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# A central limit theorem for linear Kolmogorov's birth-growth models

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# A central limit theorem for linear Kolmogorov's birth–growth models


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**Abstract**

A Poisson process in space–time is used to generate a linear Kolmogorov's birth–growth model. Points start to form on $[0, L]$ at time zero. Each newly formed point initiates two bidirectional moving frontiers of constant speed. New points continue to form on not-yet passed over parts of $[0, L]$. The whole interval will eventually be passed over by moving frontiers. Let $N_L$ be the total number of points formed. Quine and Robinson (1990) showed that if the Poisson process is homogeneous in space–time , the distribution of $(N_L - \mathbf{E}[N_L])/\sqrt{\mathbf{var}[N_L]}$ converges weakly to the standard normal distribution. In this paper a simpler argument is presented to prove this asymptotic normality of $N_L$ for a more general class of linear Kolmogorov's birth–growth models.



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## 1. Introduction

Consider the following linear random birth–growth model. Points are *formed* on $[0, L]$ according to a spatial–temporal Poisson process $\Psi_L \equiv \{(x_i, t_i) \in [0, L] \times [0, \infty)\}$. (Points are *born* at the locations $x_i$ at times $t_i$, $i = 0, 1, 2, \ldots$) Its intensity measure is $\ell \times \Lambda$, where $\ell$ is the Lebesgue measure in $\mathbb{R}$, while $\Lambda$ is an arbitrary locally finite measure on $[0, \infty)$ such that

$$\mu \equiv \int_0^\infty \exp\left\{-\int_0^t 2v(t-s)\Lambda(ds)\right\}\Lambda(dt) \quad < \quad \infty, \text{ and} \tag{1.1}$$

$$\Lambda([0, t]) \quad > \quad 0 \qquad \text{for all } t > 0. \tag{1.2}$$


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The role of these two regularity conditions will be explained the next section.

Points start to form on $[0, L]$ at time zero. Once the point $(x_i, t_i)$ is formed (born), two bidirectional *moving frontiers* commence at $x_i$. Each frontier moves at a constant speed $v$ until it meets an opposing one. The intervals passed over by moving frontiers are regarded as *covered*. New points continue to form on uncovered parts of $[0, L]$ until the whole interval is covered.

Such a birth–growth process in two-dimensions was first developed by Kolmogorov (1937) to study the growth of crystal aggregates, and then was proved to be very useful (see e.g. Chiu (1995), Cowan *et al* (1995), Evans (1945), Gilbert (1962), Johnson and Mehl (1939), Meijering (1953), Møller (1992), Okabe *et al* (1992), Quine and Robinson (1990, 1992), Stoyan *et al* (1995) and Vanderbei and Shepp (1988)).

Denote by $\Phi_L$ the spatial–temporal process of the points formed. For ease of presentation, we consider $\Phi_L$ both as a random set in $[0, L] \times [0, \infty)$ and a random measure. Let $N_L$ be the total number of points in the set $\Phi_L$. Quine and Robinson (1990) proved that the distribution of $(N_L - \mathbf{E}[N_L])/\sqrt{\mathbf{var}[N_L]}$ converges weakly to the standard normal distribution for $\Lambda(dt) = \lambda dt$. In the following a simpler proof of the asymptotic normality of $N_L$ as $L \to \infty$ will be given for the following two cases: (i) $\Lambda(\{0\}) > 0$ and (ii) $\Lambda \ll \ell$, $\Lambda([0, \varepsilon]) = O(\varepsilon)$ as $\varepsilon \to 0$ and $\Lambda([0, L]) = O(L)$ as $L \to \infty$. The latter case is a generalisation of the model of Quine and Robinson (1990). Nevertheless, the proof presented here is inspired by their approach.

## 2. Mean and variance

Although the following results hold only almost surely, for simplicity this will not be said explicitly hereinafter. Note that condition (1.2) guarantees that $N_L \geq 1$ as $L \to \infty$ and that the birth–growth process starts at time zero.

