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The signless Laplacian spectral radius of $k$-connected irregular graphs

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Abstract

Let $G$ be a $k$-connected irregular graph of order $n$, size $m$ and maximum degree $\Delta$. Let $q_1$ be the signless Laplacian spectral radius of $G$. In this article, we prove the following lower bound on $2\Delta - q_1$:

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.$$ 

Moreover, we determine similar bounds for the signless Laplacian spectral radius of proper spanning subgraphs and $k$-edge-connected graphs.

Key words: the signless Laplacian spectrum; irregular graph; $k$-connected; $k$-edge-connected

AMS 2010 MSC: 05C50.

1 Introduction

All graphs in this paper are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph of order $n$ and size $m$, where the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $|E(G)| = m$. We denote $N_G(v)$ (or $N(v)$ for short) as the set of neighbours of $v$ in $G$, and $|N_G(v)|$ as the degree of $v$. For $i \in \{1, 2, \ldots, n\}$, let $d_i = d_G(v_i) = |N_G(v_i)|$. Moreover, the maximum and minimum degree of $G$ are denoted by $\Delta$ and $\delta$, respectively. A graph $G$ is called regular if $\Delta = \delta$. The local connectivity $p(u, v)$ between distinct vertices $u$ and $v$ is the maximum number of pairwise internally disjoint $uv$-paths. A nontrivial graph $G$ is $k$-connected if $p(u, v) \geq k$ for any two distinct vertices $u$ and $v$. The connectivity $\kappa(G)$ of $G$ is the maximum value of $k$ for which $G$ is $k$-connected. For two distinct vertices $u$ and $v$, the distance between $u$ and $v$.

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$v$ is the length of the shortest path between $u$ and $v$. And the diameter of $G$, denoted by $d$, is the greatest distance between any two distinct vertices of $G$. The notations undefined in this article can be found in the book [1].

Let $A(G)$ be the adjacency matrix of $G$ and $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ (or simply $L$ and $Q$) are called the Laplacian matrix and the signless Laplacian matrix of $G$, respectively. Obviously, all the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are real numbers since these matrices are real symmetric matrices. The largest eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are denoted by $\lambda_1(G)$, $\mu_1(G)$ and $q_1(G)$ (or simply $\lambda_1$, $\mu_1$ and $q_1$) of $G$, respectively. They are called the spectral radius, the Laplacian spectral radius and the signless Laplacian spectral radius. If $G$ is a connected graph, then $Q$ is a nonnegative irreducible positive semi-definite matrix. Hence, by the Perron-Frobenius Theorem, the multiplicity of $q_1$ is one and there exists a positive eigenvector $x = (x_1, x_2, \ldots, x_n)^T$, which is called the Perron vector of $Q$, with $\sum_{i=1}^n x_i^2 = 1$, such that $Qx = q_1x$.

It is known that $q_1 \leq 2\Delta$ and equality holds if and only if $G$ is a regular graph. We want to determine the gap between $2\Delta$ and $q_1$ if $G$ is irregular. In 2012, Ning, Li and Lu proved [9, Theorem 3.2] that

$$2\Delta - q_1 > \frac{1}{n(d - \frac{1}{2})}, \quad (1.1)$$

There are several results involving $\Delta - \lambda_1$ (see [4, 5, 8, 10–12]). Recently, Chen and Hou study the spectral radius of $k$-connected irregular graphs and gave a new lower bound on $\Delta - \lambda_1$ [2, Theorem 1]:

$$\Delta - \lambda_1 > \frac{(n\Delta - 2m)k^2}{(n\Delta - 2m)(n^2 - 2(n - k)) + nk^2}. \quad (1.2)$$

In this paper, we establish a similar lower bound on $2\Delta - q_1$, which will be proved in the next section. When $k \geq \sqrt{n}$, our lower bound on $2\Delta - q_1$ is better than the bound in (1.1) and with the same arguments in this paper, we can improve the bound in (1.2), which are presented in the remarks.

