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Further results on super graceful labeling of graphs

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Further Results On Super Graceful Labeling of Graphs

Abstract

Let $G = (V(G), E(G))$ be a simple, finite and undirected graph of order p and size q. A bijection $f : V(G) \cup E(G) \to \{k, k+1, k+2, \ldots, k+p+q-1\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ is said to be a k-super graceful labeling of G. We say G is k-super graceful if it admits a k-super graceful labeling. For $k = 1$, the function f is called a super graceful labeling and a graph is super graceful if it admits a super graceful labeling. In this paper, we study the super gracefulness of complete graph, the disjoint union of certain star graphs, the complete tripartite graphs $K(1, 1, n)$, and certain families of trees. We also present four methods of constructing new super graceful graphs. In particular, all trees of order at most 7 are super graceful. We conjecture that all trees are super graceful.

Keywords: Graceful labeling; Super graceful labeling; tree.

2010 AMS Subject Classifications: 05C78.

1 Introduction

Let $G = (V, E)$ be a simple, finite and undirected graph of order $|V| = p$ and size $|E| = q$. All notation not defined in this paper can be found in [2]. An injective function $f : V \to \{1, 2, \ldots, q\}$ is called a graceful labeling of G if all the edge labels of G given by $f(uv) = |f(u) - f(v)|$ for every $uv \in E$ are distinct. This concept was first introduced by Rosa in 1967 [8]. Since then, there have been more than 1500 research papers published on graph labelings (see the dynamic survey by Gallian [3]).

In [1], the authors defined a k-sequentially additive labeling f of a graph G as a bijection from $V \cup E$ to $\{k, k+1, \ldots, k+p+q-1\}$ such that for each edge $uv \in E$, $f(uv) = f(u) + f(v)$. A graph G admitting a k-sequentially additive labeling is called a *k-sequentially additive graph.* If $k = 1$, then G is called a *simply sequentially additive graph* or an *SSA-graph.* They conjectured that all trees are SSA-graphs. More results on k-sequentially additive labeling can be found in [4, 5].

Definition 1.1. A bijection $f : V(G) \cup E(G) \to \{k, k+1, k+2, \ldots, k+p+q-1\}$ is called a *k-super graceful labeling* if $f(uv) = |f(u) - f(v)|$ for every edge uv in G. For $k = 1$, the function f is called a super graceful labeling. We say G is *super graceful* if it admits a super graceful labeling.

This is a generalization of super graceful labeling defined in [6, 7]. Among other, the authors proved that paths, cycles, complete bipartite graphs and several families of trees are super graceful. In this paper, we continue with the search for super graceful graphs. We study the super gracefulness of complete graph, the disjoint union of certain star graphs, the complete tripartite graphs $K(1, 1, n)$, and certain families of trees. We also present four methods of constructing new super graceful graphs. In particular, all trees of order at most 7 are super graceful. We conjecture that all trees are super graceful. Note that we only give the vertex labels of all the given examples.

2 New Super Graceful Graphs

In [1], the authors showed that the complete graph K_n is an SSA-graph if and only if $n \leq 3$.

Theorem 2.1. *The complete graph K_n is super graceful if and only if $n \leq 3$.*

Proof. It is easy to verify that the sufficiency holds. To prove the necessity, we show that K_n is not super graceful for $n \geq 4$. Assume that K_n admits a super graceful labeling. Let $m = n(n+1)/2$, which is the largest label. Thus, m cannot be a difference of other labels, m must be a vertex label.

Case (1). 1 is a vertex label. This means $m - 1$ is an edge label and 2 is not a vertex label. Hence $m - 2$ cannot be a difference of two vertex labels. So 1 and $m - 2$ are vertex labels. Then the difference $m - 3$

is an edge label. Note that $m-4=(m-3)-1=(m-2)-2=(m-1)-3$. Since $m-3$, 2 and $m-1$ are edge labels, $m-4$ cannot be an edge label. Thus, $m-4$ is a vertex label. This yields a contradiction since m, $m-2$ and $m-4$ are vertex labels creating 2 as an edge label twice.

Case (2). 1 is an edge label. The only way to have $m-1$ as an edge label would be an edge joining the vertices labeled 1 and m, which is impossible in this case. Thus $m-1$ is a vertex label. Hence the edge labeled by 1 is incident with the vertices labeled by m and $m-1$. It follows that $m-2$ cannot be a vertex label. Thus $m-2$ is an edge label. The edge labeled by $m-2$ must be incident with the vertices labeled by m and 2. Since $m-1$ and 2 are vertex labels, an edge which is incident with vertices labeled by $m-1$ and 2 is labeled by $m-3$. By means of this fact and together with m and 2 are served as vertex labels, 3 and 4 must be edge labels, respectively. 1, 3, 4, $m-2$ and $m-3$ are edge labels imply that $m-4$ and $m-5$ cannot be edge labels.

