max\{\lambda_2(G_1),\lambda(G_2)\}$. From Lemma 2.1, we have

$$\lambda(G'')\geq\max\{\lambda_1(\mathcal{L}_u(G_1)),\lambda_1(\mathcal{L}_v(G_2))\}.$$

From Eq. (3.4), we have for $x\geq\lambda(G_1)$, $\Phi(G'';x)\geq 0$. Then $\lambda(G'')\leq\lambda(G_1)$. Thus from Eq. (3.10), we have

$$\begin{aligned}&(d_{G_1}(u)+d_{G_2}(v))(\lambda(G'')-1)\Phi(G'';\lambda(G''))-(d_{G_1}(u)+1)(d_{G_2}(v)+1)\Phi(G;\lambda(G''))\\&=-d_{G_1}(u)d_{G_2}(v)\Phi(G_1;\lambda(G''))\Phi(G_2;\lambda(G''))\\&-\lambda(G'')(\lambda(G'')-2)\Phi(\mathcal{L}_u(G_1);\lambda(G''))\Phi(\mathcal{L}_v(G_2);\lambda(G''))\geq 0.\end{aligned}$$

So, we have $\lambda(G)\geq\lambda(G'-v_e)$. □

In [16], the authors obtained the following result on the second smallest normalized eigenvalue. Now we will give a simpler proof by using the technique of the normalized Laplacian characteristic polynomials.

Theorem 3.8 *Let $e=uv$ be a cut edge of the connected graph G and suppose that $G-uv=G_1\cup G_2$, $|V(G_i)|=n_i\geq 2$, $(i=1,2)$, $u\in V(G_1), v\in V(G_2)$. Let G' be the graph obtained from G by separating the edge uv (see Fig. 1). Then we have $\alpha(G')\geq\alpha(G)$, where $\alpha(G)$ denotes the second smallest normalized eigenvalue of G. If the equality holds, then $\alpha(G)=\min\{\alpha(G_1),\alpha(G_2)\}$.*

Proof. From Lemma 2.2, we have $\alpha(G)\leq\min\{\alpha(G_1),\alpha(G_2)\}$. If $\alpha(G')>\min\{\alpha(G_1),\alpha(G_2)\}$, we have $\alpha(G')>\alpha(G)$, the result follows. Suppose that $\alpha(G')\leq\min\{\alpha(G_1),\alpha(G_2)\}$. From Eq. (3.6), we have

$$\begin{aligned}&(-1)^{n_1+n_2-1}[(d_{G_1}(u)+d_{G_2}(v)+1)\Phi(G';\alpha(G'))-(d_{G_1}(u)+1)(d_{G_2}(v)+1)\Phi(G;\alpha(G'))]\\&=(-1)^{n_1+n_2-2}d_{G_1}(u)d_{G_2}(v)\Phi(G_1;\alpha(G'))\Phi(G_2;\alpha(G'))\geq 0.\end{aligned}$$

Then, we have $\alpha(G)\leq\alpha(G')$, and if $\alpha(G)=\alpha(G')$, then $\alpha(G')=\min\{\alpha(G_1),\alpha(G_2)\}$. □

In the following, we consider the effect on the normalized Laplacian spectral radius by separating two adjacent edges and subdividing an edge, respectively.

Theorem 3.9 *Let v_1v_2, v_2v_3 be two edges of a connected graph G with s pendant edges $v_2v_{21},\cdots,v_2v_{2s}$ at vertex v_2 and $d(v_2)=s+2$, $v_1v_3\notin E(G)$, v_1, v_2 and v_3 do not lie on C_4. Let G' be the graph obtained from G by contracting the edges v_1v_2, v_2v_3 into a new vertex (i. e. deleting edges v_1v_2, v_2v_3, and identifying vertices v_1, v_2, and v_3 into a new vertex), say v, which becomes adjacent to all the former neighbours of v_1, v_2 and of v_3, and adding two new pendent edges vv_{s+1} and vv_{s+2}. Then $\lambda(G)\geq\lambda(G')$.*

Proof. Let X be a unit eigenvector of G' corresponding to $\lambda(G')$. From $\mathcal{L}(G')X=\lambda(G')X$, we immediately have

$$X(v_{21})=X(v_{22})=\cdots=X(v_{2s})=X(v_{s+1})=X(v_{s+2}).$$

Let $V_1 = V(G) - \{v_1, v_2, v_3, v_{21}, v_{22}, \cdots, v_{2s}\}$ and $V_1' = V(G') - \{v, v_{s+1}, v_{s+2}, v_{21}, v_{22}, \cdots, v_{2s}\}$. It is obvious that $V_1 = V_1'$.