2 Main results

**Theorem 2.1** Let $G$ be a $k$-connected irregular graph ($k \geq 1$) of order $n$ ($\geq 3$), size $m$ and maximum degree $\Delta$. We have

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}. \quad (2.1)$$

**Proof.** Let $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector corresponding to $q_1$, thus $Qx = q_1x$, $x_i > 0$ and $\sum_{i=1}^n x_i^2 = 1$. Let $s$ and $u$ be vertices of $G$ such that $x_s = \max_{1 \leq i \leq n}\{x_i\}$ and
\[ x_u = \min_{1 \leq i \leq n} \{ x_i \}. \] Since \( \sum_{i=1}^{n} x_i^2 = 1 \) and \( G \) is irregular, we have \( x_s > \frac{1}{\sqrt{n}} > x_u \). Consider the following two cases:

**Case 1:** Suppose \( d_s \leq \Delta - 1 \). Since \( Qx = q_1 x \), we have

\[ q_1 x_s = d_s x_s + \sum_{j \in N(s)} x_j \leq (\Delta - 1)x_s + (\Delta - 1)x_s = 2(\Delta - 1)x_s. \]

Thus \( q_1 \leq 2\Delta - 2 \) as \( x_s > 0 \). Note that \( 1 \leq k \leq \delta \leq \Delta - 1 \leq n - 2 \) as \( G \) is irregular. We obtain \( n^2 - (\Delta - k + 2)(n - k) \geq n^2 - (n - 1 - 1 + 2)(n - k) = nk > 0 \). Therefore,

\[
\frac{(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2} \\
\leq \frac{(n\Delta - 2m)nk + nk^2}{2(n\Delta - 2m)k^2 + nk^2} \\
< \frac{n\Delta - 2m}{2(n\Delta - 2m) + n} \\
< 1.
\]

Consequently,

\[ 2\Delta - q_1 \geq 2 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}. \]

**Case 2:** Suppose \( d_s = \Delta \). For the vertex \( u \), we have

\[ q_1 x_u = d_u x_u + \sum_{j \in N(u)} x_j \geq d_u x_u + d_u x_u = 2d_u x_u. \]

Moreover, \( G \) is irregular implies that \( q_1 x_u < 2\Delta x_u \). Hence \( d_u < \Delta \) since \( x_u > 0 \).

As \( G \) is \( k \)-connected, by Menger’s Theorem, there are at least \( k \) vertex-disjoint paths between \( s \) and \( u \). We choose \( k \) paths from them such that the sum of the lengths of these \( k \) paths is as small as possible. Let \( P_1, P_2, \ldots, P_k \) be such \( k \) paths. Obviously, each of these path \( P_i \) can only contain one vertex of \( N_G(s) \). Otherwise, we can find another \( k \) paths that have a smaller sum of lengths. Then there exist at least \( \Delta - k \) vertices \( v_i \), where \( i \in \{ 1, 2, \ldots, \Delta - k \} \), such that \( v_i \in N_G(s) \) but \( v_i \notin \bigcup_{i=1}^{k} V(P_i) \). As a result,

\[ \sum_{i=1}^{k} |V(P_i)| \leq n - (\Delta - k) + 2(k - 1). \]
Following the argument of Chen [2] and Ning [9], we obtain
\[
2\Delta - q_1 = 2\Delta - x^T Q(G) x
\]
\[
= 2\Delta \sum_{i=1}^{n} x_i^2 - \sum_{ij \in E(G)} (x_i + x_j)^2
\]
\[
= 2\Delta \sum_{i=1}^{n} x_i^2 - (2 \sum_{i=1}^{n} d_i x_i^2 - \sum_{ij \in E(G)} (x_i - x_j)^2)
\]
\[
= 2 \sum_{i=1}^{n} (\Delta - d_i) x_i^2 + \sum_{ij \in E(G)} (x_i - x_j)^2
\]
\[
\geq 2(n\Delta - 2m)x_u^2 + \sum_{ij \in E(G)} (x_i - x_j)^2. \tag{2.2}
\]

Note that \(\sum_{i=1}^{k} |V(P_i)| \leq n - (\Delta - k) + 2(k - 1)\). In addition, by Cauchy-Schwarz inequality, we can prove that
\[
\sum_{ij \in E(P_i)} (x_i - x_j)^2 \geq \frac{1}{|V(P_i)| - 1} \left( \sum_{ij \in E(P_i)} (x_i - x_j) \right)^2
\]
and
\[
\sum_{t=1}^{k} \frac{1}{|V(P_t)| - 1} \geq \frac{k^2}{\sum_{t=1}^{k}(|V(P_t)| - 1)}.
\]