Finally, the edge joining the vertices labeled m and $m-1$, and the edge joining the vertices labeled $m-4$ and $m-5$ both have label 1, a contradiction. □

Theorem 2.2. *The complete tripartite graph $K(1,1,r)$ is super graceful for $r \geq 1$.*

Proof. Let $V(K(1,1,r)) = \{u_1, u_2, v_1, v_2, \ldots, v_r\}$ and $E(K(1,1,r)) = \{u_i v_j \,|\, i = 1,2, 1 \leq j \leq r\} \cup \{u_1 u_2\}$. Now, $|V(K(1,1,r))| = r+2$ and $|E(K(1,1,r))| = 2r+1$. Define a labeling $f : V(K(1,1,r)) \cup E(K(1,1,r)) \to \{1, 2, \ldots, 3r+3\}$ as follows:

$f(u_1) = 1, f(u_2) = 3r+3,\ f(v_i) = 3i+1$ for $1 \leq i \leq r$; and

$f(u_1u_2) = 3r+2,\ f(u_1v_i) = 3i, f(u_2v_i) = 3(r-i)+2$ for $1 \leq i \leq r$.

It is easy to verify that f is a bijection with $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(K(1,1,r))$. Hence, $K(1,1,r)$ is super graceful. □

Example 2.1. In Figure 1, we give the super graceful labeling of $K(1,1,5)$ according to the function defined above.

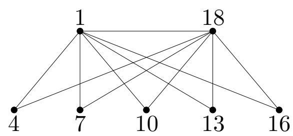


Figure 1: Super graceful labeling of $K(1,1,5)$

Problem 2.1. *Study the super gracefulness of complete tripartite graph $K(p,q,r), r \geq q \geq p \geq 1, q \geq 2$.*

Definition 2.1. Let $K(1,n_1) \cup K(1,n_2) \cup \cdots \cup K(1,n_m)$ be the disjoint union of m copies of star graphs $K(1,n_i)$ for $n_i \geq 1$ and $1 \leq i \leq m$.

Theorem 2.3. *The graph $K(1,n_1) \cup K(1,n_2) \cup \cdots \cup K(1,n_m)$ is super graceful if*

(a). *$n_i \equiv 0 \pmod{i}$.*

(b). *for $1 \leq i \leq m-1$, the largest vertex label in $K(1,n_i)$ is x implies that $n_{i+1} \equiv 0 \pmod{x+1}$.*

Proof. Let $G = K(1,n_1) \cup K(1,n_2) \cup \cdots \cup K(1,n_m)$ with $V(G) = \{u_i \,|\, 1 \leq i \leq m\} \cup \{v_{i,j} \,|\, 1 \leq i \leq m, 1 \leq j \leq n_i\}$ and $E(G) = \{u_i v_{i,j} \,|\, 1 \leq i \leq m, 1 \leq j \leq n_i\}$. Clearly, $|V(G)| = m + n_1 + n_2 + \cdots + n_m$ and $|E(G)| = n_1 + n_2 + \cdots + n_m$. Define a labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, m + 2(n_1 + n_2 + \cdots + n_m)\}$ as follows:

(a). Begin with the central vertex of each star subgraph $K(1,n_i)$.

(1). Label the vertices u_i by i for $1 \leq i \leq m$.

(2). For $K(1,n_1)$, label the edge $u_1v_{1,j}$ and the vertex $v_{1,j}$ by $m+2j-1$ and $m+2j$ respectively for $1 \le j \le n_1$. Clearly, the set of used labels in $K(1,n_1)$ is $\{1\} \cup \{m+k \,|\, 1 \le k \le 2n_1\}$.

(3). For $2 \le i \le m$, assume that the largest vertex label of $K(1,n_{i-1})$ is x. Actually, $x = m + 2(n_1 + \cdots + n_{i-1})$. Define $f(u_iv_{i,j}) = x + (\lceil j/i \rceil - 1)i + j$ and $f(v_{i,j}) = x + \lceil j/i \rceil i + j$, $1 \le j \le n_i$. Consider $j = qi + r$, where $0 \le q \le n_i/i - 1$ and $1 \le r \le i$. Then $f(v_{i,j}) = x + (2q+1)i + r$ and $f(u_iv_{i,j}) = x + 2qi + r$. Hence the set of used labels for this subcase is $\{x + 2qi + k \,|\, 1 \le k \le 2i\}$. Combining all subcases, we can see that the set of used labels is $\{x + k \,|\, 1 \le k \le 2n_i\} \cup \{i\}$.

