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# The signless Laplacian spectral radius of k-connected irregular graphs

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# The signless Laplacian spectral radius of $k$-connected irregular graphs ⋆

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**Abstract**

Let $G$ be a $k$-connected irregular graph of order $n$, size $m$ and maximum degree $\Delta$. Let $q_1$ be the signless Laplacian spectral radius of $G$. In this article, we prove the following lower bound on $2\Delta - q_1$:

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.$$

Moreover, we determine similar bounds for the signless Laplacian spectral radius of proper spanning subgraphs and $k$-edge-connected graphs.

*Key words:* the signless Laplacian spectrum; irregular graph; $k$-connected; $k$-edge-connected
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## 1 Introduction

All graphs in this paper are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph of order $n$ and size $m$, where the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $|E(G)| = m$. We denote $N_G(v)$ (or $N(v)$ for short) as the set of neighbours of $v$ in $G$, and $|N_G(v)|$ as the degree of $v$. For $i \in \{1, 2, \ldots, n\}$, let $d_i = d_G(v_i) = |N_G(v_i)|$. Moreover, the maximum and minimum degree of $G$ are denoted by $\Delta$ and $\delta$, respectively. A graph $G$ is called *regular* if $\Delta = \delta$. The *local connectivity* $p(u, v)$ between distinct vertices $u$ and $v$ is the maximum number of pairwise internally disjoint $uv$-paths. A nontrivial graph $G$ is $k$-*connected* if $p(u, v) \geq k$ for any two distinct vertices $u$ and $v$. The *connectivity* $\kappa(G)$ of $G$ is the maximum value of $k$ for which $G$ is $k$-connected. For two distinct vertices $u$ and $v$, the *distance* between $u$ and

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$v$ is the length of the shortest path between $u$ and $v$. And the *diameter* of $G$, denoted by $d$, is the greatest distance between any two distinct vertices of $G$. The notations undefined in this article can be found in the book [1].

Let $A(G)$ be the adjacency matrix of $G$ and $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ (or simply $L$ and $Q$) are called the Laplacian matrix and the signless Laplacian matrix of $G$, respectively. Obviously, all the eigenvalues of $A(G), L(G)$ and $Q(G)$ are real numbers since these matrices are real symmetric matrices. The largest eigenvalues of $A(G), L(G)$ and $Q(G)$ are denoted by $\lambda_1(G), \mu_1(G)$ and $q_1(G)$ (or simply $\lambda_1, \mu_1$ and $q_1$) of $G$, respectively. They are called the *spectral radius*, the *Laplacian spectral radius* and the *signless Laplacian spectral radius*. If $G$ is a connected graph, then $Q$ is a nonnegative irreducible positive semi-definite matrix. Hence, by the Perron-Frobenius Theorem, the multiplicity of $q_1$ is one and there exists a positive eigenvector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$, which is called the *Perron vector* of $Q$, with $\sum_{i=1}^{n} x_i^2 = 1$, such that $Q\boldsymbol{x} = q_1\boldsymbol{x}$.

It is known that $q_1 \leq 2\Delta$ and equality holds if and only if $G$ is a regular graph. We want to determine the gap between $2\Delta$ and $q_1$ if $G$ is irregular. In 2012, Ning, Li and Lu proved [9, Theorem 3.2] that

$$2\Delta - q_1 > \frac{1}{n(d - \frac{1}{4})}. \tag{1.1}$$

There are several results involving $\Delta - \lambda_1$ (see [4, 5, 8, 10–12]). Recently, Chen and Hou study the spectral radius of $k$-connected irregular graphs and gave a new lower bound on $\Delta - \lambda_1$ [2, Theorem 1]:

$$\Delta - \lambda_1 > \frac{(n\Delta - 2m)k^2}{(n\Delta - 2m)(n^2 - 2(n-k)) + nk^2}. \tag{1.2}$$

In this paper, we establish a similar lower bound on $2\Delta - q_1$, which will be proved in the next section. When $k \geq \sqrt{n}$, our lower bound on $2\Delta - q_1$ is better than the bound in (1.1) and with the same arguments in this paper, we can improve the bound in (1.2), which are presented in the remarks.