Therefore, from these two inequalities, we obtain
\[
\sum_{ij \in E(G)} (x_i - x_j)^2 \geq \sum_{t=1}^{k} \sum_{ij \in E(P_t)} (x_i - x_j)^2
\]
\[
\geq \sum_{t=1}^{k} \frac{1}{|V(P_t)| - 1} \left( \sum_{ij \in E(P_t)} (x_i - x_j) \right)^2
\]
\[
= \sum_{t=1}^{k} \frac{1}{|V(P_t)| - 1} (x_s - x_u)^2
\]
\[
\geq \frac{k^2}{\sum_{t=1}^{k}(|V(P_t)| - 1)} (x_s - x_u)^2
\]
\[
\geq \frac{k^2}{n - \Delta + 2k - 2} (x_s - x_u)^2. \tag{2.3}
\]

Combining (2.3) and (2.4), we have
\[
2\Delta - q_1 \geq 2(n\Delta - 2m)x_u^2 + \frac{k^2}{n - \Delta + 2k - 2} (x_s - x_u)^2 \tag{2.5}
\]
\[
= \left( 2(n\Delta - 2m) + \frac{k^2}{n - \Delta + 2k - 2} \right) x_u^2 - \frac{2k^2}{n - \Delta + 2k - 2} x_s x_u
\]
\[
+ \frac{k^2}{n - \Delta + 2k - 2} x_s^2.
\]

Let \(f(x_u) = (2(n\Delta - 2m) + \frac{k^2}{n - \Delta + 2k - 2})x_u^2 - \frac{2k^2}{n - \Delta + 2k - 2} x_s x_u + \frac{k^2}{n - \Delta + 2k - 2} x_s^2\). If we regard \(f(x_u)\)
as a quadratic function, then we have

\[ 2\Delta - q_1 \geq \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n - \Delta + 2k - 2) + k^2x_s^2}. \]  

(2.6)

Let

\[ C = \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2} \]

and consider the following two subcases:

**Subcase 1:** Suppose \( k = 1 \). We have

\[ 2\Delta - q_1 \geq \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n - \Delta) + 1}x_s^2 \]

and

\[ C = \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n}. \]

If \( x_u^2 \geq \frac{C}{2(n\Delta - 2m)} \), then from (2.5), we obtain

\[ 2\Delta - q_1 \geq 2(n\Delta - 2m) \cdot \frac{C}{2(n\Delta - 2m)} + \frac{k^2}{n - \Delta + 2k - 2}(x_s - x_u)^2 > C \]

since \( x_s > x_u \) as \( G \) is irregular.

If \( x_u^2 < \frac{C}{2(n\Delta - 2m)} \), then since \( \sum_{i=1}^{n} x_i^2 = 1 \), we have

\[ x_s^2 \geq \frac{1 - x_u^2}{n - 1} \geq \frac{1}{n - 1} \cdot \left(1 - \frac{C}{2(n\Delta - 2m)} \right). \]

Hence from (2.7) and \( n^2 - (\Delta + 1)(n - 1) = (n - \Delta)(n - 1) + 1 > (n - \Delta)(n - 1) \) we have

\[ 2\Delta - q_1 \geq \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n - \Delta) + 1}x_s^2 \]

\[ > \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n - \Delta) + 1} \times \frac{1}{n - 1} \]

\[ \times \left(1 - \frac{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n}{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1) + n - 1)} \right) \]

\[ = \frac{2(n\Delta - 2m)(n^2 - (\Delta + 1)(n - 1)) + n}{2(n\Delta - 2m)(n - \Delta)(n - 1) + n - 1} \]

\[ > C \times 1 \]

\[ = C. \]

**Subcase 2:** Suppose \( k \geq 2 \).
If \( x_u^2 \geq \frac{C}{2(n\Delta - 2m)} \), then the result can be obtained by using a similar argument of Subcase 1.