(b). Begin with $K(1,n_1)$.

(1). Label vertex u_1 by 1, edge $u_1v_{1,j}$ by $2j$ and vertex $v_{1,j}$ by $2j+1$ for $1 \le j \le n_1$. Clearly, the set of used labels in $K(1,n_1)$ is $\{1, 2, \ldots, 2n_1+1\}$.

(2). For $2 \le i \le m$, assume that the largest vertex label of $K(1,n_{i-1})$ is x. Actually, $x = i-1+2(n_1+\cdots+n_{i-1})$. Define $f(u_i) = x+1$, $f(u_iv_{i,j}) = (x+1)(\lceil j/(x+1) \rceil)+j$ and $f(v_{i,j}) = (x+1)(\lceil j/(x+1) \rceil+1)+j$, $1 \le j \le n_i$. Observe that the set of used labels in $K(1,n_i)$ is $\{x, x+1, x+2, ..., x+2n_i+1\}$.

Thus, in both (a) and (b) above, f is a bijection with $f(u_iv_{i,j}) = f(v_{i,j}) - f(u_i)$ for every edge $u_iv_{i,j}$ in $E(G)$. Hence, G is super graceful. $\square$

Example 2.2. In Figure 2 and Figure 3, we give the super graceful labeling of (a) $K(1,3) \cup K(1,4) \cup K(1,6) \cup K(1,4)$ and (b) $K(1,1) \cup K(1,4) \cup K(1,13)$ according to the function defined in (a) and (b) above, respectively.

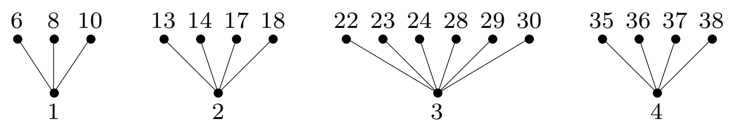


Figure 2: Super graceful labeling of $K(1,3) \cup K(1,4) \cup K(1,6) \cup K(1,4)$

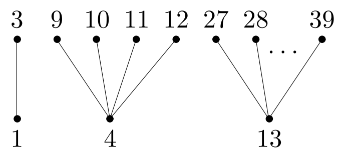


Figure 3: Super graceful labeling of $K(1,1) \cup K(1,4) \cup K(1,13)$

3 Construction of Super Graceful Graphs

In this section, we give four methods of constructing new super graceful graphs.

<u>Construction C1.</u> Let G be a graph with a super graceful labeling f having vertices u, v, w that satisfy the following conditions:

(a) $f(uv) = f(u) - f(v) = f(v) - f(w)$;

(b) w is not a neighbor of v.

By deleting the edge uv and adding a new edge vw, we obtained a new super graceful graphs.

Example 3.1. Refer to the graph $P_4(4,0,3,2)$ in Example 4.1. We let uv be the edge with label 8 such that u and v have labels 23 and 15 respectively. Let w be the vertex with label 7. We can now delete edge uv and add a new edge vw with label 8. The obtained new graph is super graceful.

<u>Construction C2.</u> Let G be a graph of order p and size q with a super graceful labeling f. Let v be a vertex of G with $f(v) = k$.

(a) Attached k pendant edges to v such that the newly added vertices are $v_1, v_2, \ldots, v_k$.

(b) For $1 \le i \le k$, label vertex v_i by $p + q + k + i$ and the corresponding pendant edge by $p + q + i$.

Clear, the newly obtained graph is super graceful.

Example 3.2. We refer to the super graceful labeling of $K(1,1,5)$ in Figure 1 and add 4 pendant edges to the vertex with label 4. Label the newly added vertices by $23, 24, 25, 26$ respectively and the corresponding pendant edges will have label $19, 20, 21, 22$. The new graph obtained is super graceful.

<u>Construction C3.</u> Let G be a graph with a super graceful labeling f.

(a) Let uv and uw be two adjacent edges such that $f(u) = f(v) - f(uv) = f(w) - f(uw)$

(b) Suppose there exists a vertex x such that $f(x) = f(v) + f(uw) = f(w) + f(uv)$.

(c) Delete edges uv and uw and add edges xv and xw.

(d) Label xv by $f(uw)$ and xw by $f(uv)$.

It is clear that the new graph obtained is also super graceful.

Example 3.3. Refer to the caterpillar $P_4(4,0,3,2)$ in Example 4.1. We let u, v, w, x be the vertices with labels $17, 19, 21, 23$ respectively. Delete the edges uv and uw with labels $2, 4$ respectively. Add 2 new edges xv and xw. Label xv and xw with $4, 2$. We have a new super graceful caterpillar.