## 2 Main results

**Theorem 2.1** *Let $G$ be a $k$-connected irregular graph ($k \geq 1$) of order $n$ ($\geq 3$), size $m$ and maximum degree $\Delta$. We have*

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}. \tag{2.1}$$

**Proof.** Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector corresponding to $q_1$, thus $Q\boldsymbol{x} = q_1\boldsymbol{x}$, $x_i > 0$ and $\sum_{i=1}^{n} x_i^2 = 1$. Let $s$ and $u$ be vertices of $G$ such that $x_s = \max_{1 \leq i \leq n}\{x_i\}$ and

$x_u = \min_{1\le i\le n}\{x_i\}$. Since $\sum_{i=1}^n x_i^2 = 1$ and $G$ is irregular, we have $x_s > \frac{1}{\sqrt{n}} > x_u$. Consider the following two cases:

**Case** 1**:** Suppose $d_s \le \Delta - 1$. Since $Q\boldsymbol{x} = q_1\boldsymbol{x}$, we have

$$q_1 x_s = d_s x_s + \sum_{j\in N(s)} x_j \le (\Delta-1)x_s + (\Delta-1)x_s = 2(\Delta-1)x_s.$$

Thus $q_1 \le 2\Delta - 2$ as $x_s > 0$. Note that $1 \le k \le \delta \le \Delta - 1 \le n-2$ as $G$ is irregular. We obtain $n^2 - (\Delta - k + 2)(n-k) \ge n^2 - (n-1-1+2)(n-k) = nk > 0$. Therefore,

$$\begin{aligned}
&\frac{(n\Delta-2m)k^2}{2(n\Delta-2m)(n^2-(\Delta-k+2)(n-k))+nk^2}\\
\le\ &\frac{(n\Delta-2m)k^2}{2(n\Delta-2m)nk+nk^2}\\
<\ &\frac{(n\Delta-2m)k^2}{2(n\Delta-2m)k^2+nk^2}\\
=\ &\frac{n\Delta-2m}{2(n\Delta-2m)+n}\\
<\ &1.
\end{aligned}$$

Consequently,

$$2\Delta - q_1 \ge 2 > \frac{2(n\Delta-2m)k^2}{2(n\Delta-2m)(n^2-(\Delta-k+2)(n-k))+nk^2}.$$

**Case** 2**:** Suppose $d_s = \Delta$. For the vertex $u$, we have

$$q_1 x_u = d_u x_u + \sum_{j\in N(u)} x_j \ge d_u x_u + d_u x_u = 2d_u x_u.$$

Moreover, $G$ is irregular implies that $q_1 x_u < 2\Delta x_u$. Hence $d_u < \Delta$ since $x_u > 0$.

As $G$ is $k$-connected, by Menger's Theorem, there are at least $k$ vertex-disjoint paths between $s$ and $u$. We choose $k$ paths from them such that the sum of the lengths of these $k$ paths is as small as possible. Let $P_1, P_2, \ldots, P_k$ be such $k$ paths. Obviously, each of these path $P_i$ can only contain one vertex of $N_G(s)$. Otherwise, we can find another $k$ paths that have a smaller sum of lengths. Then there exist at least $\Delta - k$ vertices $v_i$, where $i \in \{1, 2, \ldots, \Delta - k\}$, such that $v_i \in N_G(s)$ but $v_i \notin \bigcup_{i=1}^k V(P_i)$. As a result,

$$\sum_{i=1}^k |V(P_i)| \le n - (\Delta - k) + 2(k-1).$$

Following the argument of Chen [2] and Ning [9], we obtain

$$\begin{aligned}
2\Delta - q_1 &= 2\Delta - \boldsymbol{x}^T Q(G)\boldsymbol{x} \\
&= 2\Delta \sum_{i=1}^{n} x_i^2 - \sum_{ij\in E(G)} (x_i + x_j)^2 \\
&= 2\Delta \sum_{i=1}^{n} x_i^2 - (2\sum_{i=1}^{n} d_i x_i^2 - \sum_{ij\in E(G)} (x_i - x_j)^2) \\
&= 2\sum_{i=1}^{n} (\Delta - d_i) x_i^2 + \sum_{ij\in E(G)} (x_i - x_j)^2 \qquad (2.2)\\
&\geq 2(n\Delta - 2m)x_u^2 + \sum_{ij\in E(G)} (x_i - x_j)^2. \qquad (2.3)
\end{aligned}$$