Slivnyak (1962) showed that the reduced Palm distribution of a Poisson process is the same as the distribution of the Poisson process (see e.g. Stoyan *et al* (1995)). Applying this result on the Mecke–Campbell (or called refined Campbell) theorem (Mecke, 1967) yields

$$\begin{aligned}\mathbf{E} &\sum_{(x_0,t_0),\ldots,(x_n,t_n)\in\Psi_L}^{\neq} f((x_0,t_0),\ldots,(x_n,t_n),\Psi_L) \\ &= \int\int\cdots\int\int \mathbf{E} f((x_0,t_0),\ldots,(x_n,t_n),\Psi_L\cup\{(x_0,t_0),\ldots,(x_n,t_n)\}) \\ &\qquad \times \ell(dx_0)\Lambda(dt_0)\cdots\ell(dx_n)\Lambda(dt_n), \qquad (2.1)\end{aligned}$$

where $\sum^{\neq}$ denotes the summation over $(n+1)$-tuples of $n+1$ distinct spatial–temporal points (see also Møller (1992) Eq. (3.1)). This equation will be used to obtain the mean and variance of $N_L$.

Let $\Xi(\Psi_L, t)$ denote the random region in $(-\infty, \infty) \times [0, \infty)$ which is *covered* just before time $t$ by the $\Psi_L$-generated birth–growth process. It can be written as

$$\Xi(\Psi_L, t) \equiv \bigcup_{(x,s)\in\Psi_L, s<t} \{(y,r) : y \in [x-(r-s)v, x+(r-s)v], r \geq s\}.$$

For each point $(x, t)$ in $\Psi_L$, these three events $\{(x, t) \notin \Xi(\Psi_L, t)\}$, $\{(x, t) \notin \Xi(\Psi_L \setminus \{(x, t)\}, t)\}$ and $\{(x, t) \in \Phi_L\}$ are equivalent, since the first two events imply that at time $t$ the position $x$ has not yet been covered by the $\Psi_L$-generated birth–growth process, and consequently a point is formed at $(x,t)$. Therefore, we have

$$\mathbf{E}[N_L] = \mathbf{E}\left[\sum_{(x,t)\in\Psi_L} \mathbf{1}\left((x,t) \notin \Xi(\Psi_L,t)\right)\right],$$

where $\mathbf{1}(\cdot)$ denotes the indicator function, and so by (2.1),

$$\mathbf{E}[N_L] = \int_0^\infty \int_0^L \mathbf{E}\left[\mathbf{1}\left((x,t) \notin \Xi(\Psi_L,t)\right)\right] \ell(dx)\Lambda(dt). \tag{2.2}$$

To evaluate this integral, one has to note that no point of $\Psi_L$ lies outside the region $[0,L]\times[0,\infty)$, and so the equality

$$\mathbf{E}\left[\mathbf{1}\left((x,t) \notin \Xi(\Psi_L,t)\right)\right] = \exp\left\{-\int_0^t 2v(t-s)\Lambda(ds)\right\}$$

holds only when $vt \le x \le L-vt$ and $0 \le t \le L/2v$. But nevertheless, when $L\to\infty$,

$$\mathbf{E}[N_L] \sim \mu L, \tag{2.3}$$

where $\mu$ has been defined and assumed to be finite in condition (1.1).

**Remark.** Suppose that the birth–growth process took place on the whole real line $(-\infty,\infty)$ instead of the interval $[0,L]$. Denote the collection of all points that would be born at uncovered positions by $\Phi$. Then $\mu$ is the intensity of the point process which is the projection of $\Phi$ onto $(-\infty,\infty)$.

Similarly, for all $(x_0,t_0)$ and $(x_1,t_1)$ in $\Psi_L$, consider

$$\begin{aligned} f((x_0,t_0),(x_1,t_1),\Psi_L) &= \mathbf{1}((x_0,t_0)\notin\Xi(\Psi_L,t_0))\mathbf{1}((x_1,t_1)\notin\Xi(\Psi_L,t_1)) \\ &= \mathbf{1}((x_0,t_0)\in\Phi_L)\mathbf{1}((x_1,t_1)\in\Phi_L) \end{aligned}$$

Then obviously,

$$\mathbf{E}[N_L(N_L-1)] = \mathbf{E}\sum_{(x_0,t_0),(x_1,t_1)\in\Psi_L}^{\neq} f((x_0,t_0),(x_1,t_1),\Psi_L)$$