<u>Construction C4.</u> Begin with vertices u_i $(0 \le i \le n)$.

(a) Label u_i by $1 + id$ for $d \ge 2$.

(b) For $1 \le j \le d - 1$, add a vertex v_j and join it to each of u_i.

(c) Label edge $u_i v_j$ by $j + 1 + (n - i)d$ and vertex v_j by $2 + j + nd$.

(d) Delete edge $u_i v_j$ if its label is also one of the vertex labels.

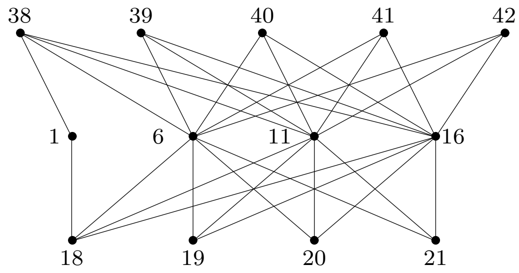
(e) For $k = 1, 2, \ldots$, introduce d new vertices with labels $(1 + k)dn + dk + j + k + 1$, $j = 1, \ldots, d$. Join each of them to $u_i, i = 0, 1, \ldots, n$. The induced edge labels are $(1 + k)dn + d(k - i) + j + k$.

(f) Delete each new edge in (e) if its label is one of the new vertex labels.

It is easy to verify that the bipartite graph we get now is super graceful.

Note: We can choose not to perform Steps (e) and (f). If we do perform Steps (e) and (f), the common new vertex labels and new edge labels introduced in part (e) are those of the new vertices except the last one. Thus edges with these labels are to be deleted.

Example 3.4. In Figure 4, we give an example for $n = 3, d = 5$. In Step (a), vertices u_0 to u_3 are labeled with $1, 6, 11, 16$ respectively. In Steps (b) and (c), we add vertices v_1 to v_4 that are labeled with 18 to 21 consecutively. In Step (d), we delete the edge joining vertices u_0 to v_2, v_3, v_4. We then perform Steps (e) and (f) by taking $k = 1$. Thus, we add 5 more vertices that are labeled with 38 to 42 consecutively.



$n = 3, d = 5$

Figure 4: A super graceful graph under Construction C4

Since this construction method will give us infinitely many connected super graceful bipartite graphs, we then have

Problem 3.1. *Study the super gracefulness of connected bipartite graphs.*

4 Super graceful trees

We now investigate the super gracefulness of some families of trees.

A *caterpillar graph* is a tree in which all the vertices are within distance 1 of a central path P_n for $n \geq 1$. A caterpillar graph of order greater than 1 is a star graph when $n = 1$, which is $K(1, r)$ for some $r \geq 1$. When $n \geq 2$. A caterpillar graph is obtained from a path $P_n = u_1u_2 \cdots u_n$ by attaching $m_i \geq 0$ pendant vertices $v_{i,j}$ $(1 \leq j \leq m_i)$ to each u_i. We shall denote this caterpillar graph by $P_n(m_1, m_2, \ldots, m_n)$. In [6, 7], the authors showed that $P_n(1, 2, \ldots, n)$, P_n and $P_n(m, m, \ldots, m)$ for $n, m \geq 1$ are super graceful. We now show that $P_n(m_1, m_2, \ldots, m_n)$ is super graceful for $n \geq 2$, $m_i \geq 0$ and $1 \leq i \leq n$, i.e., all caterpillar graphs are super graceful.

Theorem 4.1. *The graph $P_n(m_1, m_2, \ldots, m_n)$ is super graceful for $n \geq 2, m_i \geq 0$.*