Note that $\sum_{i=1}^{k} |V(P_i)| \leq n-(\Delta-k)+2(k-1)$. In addition, by Cauchy-Schwarz inequality, we can prove that

$$\sum_{ij\in E(P_t)} (x_i - x_j)^2 \geq \frac{1}{|V(P_t)|-1}\Big(\sum_{ij\in E(P_t)} (x_i - x_j)\Big)^2$$

and

$$\sum_{t=1}^{k} \frac{1}{|V(P_t)|-1} \geq \frac{k^2}{\sum_{t=1}^{k}(|V(P_t)|-1)}.$$

Therefore, from these two inequalities, we obtain

$$\begin{aligned}
\sum_{ij\in E(G)} (x_i - x_j)^2 &\geq \sum_{t=1}^{k} \sum_{ij\in E(P_t)} (x_i - x_j)^2 \\
&\geq \sum_{t=1}^{k} \frac{1}{|V(P_t)|-1}\Big(\sum_{ij\in E(P_t)} (x_i - x_j)\Big)^2 \\
&= \sum_{t=1}^{k} \frac{1}{|V(P_t)|-1}(x_s - x_u)^2 \\
&\geq \frac{k^2}{\sum_{t=1}^{k}(|V(P_t)|-1)}(x_s - x_u)^2 \\
&\geq \frac{k^2}{n-\Delta+2k-2}(x_s - x_u)^2. \qquad (2.4)
\end{aligned}$$

Combining (2.3) and (2.4), we have

$$\begin{aligned}
2\Delta - q_1 &\geq 2(n\Delta - 2m)x_u^2 + \frac{k^2}{n-\Delta+2k-2}(x_s - x_u)^2 \qquad (2.5)\\
&= \Big(2(n\Delta - 2m) + \frac{k^2}{n-\Delta+2k-2}\Big)x_u^2 - \frac{2k^2}{n-\Delta+2k-2}x_s x_u \\
&\quad + \frac{k^2}{n-\Delta+2k-2}x_s^2.
\end{aligned}$$

Let $f(x_u) = (2(n\Delta-2m) + \frac{k^2}{n-\Delta+2k-2})x_u^2 - \frac{2k^2}{n-\Delta+2k-2}x_s x_u + \frac{k^2}{n-\Delta+2k-2}x_s^2$. If we regard $f(x_u)$

as a quadratic function, then we have

$$2\Delta - q_1 \geq \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n - \Delta + 2k - 2) + k^2} x_s^2. \tag{2.6}$$

Let

$$C = \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}$$

and consider the following two subcases:

**Subcase 1:** Suppose $k = 1$. We have

$$2\Delta - q_1 \geq \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n - \Delta) + 1} x_s^2 \tag{2.7}$$

and

$$C = \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n}.$$

If $x_u^2 \geq \frac{C}{2(n\Delta - 2m)}$, then from (2.5), we obtain

$$2\Delta - q_1 \geq 2(n\Delta - 2m) \cdot \frac{C}{2(n\Delta - 2m)} + \frac{k^2}{n - \Delta + 2k - 2}(x_s - x_u)^2 > C$$

since $x_s > x_u$ as $G$ is irregular.

If $x_u^2 < \frac{C}{2(n\Delta - 2m)}$, then since $\sum_{i=1}^{n} x_i^2 = 1$, we have

$$x_s^2 \geq \frac{1 - x_u^2}{n - 1} > \frac{1}{n - 1} \cdot \left(1 - \frac{C}{2(n\Delta - 2m)}\right).$$

Hence from (2.7) and $n^2 - (\Delta + 1)(n - 1) = (n - \Delta)(n - 1) + 1 > (n - \Delta)(n - 1)$ we have

$$\begin{aligned}
2\Delta - q_1 &\geq \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n - \Delta) + 1} x_s^2 \\
&> \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n - \Delta) + 1} \times \frac{1}{n - 1} \\
&\quad \times \left(1 - \frac{1}{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n}\right) \\
&= \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n} \\
&\quad \times \frac{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n - 1}{2(n\Delta - 2m)(n - \Delta)(n - 1) + n - 1} \\
&> C \times 1 \\
&= C.
\end{aligned}$$

**Subcase 2:** Suppose $k \geq 2$.

If $x_u^2 \geq \frac{C}{2(n\Delta-2m)}$, then the result can be obtained by using a similar argument of Subcase 1.