Since \( d_u \geq k \). We can choose \( k - 1 \) vertices from \( N_G(u) \), denoted by \( u_1, u_2, \ldots, u_{k-1} \), such that \( s \notin \{u_1, u_2, \ldots, u_{k-1}\} \). If \( \sum_{t=1}^{k-1} x_{ut}^2 > C \cdot (1 + \frac{k-1}{2(n\Delta - 2m)}) \), then by (2.3) and the method to derive (2.6), we obtain:

\[
2\Delta - q_1 \geq 2(n\Delta - 2m)x_u^2 + \sum_{t=1}^{k-1} (x_{ut} - x_u)^2
\]
\[
= \sum_{t=1}^{k-1} \left( \frac{2(n\Delta - 2m)}{k-1} x_u^2 + (x_{ut} - x_u)^2 \right)
\]
\[
\geq \sum_{t=1}^{k-1} \frac{2(n\Delta - 2m)}{2(n\Delta - 2m) + k-1} x_{ut}^2
\]
\[
= \frac{2(n\Delta - 2m)}{2(n\Delta - 2m) + k-1} \cdot \sum_{t=1}^{k-1} x_{ut}^2
\]
\[
= \frac{2(n\Delta - 2m)}{2(n\Delta - 2m) + k-1} \cdot \frac{2(n\Delta - 2m) + k-1}{2(n\Delta - 2m)} \cdot C
\]
\[
= C.
\]

It remains to show that our result is valid when \( x_u^2 < \frac{C}{2(n\Delta - 2m)} \) and \( \sum_{t=1}^{k-1} x_{ut}^2 \leq C \cdot (1 + \frac{k-1}{2(n\Delta - 2m)}) \). Using \( \sum_{i=1}^{u} x_i^2 = 1 \) again, we have

\[
x_s^2 \geq \frac{1}{n-k} \left( 1 - x_u^2 - \sum_{i=1}^{k-1} x_{ui}^2 \right) > \frac{1}{n-k} \left( 1 - \frac{2(n\Delta - 2m) + k}{2(n\Delta - 2m)} \cdot C \right).
\]

Therefore, from (2.6),

\[
2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n - \Delta + 2k - 2) + k^2} \cdot \frac{1}{n-k} \left( 1 - \frac{2(n\Delta - 2m) + k}{2(n\Delta - 2m)} \cdot C \right)
\]
\[
= \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}
\]
\[
= \frac{2k^2}{2(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.
\]

This completes the proof.

**Remark 1** Note that the bound in (2.1) increases when \( n\Delta - 2m \) increases. It is obvious that \( n\Delta - 2m \geq 1 \) as \( G \) is irregular. Hence from Theorem 2.1 we can easily find that

\[
2\Delta - q_1 > \frac{2k^2}{2(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.
\]

Let

\[
f(k) = \frac{2k^2}{2(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.
\]
Then

\[ f(k) \geq \frac{2k^2}{2(n^2 - 3(n-k)) + nk^2} = \frac{1}{n/2 + (n^2 - 3n)/k^2 + 3/k}. \]

And \( \frac{1}{n/2+(n^2-3n)/k^2+3/k} \) is also an increasing function on \( k \). Thus when \( k \geq \sqrt{n} \), we have

\[ f(k) \geq \frac{1}{(3n)/2 + 3/\sqrt{n} - 3} > \frac{1}{(7n)/4} \geq \frac{1}{n(d-1/4)}. \]

The second inequality holds as \( n \geq 3 \) and the third holds since \( G \) is irregular, which implies \( d \geq 2 \). Therefore, when \( k \geq \sqrt{n} \), the bound in (2.1) is better than the bound in (1.1).

**Remark 2** In the proof of Theorem 2.1, we use the fact that there are at least \( \Delta - k \) vertices of \( G \) that do not belong to the subgraph \( H = G[\bigcup_{t=1}^k V(P_t)] \). The lower bound on \( 2\Delta - q_1 \) can be improved when the vertices outside the subgraph \( H \) are increased. In fact, one can show that if there are \( l \) vertices outside the subgraph \( H \), then

\[ 2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (l + 2)(n - k)) + nk^2}. \]  

The bound in (2.1) is a particular case \( (l = \Delta - k) \) of the bound in (2.8).