Let Y be a valuation of G defined by

$$\begin{cases} Y(v_1) = \sqrt{\frac{d_G(v_1)}{d_{G'}(v)}} X(v); \\ Y(v_3) = \sqrt{\frac{d_G(v_3)}{d_{G'}(v)}} X(v); \\ Y(v_2) = \sqrt{s+2} X(v_{s+1}); \\ Y(v_{2i}) = \sqrt{\frac{1}{d_{G'}(v)}} X(v), \ i = 1, \ldots, s; \\ Y(w) = X(w), \ w \in V_1. \end{cases}$$

Note that $d_{G'}(v) = d_G(v_1) + d_G(v_3) + s$. Thus we have

$$\begin{aligned} Y^T Y &= \frac{d_G(v_1) + d_G(v_3) + s}{d_{G'}(v)} X^2(v) + (s+2) X^2(v_{s+1}) + \Sigma_{w \in V_1} X^2(w) \\ &= X^2(v) + (s+2) X^2(v_{s+1}) + \Sigma_{w \in V_1'} X^2(w) \\ &= X^T X = 1 \end{aligned}$$

and

$$Y^T \mathcal{L}(G) Y = X^T \mathcal{L}(G') X.$$

From Eq. (1.1), we have

$$\lambda(G) - \lambda(G') \geq \frac{Y^T \mathcal{L}(G) Y}{Y^T Y} - X^T \mathcal{L}(G') X = 0$$

Therefore, we have $\lambda(G) \geq \lambda(G')$. □

A subdivision of a connected graph G with at least two vertices is a graph obtained from G by removing some edge $e = uv$ and adding a new vertex w and edges uw and vw (i.e., subdividing an edge uv into two new edges uw and vw). Subdivision graphs, with many vertices subdividing each edge of the original graph, and their spectra are particularly important in the study of thermodynamic properties of crystalline solids (see [13]).

From Lemma 2.3 and Theorem 3.9, we have the following

Theorem 3.10 *Let uv be an edge of the connected graph G, and G' be the graph obtained from G by subdividing the edge uv into 3 new edges. Then we have $\lambda(G) \leq \lambda(G')$.*

4 The smallest normalized Laplacian spectral radius of non-bipartite unicyclic graphs

As an application of the results, in the following, we give the smallest normalized Laplacian spectral radius of non-bipartite unicyclic graphs with fixed order.

Lemma 4.1 (**[10]**) *Suppose that u, v are two distinct vertices of a connected non-bipartite graph G, and $vv_1, vv_2, \cdots, vv_s$ are s ($s \geq 1$) pendant edges of G. Let X be a unit eigenvector of G corresponding to $\lambda(G)$. Let*

$$G_u = G - vv_1 - vv_2 - \cdots - vv_s + uv_1 + uv_2 + \cdots + uv_s.$$

If $\frac{1}{\sqrt{d(u)}}|x(u)| \geq \frac{1}{\sqrt{d(v)}}|x(v)|$, then $\lambda(G_u) \geq \lambda(G)$. Further, if $X(u) \neq 0$ or $X(v) \neq 0$, then $\lambda(G_u) > \lambda(G)$.

Theorem 4.2 *Let G be a non-bipartite unicyclic graph on n vertices with girth $g \geq 3$. Then $\lambda(G) \geq \lambda(C_{3;\ \lfloor\frac{n-3}{3}\rfloor, \lceil\frac{n-3}{3}\rceil, n-3-\lfloor\frac{n-3}{3}\rfloor-\lceil\frac{n-3}{3}\rceil})$.*

Proof. From Corollary 3.6, the graph $C_{g;\ n_1,n_2,\cdots,n_g}$ minimizes the normalized Laplacian spectral radius over the class of unicyclic graphs with girth g, where $C_{g;\ n_1,n_2,\cdots,n_g}$ is the unicyclic graph obtained from $C_g:\ v_1v_2\cdots v_gv_1$ by attaching n_i pendent edges at v_i ($1 \leq i \leq g$), respectively, and $n_1 + n_2 + \cdots + n_g = n - g$. From Theorem 3.9, there exists some unicyclic graph $C_{3;\ s,t,r}$ such that $\lambda(G) \geq \lambda(C_{g;\ n_1,n_2,\cdots,n_g}) \geq \lambda(C_{3;\ s,t,r})$, where $s + t + r = n - 3$. If $s \geq t + 2$, we now prove that $\lambda(C_{3;\ s,t,r}) > \lambda(C_{3;\ s-1,t+1,r})$. Suppose that $C_{3;\ s-1,t+1,r}$ is obtained from $C_3 : uvwu$ by attaching $s - 1, t + 1, r$ pendant vertices at vertices u, v, w, respectively. Let X be a unit eigenvector of $C_{3;\ s-1,t+1,r}$ corresponding to $\lambda(C_{3;\ s-1,t+1,r})$. It is impossible that both $X(u) = 0$ and $X(v) = 0$. Otherwise, we have $X = 0$ from Lemma 3.1, a contradiction. If $\frac{1}{\sqrt{s+1}}|X(u)| \geq \frac{1}{\sqrt{t+3}}|X(v)|$, then from Lemma 4.1, we have $\lambda(C_{3;\ s,t,r}) > \lambda(C_{3;\ s-1,t+1,r})$. If $\frac{1}{\sqrt{s+1}}|X(u)| < \frac{1}{\sqrt{t+3}}|X(v)|$, then also from Lemma 4.1, we have $\lambda(C_{3;\ t,s,r}) > \lambda(C_{3;\ s-1,t+1,r})$. Note that $C_{3;\ t,s,r} = C_{3;\ s,t,r}$. This completes the proof. □

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