Since $d_u \geq k$. We can choose $k-1$ vertices from $N_G(u)$, denoted by $u_1, u_2, \ldots, u_{k-1}$, such that $s \notin \{u_1, u_2, \ldots, u_{k-1}\}$. If $\sum_{t=1}^{k-1} x_{u_t}^2 > C \cdot (1 + \frac{k-1}{2(n\Delta-2m)})$, then by (2.3) and the method to derive (2.6), we obtain

$$\begin{aligned}
2\Delta - q_1 &\geq 2(n\Delta - 2m)x_u^2 + \sum_{t=1}^{k-1}(x_{u_t} - x_u)^2 \\
&= \sum_{t=1}^{k-1}\Big(\frac{2(n\Delta-2m)}{k-1}x_u^2 + (x_{u_t} - x_u)^2\Big) \\
&\geq \sum_{t=1}^{k-1}\frac{2(n\Delta-2m)}{2(n\Delta-2m)+k-1}x_{u_t}^2 \\
&= \frac{2(n\Delta-2m)}{2(n\Delta-2m)+k-1}\cdot\sum_{t=1}^{k-1}x_{u_t}^2 \\
&> \frac{2(n\Delta-2m)}{2(n\Delta-2m)+k-1}\cdot\frac{2(n\Delta-2m)+k-1}{2(n\Delta-2m)}\cdot C \\
&= C.
\end{aligned}$$

It remains to show that our result is valid when $x_u^2 < \frac{C}{2(n\Delta-2m)}$ and $\sum_{t=1}^{k-1} x_{u_t}^2 \leq C \cdot (1 + \frac{k-1}{2(n\Delta-2m)})$. Using $\sum_{i=1}^{n} x_i^2 = 1$ again, we have

$$x_s^2 \geq \frac{1}{n-k}\Big(1 - x_u^2 - \sum_{t=1}^{k-1}x_{u_t}^2\Big) > \frac{1}{n-k}\Big(1 - \frac{2(n\Delta-2m)+k}{2(n\Delta-2m)}\cdot C\Big).$$

Therefore, from (2.6),

$$\begin{aligned}
2\Delta - q_1 &> \frac{2(n\Delta-2m)k^2}{2(n\Delta-2m)(n-\Delta+2k-2)+k^2}\cdot\frac{1}{n-k}\cdot\Big(1 - \frac{2(n\Delta-2m)+k}{2(n\Delta-2m)}\cdot C\Big) \\
&= \frac{2(n\Delta-2m)k^2}{2(n\Delta-2m)(n^2-(\Delta-k+2)(n-k))+nk^2} \\
&= C.
\end{aligned}$$

This completes the proof.

**Remark 1** Note that the bound in (2.1) increases when $n\Delta - 2m$ increases. It is obvious that $n\Delta - 2m \geq 1$ as $G$ is irregular. Hence from Theorem 2.1 we can easily find that

$$2\Delta - q_1 > \frac{2k^2}{2(n^2-(\Delta-k+2)(n-k))+nk^2}.$$

Let

$$f(k) = \frac{2k^2}{2(n^2-(\Delta-k+2)(n-k))+nk^2}.$$

Then

$$f(k) \geq \frac{2k^2}{2(n^2 - 3(n-k)) + nk^2} = \frac{1}{n/2 + (n^2 - 3n)/k^2 + 3/k}.$$

And $\frac{1}{n/2+(n^2-3n)/k^2+3/k}$ is also an increasing function on $k$. Thus when $k \geq \sqrt{n}$, we have

$$f(k) \geq \frac{1}{(3n)/2 + 3/\sqrt{n} - 3} > \frac{1}{(7n)/4} \geq \frac{1}{n(d - 1/4)}.$$

The second inequality holds as $n \geq 3$ and the third holds since $G$ is irregular, which implies $d \geq 2$. Therefore, when $k \geq \sqrt{n}$, the bound in (2.1) is better than the bound in (1.1).

**Remark 2** In the proof of Theorem 2.1, we use the fact that there are at least $\Delta - k$ vertices of $G$ that do not belong to the subgraph $H = G[\bigcup_{t=1}^{k} V(P_t)]$. The lower bound on $2\Delta - q_1$ can be improved when the vertices outside the subgraph $H$ are increased. In fact, one can show that if there are $l$ vertices outside the subgraph $H$, then

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (l+2)(n-k)) + nk^2}. \tag{2.8}$$

The bound in (2.1) is a particular case ($l = \Delta - k$) of the bound in (2.8).