By applying (2.1) we have

$$\begin{aligned} \mathbf{E}[N_L(N_L-1)] = \int_0^\infty\int_0^\infty\int_0^L\int_0^L &\mathbf{E}f((x_0,t_0),(x_1,t_1),\Psi_L\cup\{(x_0,t_0),(x_1,t_1)\}) \\ &\times\ell(dx_0)\ell(dx_1)\Lambda(dt_0)\Lambda(dt_1). \end{aligned}$$

By definition of $\Xi$,

$$\begin{aligned} \{(x_1,t_1)\notin\Xi(\Psi_L\cup\{(x_0,t_0),(x_1,t_1)\},t)\} &= \{(x_1,t_1)\notin\Xi(\Psi_L\cup\{(x_0,t_0)\},t)\} \\ &\subseteq \{(x_1,t_1)\notin\Xi(\Psi_L,t)\}. \end{aligned}$$

So

$$\begin{aligned} &\mathbf{E}f((x_0,t_0),(x_1,t_1),\Psi_L\cup\{(x_0,t_0),(x_1,t_1)\}) \\ &\qquad\le \mathbf{E}\left[\mathbf{1}\left((x_0,t_0)\notin\Xi(\Psi_L,t)\right)\mathbf{1}\left((x_1,t_1)\notin\Xi(\Psi_L,t)\right)\right]. \end{aligned}$$

Consequently from (2.2) and (2.3), $\mathbf{E}[N_L(N_L-1)] \le \mathbf{E}[N_L]^2$ and

$$\sigma^2 \equiv \lim_{L\to\infty} \frac{\mathbf{var}[N_L]}{L} \le \mu.$$

More directly, $\sigma^2$ can be obtained from the following equation:

$$\begin{aligned}&(\sigma^2+\mu^2-\mu)\\ &= \lim_{L\to\infty}\int_0^\infty\int_0^L\int_0^\infty\int_{|x_1-x_0|<v|t_1-t_0|} \exp\{-J(x_0,t_0,x_1,t_1)\}\,\ell(dx_0)\Lambda(dt_0)\ell(dx_1)\Lambda(dt_1)/L,\end{aligned}$$

where

$$\begin{aligned}&J(x_0,t_0,x_1,t_1)\\ &\quad= \int_0^{\max(t_0,t_1)} 2v\left([t_1-s]_+ + [t_0-s]_+ - [\frac{|x_0-x_1|+v(t_0+t_1)}{2v} - s]_+\right)\Lambda(ds),\end{aligned}$$

and $[x]_+ = \max(x,0)$.

## 3. Central Limit Theorem

Throughout this section it is assumed that either (i) $\Lambda(\{0\}) > 0$ or (ii) $\Lambda \ll \ell$, $\Lambda([0,\varepsilon]) = O(\varepsilon)$ as $\varepsilon \to 0$ and $\Lambda([0,L]) = O(L)$ as $L\to\infty$.

Let $T_1 \equiv \inf\{t_i : (x_i,t_i) \in \Psi_L\}$ denote the birth-time of the first born (formed) point. Since $N_L \ge 1$ as $L\to\infty$, to show the asymptotic normality of $N_L$, we can assume without loss of generality that $\Psi_L \ne \emptyset$, and so $T_1 < \infty$ is well-defined. Let $U$ follow the uniform distribution on $[0,L]$. Thus, when there is a unique first point in $\Psi_L$, the spatial–temporal coordinates of this point follows the same distribution as $(U,T_1)$. When there are more than one points born at $T_1$ (e.g. when $\Lambda$ is concentrated at $\{0\}$, and so $T_1 = 0$), $(U,T_1)$ can be regarded as the spatial–temporal coordinates of a randomly chosen one of them.