Proof. Define a labeling $f : V(P_n(m_1, m_2, \ldots, m_n)) \cup E(P_n(m_1, m_2, \ldots, m_n)) \to \{1, 2, 3, \ldots, 2(n + m_1 + m_2 + \cdots + m_n) - 1\}$ as follows:
$f(u_iu_{i+1}) = 2(n + m_{i+1} + m_{i+2} + \cdots + m_n) - 2i$ for $1 \leq i \leq n - 1$, and
$f(u_iv_{i,j}) = 2(n + m_i + m_{i+1} + \cdots + m_n) - 2(i + j + 1)$ for $1 \leq i \leq n$ and $1 \leq j \leq m_i$.
For odd i:
$f(u_1) = 2(n + m_1 + \cdots + m_n) - 1$,
$f(v_{1,j}) = 2j - 1$, $1 \leq j \leq m_1$,
$f(u_i) = 2(n + m_1 + m_2 + \cdots + m_n) - 2(m_2 + m_4 + \ldots + m_{i-1}) - i$ for $3 \leq i \leq n$,
$f(v_{i,j}) = 2(m_1 + m_3 + \cdots + m_{i-2}) + i + 2j - 2$ for $3 \leq i \leq n$, $1 \leq j \leq m_i$.
For even i:
$f(u_i) = 2(m_1 + m_3 + \cdots + m_{i-1}) + i - 1$ for $2 \leq i \leq n$,
$f(v_{2,j}) = 2(n + m_1 + \cdots + m_n) - 2j - 1$, for $1 \leq j \leq m_2$,
$f(v_{i,j}) = 2(n + m_1 + m_2 + \cdots + m_n) - 2(m_2 + m_4 + \ldots + m_{i-2}) - i - 2j + 1$ for $4 \leq i \leq n$, $1 \leq j \leq m_i$.

It can be verified that f is a bijection with $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(P_n(m_1, m_2, \ldots, m_n))$. Hence, $P_n(m_1, m_2, \ldots, m_n)$ is super graceful. □

Example 4.1. In Figure 5, we give the super graceful labeling of $P_4(4, 0, 3, 2)$ according to the function defined above.

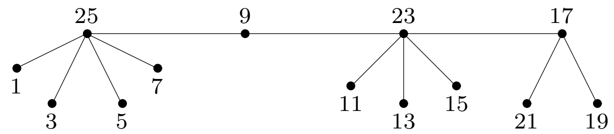


Figure 5: Super graceful labeling of $P_4(4, 0, 3, 2)$

Note that if $n = 1$ or 2, we get that both the star graph and double star graph are super graceful.

Definition 4.1. Given $t \geq 3$ paths of length $n_j \geq 1$ with an end vertex $v_{j,1}$ $(1 \leq j \leq t)$. A *spider graph* $SP(n_1, n_2, n_3, \ldots, n_t)$ is the one-point union of the t paths at vertex $v_{j,1}$.

In [6], the authors showed that $SP(n, n, \ldots, n)$, $n \geq 1$ is super graceful. We now show that many other families of spider graphs are also super graceful. For simplicity, we shall use a^n to denote a sequence of length n in which all items are a, where $a, n \geq 1$.

Theorem 4.2. *The following spider graphs are super graceful.*

(a) $SP(1^n, k^m)$*,* $n \geq 1$*,* $k, m \geq 2$*;*

(b) $SP(2^n, 3^2)$*,* $n \geq 1$*;*

(c) $SP(2, 3^n)$*,* $n \geq 1$*;*

(d) $SP(1^n, 2, 4)$*,* $n \geq 1$*;*

(e) $SP(2, k, n)$*,* $n \geq k \geq 2$*,* $2 \leq k \leq 8$*.*

Proof. (a) Let $V(SP(1^n, k^m)) = \{u\} \cup \{w_a \,|\, 1 \leq a \leq n\} \cup \{v_{i,j} \,|\, 1 \leq i \leq m, 1 \leq j \leq k\}$ and $E(SP(1^n, k^m)) = \{uw_a \,|\, 1 \leq a \leq n\} \cup \{uv_{i,1} \,|\, 1 \leq i \leq m\} \cup \{v_{i,j}v_{i,j+1} \,|\, 1 \leq i \leq m, 1 \leq j \leq m - 1\}$. Note that $|V(SP(1^n, k^m))| \cup |E(SP(1^n, k^m))| = 2mk + 2n + 1$. Define a labeling $f : V(SP(1^n, k^m)) \cup E(SP(1^n, k^m)) \to \{1, 2, \ldots, 2mk + 2n + 1\}$ as follows:

Case (1) k is odd.

$f(u) = 1$, $f(w_a) = 2mk + 2a + 1$ for $1 \leq a \leq n$,

$f(v_{i,j}) = (i - 1)k + j + 1$ for odd i and even j,

$f(v_{i,j}) = (2m - i + 1)k - j + 2$ for odd i, j,

$f(v_{i,j}) = ik - j + 2$ for even i and odd j,

$f(v_{i,j}) = (2m - i + 1)k + j - 4$ for even i, j.

Case (2) k is even.

$f(u) = 1$, $f(w_a) = 2mk + 2a + 1$ for $1 \leq a \leq n$,

$f(v_{i,j}) = (i - 1)k + j + 1$ for odd i and even j,

$f(v_{i,j}) = (2m - i + 1)k - j + 2$ for odd i, j,

$f(v_{i,j}) = ik - j + 2$ for even i and odd j,

$f(v_{i,j}) = (2m - i + 1)k + j - 5$ for even i, j.