**Remark 3** Using the same arguments, we can prove that

$$\Delta - \lambda_1 > \frac{(n\Delta - 2m)k^2}{(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n-k)) + nk^2}$$

which improves the bound in (1.2). Similar to (2.8), if there are $l$ vertices outside the subgraph $H$, then

$$\Delta - \lambda_1 > \frac{(n\Delta - 2m)k^2}{(n\Delta - 2m)(n^2 - (l+2)(n-k)) + nk^2}.$$

To consider the signless Laplacian spectral radius of a subgraph, we begin with the following lemma.

**Lemma 2.2** [7, Theorem 2.1] *Let $G$ be a graph on $n$ vertices and $m$ edges and let $e$ be an edge of $G$. Let $q_1, q_2, \ldots, q_n$ ($q_1 \geq q_2 \geq \ldots \geq q_n$) and $s_1, s_2, \ldots, s_n$ ($s_1 \geq s_2 \geq \ldots \geq s_n$) be the signless Laplacian spectra of $G$ and of $G - e$, respectively. Then*

$$0 \leq s_n \leq q_n \leq \ldots \leq s_2 \leq q_2 \leq s_1 \leq q_1.$$

Let $G$ be an irregular connected graph of order $n$ and maximum degree $\Delta$. We cannot always find a $\Delta$-regular graph $G'$ such that $G$ is a proper spanning subgraph of $G'$. In fact, if $n\Delta$ is an odd number, then the $\Delta$-regular graph $G'$ cannot be found as $n\Delta = 2m \equiv 0$

(mod 2) for $G'$. Thus if $G$ is a proper spanning subgraph of $\Delta$-regular graph $G'$, then we can prove the following theorem.

**Theorem 2.3** *Let $G$ be a proper spanning subgraph of a $\Delta$-regular $k$-connected graph $G'$ of order $n$. If $k \geq 2$, then*

$$2\Delta - q_1(G) > \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta+2k-4)+(n+1)(k-1)^2}.$$

**Proof.** Lemma 2.2 implies that the signless Laplacian spectral radius cannot be increased by deleting an edge. Thus we may assume that $G = G' - e$ for some edge $e$ of $G'$. Let $e = uv$, we have $d_G(u) = d_G(v) = \Delta - 1$ and $d_G(w) = \Delta$ for other vertices $w$. Since $G$ is connected when $k \geq 2$, there exist a Perron vector, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ with $x_i > 0$, corresponding to $q_1(G)$. Let $s$ be a vertex of $G$ such that $x_s = \max_{1\leq i\leq n}\{x_i\}$. Hence

$$q_1(G)x_s = d_G(s)x_s + \sum_{j\in N_G(s)} x_j \leq d_G(s)x_s + d_G(s)x_s = 2d_G(s)x_s$$

which means that $q_1(G) \leq 2d_G(s)$. Moreover, using the fact that $\lambda_1(G) > \frac{2|E(G)|}{n} = \frac{n\Delta-2}{n}$ (see [6]) together with $2 \leq k \leq n-1$, we have

$$q_1(G) \geq 2\lambda_1(G) > 2\frac{2|E(G)|}{n} = 2\frac{n\Delta-2}{n} = 2\Delta - \frac{4}{n} \geq 2\Delta - \frac{4}{3} > 2\Delta - 2.$$

Combining this with $q_1(G) \leq 2d_G(s)$, we obtain $d_G(s) > \Delta - 1$. Therefore, $d_G(s) = \Delta$ and so $s \neq u$ and $s \neq v$.