Denote 'has the same distribution as' by '$\stackrel{d}{=}$'. As it can be seen from Figure 1, when the first point is born at $(U,T_1)$, $[0,L]$ is divided into two intervals of length $U$ and $L-U$. A moving frontier commences at an endpoint of each interval. Therefore, $N_L$ is equal to one plus the total number of points of $\Phi_L$ in the two right-angled triangles with bases length $U$ and $L-U$ on $[0,L]$ shown in Figure 1, i.e.

$$N_L \stackrel{d}{=} 1 + R_U^{(T_1)} + R_{L-U}^{(T_1)}, \tag{3.1}$$

where $R_U^{(T_1)}$ denotes the total number of points of $\Phi_L$ which are born in the right-angled triangle based on $[0,U)\times\{T_1\}$ and height $U/v$. Denote this right-angled triangle by $A_U^{(T_1)}$. Thus, $R_U^{(T_1)} = \Phi_L(A_U^{(T_1)})$. Moreover, moving frontiers in one of these two triangles cannot pass into the other one, and $\Psi_L$ is a Poisson process. Thus, conditional on $\{U=u\}$, where $0<u<L$, the random variables $R_U^{(T_1)}$ and $R_{L-U}^{(T_1)}$ in (3.1) are independent. Note that only the position on the time-axis $\{T_1\}$ and the length $U$ of the base of the triangle but not the exact position of this interval on $[0,L]$ are essential to the distribution of $R_U^{(T_1)}$, with the understanding that the open endpoint of the base $[0,U)$ (i.e. $\{U\}$) contains a point of $\Phi_L$.

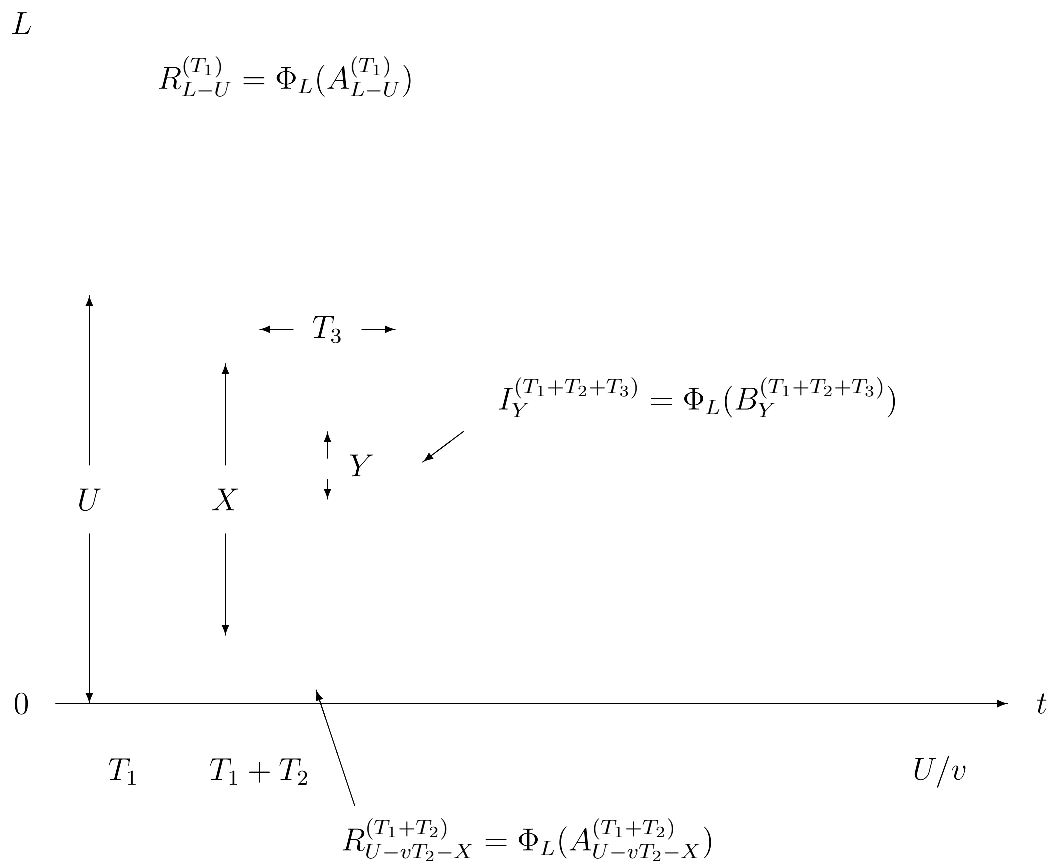


Figure 1 (after Quine and Robinson (1990)).