It can be verified that f is a bijection with all the vertex labels being odd and the edge labels being even such that for each edge uv, $f(uv) = |f(u) - f(v)|$. Hence, $SP(1^n, k^m)$ is super graceful.

In Figure 6, we give the super graceful labeling of $SP(1^n, 4^3)$, $SP(1^n, 5^3)$, $SP(1^n, 4^4)$, $SP(1^n, 5^4)$ as defined above.

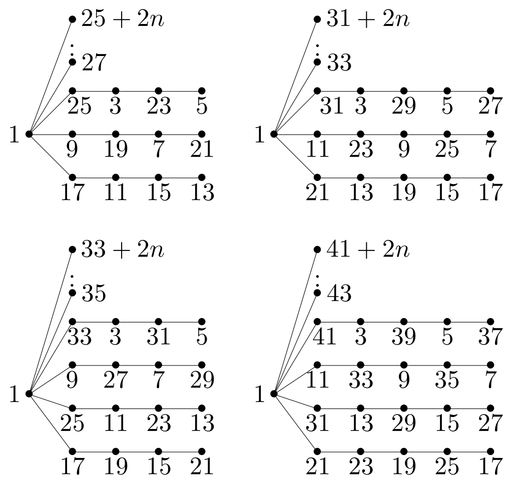


Figure 6: $SP(1^n, 4^3)$, $SP(1^n, 5^3)$, $SP(1^n, 4^4)$, $SP(1^n, 5^4)$ are super graceful.

(b) Let $V(SP(2^n, 3^2)) = \{u\} \cup \{w_{j,k} \,|\, 1 \le j \le 2, 1 \le k \le 3\} \cup \{v_{i,1}, v_{i,2} \,|\, 1 \le i \le n\}$ and $E(SP(3^2, 2^n)) = \{uw_{1,1}, uw_{2,1}\} \cup \{w_{j,k}w_{j,k+1} \,|\, 1 \le j, k \le 2\} \cup \{uv_{i,1}, v_{i,1}v_{i,2} \,|\, 1 \le i \le n\}$. Note that $|V(SP(2^n, 3^2)) \cup E(SP(2^n, 3^2))| = 4n + 13$. Define a labeling $f : V(SP(2^n, 3^2)) \cup E(SP(2^n, 3^2)) \to \{1, 2, \dots, 4n + 13\}$ as follows:

$f(u) = 3, f(w_{1,1}) = 4n+11, f(w_{1,2}) = 1, f(w_{1,3}) = 4n+13, f(w_{2,1}) = 4n+9, f(w_{2,2}) = 5, f(w_{2,3}) = 4n+7,$

$f(v_{i,1}) = 2i + 5$ for odd i, and $f(v_{i,1}) = 4n + 7 - 2i$ for even i,

$f(v_{i,2}) = 4n + 7 - 2i$ for odd i, and $f(v_{2,1}) = 2i + 5$ for even i.

It can be verified that f is a bijection with all the vertex labels being odd and the edge labels being even such that for each edge uv, $f(uv) = |f(u) - f(v)|$.

In Figure 7, we give the super graceful labeling of $SP(2^3, 3^2)$ and $SP(2^4, 3^2)$ as defined above.

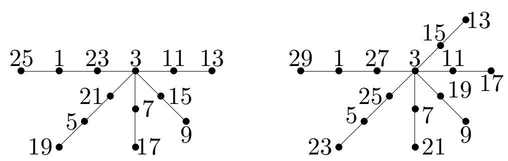


Figure 7: $SP(2^3, 3^2)$ and $SP(2^4, 3^2)$ are super graceful

(c) Let $V(SP(2, 3^n)) = \{u, v, w\} \cup \{v_{i,j} \,|\, 1 \le i \le n, 1 \le j \le 3\}$ and $E(SP(2, 3^n)) = \{uv, vw\} \cup \{uv_{i,1} \,|\, 1 \le i \le n\} \cup \{v_{i,j}v_{i,j+1} \,|\, 1 \le i \le n, 1 \le j \le 2\}$. Note $SP(2, 3)$ is a super graceful path. For $n \ge 2$, we note that $|V(SP(2, 3^n)) \cup E(SP(2, 3^n)| = 3n + 3 + 3n + 2 = 6n + 5$. Define a labeling $f : V(SP(2, 3^n)) \cup E(SP(2, 3^n)) \to \{1, 2, 3, \dots, 6n + 5\}$ as follows:

Case (1). $n \ge 3$ is odd. We have

$f(u) = 3, f(v) = 3n + 4, f(w) = 3n + 2, f(v_{1,1}) = 6n + 3, f(v_{1,2}) = 1, f(v_{1,3}) = 6n + 5,$

$f(v_{i,1}) = 6n + 7 - 3i$ for even i, $f(v_{i,1}) = 3i - 2$ for odd $i \ge 3$,

$f(v_{i,2}) = 3i - 1$ for even i, $f(v_{i,2}) = 6n + 6 - 3i$ for odd $i \ge 3$,

$f(v_{i,3}) = 6n + 5 - 3i$ for even i, $f(v_{i,3}) = 3i$ for odd $i \geq 3$.