In addition, formula (2.2) gives

$$\begin{aligned} 2\Delta - q_1(G) &= 2\sum_{i=1}^{n}(\Delta - d_G(v_i))x_i^2 + \sum_{ij\in E(G)}(x_i - x_j)^2 \\ &= 2(x_u^2 + x_v^2) + \sum_{ij\in E(G)}(x_i - x_j)^2. \end{aligned} \tag{2.9}$$

With the same argument as Chen in [2], we know that

$$\sum_{ij\in E(G)}(x_i - x_j)^2 \geq \frac{(k-1)^2}{n-\Delta+2k-4}(x_s - x_u)^2.$$

Hence, similar to the proof of (2.6), we have

$$\begin{aligned} 2\Delta - q_1(G) &\geq 2(x_u^2 + x_v^2) + \frac{(k-1)^2}{n-\Delta+2k-4}(x_s - x_u)^2 \\ &> 2x_u^2 + \frac{(k-1)^2}{n-\Delta+2k-4}(x_s - x_u)^2 \\ &\geq \frac{2(k-1)^2}{2(n-\Delta+2k-4)+(k-1)^2}x_s^2. \end{aligned} \tag{2.10}$$

Define

$$C' = \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta+2k-4)+(n+1)(k-1)^2}.$$

If $x_u^2 + x_v^2 > \frac{C'}{2}$, then from (2.9),

$$2\Delta - q_1(G) = 2(x_u^2 + x_v^2) + \sum_{ij\in E(G)} (x_i - x_j)^2 > 2\frac{C'}{2} + \sum_{ij\in E(G)} (x_i - x_j)^2 \geq C'.$$

Since $d_G(u) = \Delta - 1$, it is possible to choose $\Delta - 2$ vertices $\{u_1, u_2, \ldots, u_{\Delta-2}\}$ from $N_G(u)$ such that $s \notin \{u_1, u_2, \ldots, u_{\Delta-2}\}$. Hence if $\sum_{t=1}^{\Delta-2} x_{u_t}^2 \geq \frac{\Delta}{2} C'$, then by (2.9) again, we have

$$\begin{aligned}
2\Delta - q_1(G) &> 2x_u^2 + \sum_{t=1}^{\Delta-2} (x_{u_t} - x_u)^2 \\
&= \sum_{t=1}^{\Delta-2} \left( \frac{2}{\Delta-2} x_u^2 + (x_{u_t} - x_u)^2 \right) \\
&\geq \sum_{t=1}^{\Delta-2} \frac{\frac{2}{\Delta-2}}{1 + \frac{2}{\Delta-2}} x_{u_t}^2 \\
&\geq \frac{2}{\Delta}\frac{\Delta}{2} C' \\
&= C'.
\end{aligned}$$

The remaining case is $x_u^2 + x_v^2 \leq \frac{C'}{2}$ and $\sum_{t=1}^{\Delta-2} x_{u_t}^2 < \frac{\Delta}{2} C'$. Obviously,

$$x_s^2 \geq \frac{1 - x_u^2 - x_v^2 - \sum_{t=1}^{\Delta-2} x_{u_t}^2}{n-\Delta} > \frac{1}{n-\Delta}\left(1 - \frac{C'}{2} - \frac{\Delta}{2}C'\right) = \frac{1}{n-\Delta}\left(1 - \frac{\Delta+1}{2}C'\right),$$

and from (2.10) we obtain

$$\begin{aligned}
2\Delta - q_1(G) &> \frac{2(k-1)^2}{2(n-\Delta+2k-4)+(k-1)^2} \times \frac{1}{n-\Delta} \times \left(1 - \frac{\Delta+1}{2}C'\right) \\
&= \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta+2k-4)+(n+1)(k-1)^2} \\
&= C'.
\end{aligned}$$

This completes the proof.

**Remark 4** It can be shown that Theorem 2.3 holds for any proper subgraph no matter whether the proper subgraph is spanning or not. Indeed, if $G$ is a proper subgraph but not a spanning subgraph of $G'$, then we can construct a new graph $G''$ by adding some isolated vertices such that the order of $G''$ is the same as the order of $G'$. For $G''$, we know that $q_1(G'') = q_1(G)$. And obviously, $G''$ is a proper spanning subgraph of $G'$. Hence Theorem 2.3 holds for any proper subgraph $G$.

The most critical condition in estimating the value of $\sum_{ij\in E(G)}(x_i - x_j)^2$ in the proof of Theorem 2.1 and Theorem 2.3 is that we can use each edge of $G$ at most one time. As a consequence, if a graph $G$ is $k$-edge-connected, then this condition is satisfied because we can find pairwise edge-disjoint paths by Menger's Theorem. Therefore we have

**Theorem 2.4** *Let $G$ be a $k$-edge-connected irregular graph ($k \geq 1$) of order $n$ ($\geq 3$), size $m$ and maximum degree $\Delta$. Then we have*

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.$$

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