If $R^{(T_1)}_U \geq 1$, i.e. there is another point $(X, T_1 + T_2)$ of $\Phi_L$ in the triangle $A^{(T_1)}_U$, then by a similar argument as above, $R^{(T_1)}_U$ equals one plus the total number of points in a right-angled and an isosceles triangle. More precisely,

$$R^{(T_1)}_U \stackrel{d}{=} \mathbf{1}(T_2 < \frac{U}{v}) \left(1 + R^{(T_1+T_2)}_{U-vT_2-X} + I^{(T_1+T_2)}_X\right), \tag{3.2}$$

where $I^{(T_1+T_2)}_X$ denotes the total number of points of $\Phi_L$ which are born in the isosceles triangle with base $(0, X) \times \{T_1 + T_2\}$ and height $X/(2v)$ shown in Figure 1. Denote this isosceles triangle by $B^{(T_1+T_2)}_X$, i.e. $I^{(T_1+T_2)}_X = \Phi_L(B^{(T_1+T_2)}_X)$. Define $\inf\{t_i : (x_i, t_i) \in \emptyset\} = \infty$. Then $X$ in (3.2) is uniform on $(0, U - vT_2)$, where $T_2 \equiv \min\{U/v, \inf\{t_i : (x_i, t_i) \in \Psi_L \cap A^{(T_1)}_U\}\}$. Furthermore, conditional on $\{U = u, X = x, T_2 = t_2\}$, where $0 < u < L$, $0 < x < u - vt_2$ and $0 \leq t_2 < u/v$, the random variables $R^{(T_1+T_2)}_{U-vT_2-X}$ and $I^{(T_1+T_2)}_X$ are independent. Similar to $R^{(T_1)}_U$, only $T_1 + T_2$ and $X$ but not the exact position of the interval on $[0, L]$ on which the isosceles triangle based are essential to the distribution of $I^{(T_1+T_2)}_X$, with the understanding that each of the two endpoints of the base $(0, X)$ (i.e. $\{0\}$ and $\{X\}$) contains a point of $\Phi_L$.

Applying this kind of argument again to the isosceles triangle $B^{(T_1+T_2)}_X$ yields

$$I^{(T_1+T_2)}_X \stackrel{d}{=} \mathbf{1}(T_3 < \frac{X}{2v}) \left(1 + I^{(T_1+T_2+T_3)}_Y + I^{(T_1+T_2+T_3)}_{X-2vT_3-Y}\right), \tag{3.3}$$

where $Y$ is uniform on $(0, X - 2vT_3)$, $T_3 \equiv \min\{X/(2v), \inf\{t_i : (x_i, t_i) \in \Psi_L \cap B_X^{(T_1+T_2)}\}\}$, and conditional on $\{U = u, X = x, Y = y, T_2 = t_2, T_3 = t_3\}$, where $0 < u < L$, $0 < x < u - vt_2$, $0 < y < x - 2vt_3$, $0 \le t_2 < u/v$ and $0 \le t_3 < x/(2v)$, the random variables $I_Y^{(T_1+T_2+T_3)}$ and $I_{X-2vT_3-Y}^{(T_1+T_2+T_3)}$ are independent.

**Lemma 3.1** *Suppose $Z_L$ is a positive and finite random variable with finite mean and variance for each $L$. If the Laplace transform has the form*

$$\mathbf{E}[\exp\{-\xi Z_L\}] = \exp\{L\alpha(\xi) + \beta(\xi)\}$$

*for some real-valued functions $\alpha$ and $\beta$ such that $\alpha(\xi)$ and $\beta(\xi)$ are bounded for each fixed $\xi \in [0, \infty)$, then the distribution of $(Z_L - \mathbf{E}[Z_L])/\sqrt{\mathbf{var}[Z_L]}$ converges weakly to the standard normal distribution as $L \to \infty$.*

**Proof.** As $\mathbf{E}[Z_L]$ and $\mathbf{var}[Z_L]$ exist and are finite,

$$\begin{aligned} \alpha(\xi) &= \alpha'(0)\xi + \frac{\alpha''(0)}{2}\xi^2 + o(\xi^2), \\ \beta(\xi) &= \beta'(0)\xi + \frac{\beta''(0)}{2}\xi^2 + o(\xi^2). \end{aligned}$$