Case (2). $n \geq 2$ is even. We have

$f(u) = 3, f(v) = 3n + 1, f(w) = 3n + 3, f(v_{1,1}) = 6n + 3, f(v_{1,2}) = 1, f(v_{1,3}) = 6n + 5,$

$f(v_{i,1}) = 6n + 7 - 3i$ for even i, $f(v_{i,1}) = 3i - 2$ for odd $i \geq 3$,

$f(v_{i,2}) = 3i - 1$ for even i, $f(v_{i,2}) = 6n + 6 - 3i$ for odd $i \geq 3$,

$f(v_{i,3}) = 6n + 5 - 3i$ for even i, $f(v_{i,3}) = 3i$ for odd $i \geq 3$.

It can be verified that f is a bijection with all the vertex labels being odd and the edge labels being even such that for each edge uv, $f(uv) = |f(u) - f(v)|$.

In Figure 8, we give the super graceful labeling of $SP(2, 3^3)$ and $SP(2, 3^4)$ as defined above.

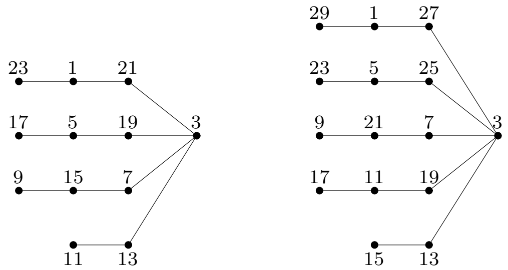


Figure 8: $SP(2, 3^3)$ and $SP(2, 3^4)$ are super graceful

(d) We have $|V(SP(1^n, 2, 4))| + |E(SP(1^n, 2, 4))| = 13 + 2n$. It is easy to verify that the labeling of the graph $SP(1^n, 2, 4)$ as shown in Figure 9 is super graceful.

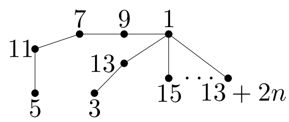


Figure 9: $SP(1^n, 2, 4)$ is super graceful

(e) We provide the proof for $k = 2, 3$. In a similar way, it is easy to verify that the result holds for $4 \leq k \leq 8$. We let $V(SP(2, 2, n)) = \{x, w_1, w_2, u_1, u_2\} \cup \{v_i \,|\, 1 \leq i \leq n\}$ and $E(SP(2, 2, n)) = \{xw_1, w_1w_2, xu_1, u_1u_2, xv_1\} \cup \{v_iv_{i+1} \,|\, 1 \leq i \leq n-1\}$. Note that $|V(SP(2, 2, n)| + |E(SP(2, 2, n))| = 2n + 9$. Define a labeling $f : V(SP(2, 2, n)) \cup E(SP(2, 2, n)) \to \{1, 2, 3, \ldots, 2n + 9\}$ as follows:

Case (1). n is odd. We have

$f(x) = n + 10, f(w_1) = n + 2, f(w_2) = n + 8, f(u_1) = n + 6, f(u_2) = n + 4,$

$f(v_i) = n + 1 - i$ for odd $1 \leq i \leq n$, and $f(v_i) = n + 10 + i$ for even $1 \leq i \leq n$.

Case (2). n is even. We have

$f(x) = n + 1, f(w_1) = n + 9, f(w_2) = n + 3, f(u_1) = n + 5, f(u_2) = n + 7,$

$f(v_i) = n + 10 + i$ for odd $1 \leq i \leq n$, and $f(v_i) = n + 1 - i$ for even $1 \leq i \leq n$.