**Remark 3** Using the same arguments, we can prove that

\[ \Delta - \lambda_1 > \frac{(n\Delta - 2m)k^2}{(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2} \]

which improves the bound in (1.2). Similar to (2.8), if there are \( l \) vertices outside the subgraph \( H \), then

\[ \Delta - \lambda_1 > \frac{(n\Delta - 2m)k^2}{(n\Delta - 2m)(n^2 - (l + 2)(n - k)) + nk^2}. \]

To consider the signless Laplacian spectral radius of a subgraph, we begin with the following lemma.

**Lemma 2.2** [7, Theorem 2.1] Let \( G \) be a graph on \( n \) vertices and \( m \) edges and let \( e \) be an edge of \( G \). Let \( q_1, q_2, \ldots, q_n \) \((q_1 \geq q_2 \geq \ldots \geq q_n)\) and \( s_1, s_2, \ldots, s_n \) \((s_1 \geq s_2 \geq \ldots \geq s_n)\) be the signless Laplacian spectra of \( G \) and of \( G-e \), respectively. Then

\[ 0 \leq s_n \leq q_n \leq \ldots \leq s_2 \leq q_2 \leq s_1 \leq q_1. \]

Let \( G \) be an irregular connected graph of order \( n \) and maximum degree \( \Delta \). We cannot always find a \( \Delta \)-regular graph \( G' \) such that \( G \) is a proper spanning subgraph of \( G' \). In fact, if \( n\Delta \) is an odd number, then the \( \Delta \)-regular graph \( G' \) cannot be found as \( n\Delta = 2m \equiv 0 \)
(mod 2) for $G'$. Thus if $G$ is a proper spanning subgraph of $\Delta$-regular graph $G'$, then we can prove the following theorem.

**Theorem 2.3** Let $G$ be a proper spanning subgraph of a $\Delta$-regular $k$-connected graph $G'$ of order $n$. If $k \geq 2$, then

$$2\Delta - q_1(G) > \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta + 2k - 4) + (n+1)(k-1)^2}.$$  

**Proof.** Lemma 2.2 implies that the signless Laplacian spectral radius cannot be increased by deleting an edge. Thus we may assume that $G = G' - e$ for some edge $e$ of $G'$. Let $e = uv$, we have $d_G(u) = d_G(v) = \Delta - 1$ and $d_G(w) = \Delta$ for other vertices $w$. Since $G$ is connected when $k \geq 2$, there exist a Perron vector, $x = (x_1, x_2, \ldots, x_n)^t$ with $x_i > 0$, corresponding to $q_1(G)$. Let $s$ be a vertex of $G$ such that $x_s = \max_{1 \leq i \leq n} \{x_i\}$. Hence

$$q_1(G)x_s = d_G(s)x_s + \sum_{j \in N_G(s)} x_j \leq d_G(s)x_s + d_G(s)x_s = 2d_G(s)x_s$$

which means that $q_1(G) \leq 2d_G(s)$. Moreover, using the fact that $\lambda_1(G) > \frac{2|E(G)|}{n} = \frac{n\Delta - 2}{n}$ (see [6]) together with $2 \leq k \leq n - 1$, we have

$$q_1(G) \geq 2\lambda_1(G) > \frac{2|E(G)|}{n} = \frac{2n\Delta - 2}{n} = 2\Delta - \frac{4}{n} \geq 2\Delta - \frac{4}{3} > 2\Delta - 2.$$  

Combining this with $q_1(G) \leq 2d_G(s)$, we obtain $d_G(s) > \Delta - 1$. Therefore, $d_G(s) = \Delta$ and so $s \neq u$ and $s \neq v$.

In addition, formula (2.2) gives

$$2\Delta - q_1(G) = 2\sum_{i=1}^{n} (\Delta - d_G(v_i))x_i^2 + \sum_{ij \in E(G)} (x_i - x_j)^2$$

$$= 2(x_u^2 + x_v^2) + \sum_{ij \in E(G)} (x_i - x_j)^2. \quad (2.9)$$

With the same argument as Chen in [2], we know that

$$\sum_{ij \in E(G)} (x_i - x_j)^2 \geq \frac{(k-1)^2}{n - \Delta + 2k - 4} (x_s - x_u)^2.$$  