We now let $V(SP(2, 3, n)) = \{x, w_1, w_2, u_1, u_2, u_3\} \cup \{v_i \,|\, 1 \leq i \leq n\}$ and $E(SP(2, 3, n)) = \{xw_1, w_1w_2, xu_1, u_1u_2, u_2u_3, xv_1\} \cup \{v_iv_{i+1} \,|\, 1 \leq i \leq n - 1\}$. Note that $|V(SP(2, 3, n)| + |E(SP(2, 3, n))| = 2n + 11$. Define a labeling $f : V(SP(2, 3, n)) \cup E(SP(2, 3, n)) \to \{1, 2, 3, \ldots, 2n + 11\}$ as follows:

Case (1). n is odd. We have

$f(x) = n+12, f(w_1) = n+2, f(w_2) = n+10, f(u_1) = n+6, f(u_2) = n+8, f(u_3) = n+4,$
$f(v_i) = n+1-i$ for odd $1 \le i \le n$, and $f(v_i) = n+12+i$ for even $1 \le i \le n$.

Case (2). n is even. We have

$f(x) = n+1, f(w_1) = n+11, f(w_2) = n+3, f(u_1) = n+7, f(u_2) = n+5, f(u_3) = n+9,$
$f(v_i) = n+12+i$ for odd $1 \le i \le n$, and $f(v_i) = n+1-i$ for even $1 \le i \le n$.

It can be verified that f is a bijection with all the vertex labels being odd and the edge labels being even such that for each edge uv, $f(uv) = |f(u) - f(v)|$. □

Additionally, it is not difficult to see that $SP(2, k, n)$ is super graceful if we can prove the followings:

Problem 4.1. *Let x be any real number.*

1) Given the sequence $x+2i, i = 0, 1, 2, ..., m-1, m+1$, $m \ge 2$, we can always arrange the $m+1$ numbers as a sequence so that the difference of the m pairs of consecutive numbers are evens from 2 to $2m$.

2) Given the sequence $x+2i, i = 0, 2, 3, .., m+1$, $m \ge 2$, we can always arrange the $m+1$ numbers as a sequence so that the difference of the m pairs of consecutive numbers are evens from 2 to $2m$.

Note that the spider in Case (e) is also known as lobsters. Observe that we always label the end-vertex, v_n, of the longest path with 1. If we identify the central vertex of a star $K(1, s), s \ge 2$, to v_n, then the new graph is also super graceful by labeling the s end-vertices of $K(1, s)$ with $2n + 2k + 5 + 2i$ for $i = 1, 2, 3, \ldots, s$. Hence, we have obtained new families of super graceful lobsters.

Consider another lobster tree, $LT(n)$, having a longest path $P_n = u_1u_2 \ldots u_n$ ($n \ge 6$) and two P_3 subpaths u_3wx and $u_{n-2}yz$. For $6 \le n \le 12$, we obtained that the tree is super graceful. The labelings of the vertices in the order of $u_1, u_2, \ldots, u_n; w, x, y, z$ are given below:

$n = 6 : 1, 19, 3, 13, 7, 15; 17, 5, 9, 11$
$n = 7 : 1, 21, 3, 19, 9, 15, 7; 17, 5, 13, 11$
$n = 8 : 1, 23, 3, 21, 5, 19, 7, 15; 13, 9, 17, 11$
$n = 9 : 1, 25, 3, 23, 9, 19, 7, 11, 17; 21, 5, 15, 13$
$n = 10 : 1, 27, 3, 25, 5, 23, 7, 21, 9, 15; 13, 17, 19, 11$
$n = 11 : 1, 29, 3, 27, 5, 25, 7, 23, 9, 21, 15; 13, 17, 11, 19$
$n = 12 : 1, 31, 3, 29, 5, 27, 7, 25, 9, 23, 11, 19; 13, 17, 21, 15$

The case $n = 8$ is shown in Figure 10 below:

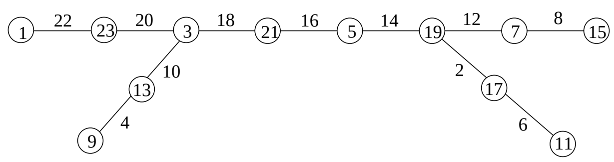


Figure 10: $LT(8)$ is super graceful

Problem 4.2. *Show that the lobster tree $LT(n)$ is super graceful for all $n \ge 6$.*

Moreover, we may consider case (a) of Theorem 4.2 for $n = 0$. That means we may ignore all the vertices w_a's. By using the same labeling and combining with Theorem 2.3 for the case $m = 1$, we get the result of Perumal *et al.* [6].

Corollary 4.3. *The spider graphs $SP(k^m)$ are super graceful, where $k \ge 1$ and $m \ge 2$.*

Combining with Theorems 2.3, 4.1, 4.2 and Corollary 4.3, we can conclude that all trees of order at most 7 are super graceful.

Note that the Construction C2 can be used repeatedly to any super graceful tree to create infinitely many super graceful trees. Hence, we end this paper with the following conjecture.

Conjecture 4.1. *All trees are super graceful.*

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