Hence, similar to the proof of (2.6), we have

$$2\Delta - q_1(G) \geq 2(x_u^2 + x_v^2) + \frac{(k-1)^2}{n - \Delta + 2k - 4} (x_s - x_u)^2$$

$$> 2x_u^2 + \frac{(k-1)^2}{n - \Delta + 2k - 4} (x_s - x_u)^2$$

$$\geq \frac{2(k-1)^2}{2(n - \Delta + 2k - 4) + (k-1)^2 x_s^2}. \quad (2.10)$$
Define
\[ C' = \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta + 2k - 4) + (n+1)(k-1)^2}. \]
If \( x_u^2 + x_v^2 > \frac{C'}{2} \), then from (2.9),
\[
2\Delta - q_1(G) = 2(x_u^2 + x_v^2) + \sum_{ij \in E(G)} (x_i - x_j)^2 > 2 \frac{C'}{2} + \sum_{ij \in E(G)} (x_i - x_j)^2 \geq C'.
\]
Since \( d_G(u) = \Delta - 1 \), it is possible to choose \( \Delta - 2 \) vertices \( \{u_1, u_2, \ldots, u_{\Delta-2}\} \) from \( N_G(u) \) such that \( s \notin \{u_1, u_2, \ldots, u_{\Delta-2}\} \). Hence if \( \sum_{t=1}^{\Delta-2} x_u^2 \geq \frac{2}{\Delta-2} C' \), then by (2.9) again, we have
\[
2\Delta - q_1(G) > 2x_u^2 + \sum_{t=1}^{\Delta-2} (x_u - x_{u_t})^2
\]
\[
= \sum_{t=1}^{\Delta-2} \left( \frac{2}{\Delta-2} x_u^2 + (x_u - x_{u_t})^2 \right)
\]
\[
\geq \sum_{t=1}^{\Delta-2} \frac{2}{\Delta-2} x_u^2
\]
\[
\geq \frac{2}{\Delta-2} \frac{\Delta}{2} C'
\]
\[
= C'.
\]
The remaining case is \( x_u^2 + x_v^2 \leq \frac{C'}{2} \) and \( \sum_{t=1}^{\Delta-2} x_u^2 < \frac{2}{\Delta-2} C' \). Obviously,
\[
x_u^2 \geq \frac{1 - x_u^2 - x_v^2 - \sum_{t=1}^{\Delta-2} x_u^2}{n-\Delta} > \frac{1}{n-\Delta} \left( 1 - \frac{C'}{2} - \frac{\Delta}{2} C' \right) = \frac{1}{n-\Delta} \left( 1 - \frac{\Delta + 1}{2} C' \right),
\]
and from (2.10) we obtain
\[
2\Delta - q_1(G) > \frac{2(k-1)^2}{2(n-\Delta + 2k - 4) + (k-1)^2} \times \frac{1}{n-\Delta} \times \left( 1 - \frac{\Delta + 1}{2} C' \right)
\]
\[
= \frac{2(k-1)^2}{2(n-\Delta)(n-\Delta + 2k - 4) + (n+1)(k-1)^2}
\]
\[
= C'.
\]
This completes the proof.

**Remark 4** It can be shown that Theorem 2.3 holds for any proper subgraph no matter whether the proper subgraph is spanning or not. Indeed, if \( G \) is a proper subgraph but not a spanning subgraph of \( G' \), then we can construct a new graph \( G'' \) by adding some isolated vertices such that the order of \( G'' \) is the same as the order of \( G' \). For \( G'' \), we know that \( q_1(G'') = q_1(G) \). And obviously, \( G'' \) is a proper spanning subgraph of \( G' \). Hence Theorem 2.3 holds for any proper subgraph \( G \).

The most critical condition in estimating the value of \( \sum_{ij \in E(G)} (x_i - x_j)^2 \) in the proof of Theorem 2.1 and Theorem 2.3 is that we can use each edge of \( G \) at most one time. As a
consequence, if a graph $G$ is $k$-edge-connected, then this condition is satisfied because we can find pairwise edge-disjoint paths by Menger’s Theorem. Therefore we have

**Theorem 2.4** Let $G$ be a $k$-edge-connected irregular graph $(k \geq 1)$ of order $n \geq 3$, size $m$ and maximum degree $\Delta$. Then we have

$$2\Delta - q_1 > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - (\Delta - k + 2)(n - k)) + nk^2}.$$  

**References**