Then for each fixed $\xi$,

$$\mathbf{E}\left[\exp\left\{-\xi\left(\frac{Z_L - \mathbf{E}[Z_L]}{\sqrt{\mathbf{var}[Z_L]}}\right)\right\}\right] = \exp\left\{\frac{\xi^2}{2} + Lo\left(\frac{\xi^2}{L}\right)\right\} \to \exp\left\{\frac{\xi^2}{2}\right\}$$

as $L \to \infty$, and the result follows. □

**Lemma 3.2** *(a) As $L \to \infty$, $T_1$, $T_2$ and $T_3$ converge in probability to zero.*

*(b) As $L \to \infty$, for each $0 < u, x, y \le 1$, $R_{uL}^{(T_1)} - R_{uL}^{(0)}$, $R_{x(uL-vT_2)}^{(T_1+T_2)} - R_{xuL}^{(0)}$, $I_{x(uL-vT_2)}^{(T_1+T_2)} - I_{xuL}^{(0)}$ and $I_{y[x(uL-vT_2)-2vT_3]}^{(T_1+T_2+T_3)} - I_{yxuL}^{(0)}$ converge in probability to zero.*

**Proof.** (a) For each $\varepsilon > 0$, $\{T_1 > \varepsilon\}$ is equivalent to $\{\Psi_L([0, L] \times [0, \varepsilon]) = 0\}$. Thus

$$\mathbf{P}\{T_1 > \varepsilon\} = \exp\{-L\Lambda([0, \varepsilon])\} \to 0$$

as $L \to \infty$. The convergence in probability to zero for $T_2$ and $T_3$ can be proved similarly.

(b) For $\Lambda(\{0\}) > 0$, the statement is obvious, since $T_1$, $T_2$, $T_3$ converge almost surely to zero. Hence it suffices to consider only the case that $\Lambda \ll \ell$, $\Lambda([0, \varepsilon]) = O(\varepsilon)$ as $\varepsilon \to 0$ and $\Lambda([0, L]) = O(L)$ as $L \to \infty$.

Clearly

$$\mathbf{E}\left[\left|R_{uL}^{(T_1)} - R_{uL}^{(0)}\right|\right] \le \mathbf{E}[uL\Lambda([0, T_1])] + \mathbf{E}[vT_1\Lambda([0, uL/v])].$$

Thus, if $\mathbf{E}[LT_1]$ converges to zero, the result follows. Since $LT_1$ converges to zero in probability and is uniformly integrable, it converges in mean to zero.

The convergence in probability to zero of the other random variables can be proved similarly. □

**Remark.** Lemma 3.2(b) is quite similar to Slutzky's theorem for convergence in probability. However, $R_{uL}^{(\cdot)}$ is not a Borel function on $\mathbb{R}$ but a random variable, and so a proof for the statement is necessary. Moreover, Lemma 3.2(b) implies that the conditional distribution of $N_L$, conditional on the event $\{T_1 = 0, T_2 = 0, T_3 = 0\}$, converges weakly to the unconditional distribution of $N_L$ as $L \to \infty$.

**Lemma 3.3** *For each real $\xi$ and positive $x$ and $L$ denote*

$$\mathcal{L}_{xL}^{(I)}(\xi) \equiv \mathbf{E}[\exp\{-\xi(I_{xL}^{(0)} + 1)\}].$$

*Conditional on the event $\{T_1 = 0, T_2 = 0, T_3 = 0, U = uL, X = xL\}$, where $0 < u < 1$ and $0 < x < u$,*

$$\mathcal{L}_{xL}^{(I)}(\xi) = \exp\{xL\alpha(\xi)\}$$

*for some real-valued function $\alpha$ such that $\alpha(\xi)$ is finite for each $\xi \in [0, \infty)$.*

**Proof.** Since

$$\mathbf{P}\left\{I_{xL}^{(0)} = k\right\} \le \frac{\lambda_L^k e^{-\lambda_L}}{k!}$$

for $k = 0,\ 1,\ 2,\ \ldots$, where $\lambda_L = \int_0^{L/(2v)} (L - 2vt)\Lambda(dt)$, $\mathcal{L}_{xL}^{(I)}(\cdot)$ exists and is bounded above by 1 on $[0, \infty)$.

From (3.3), conditional on $\{T_1 = 0, T_2 = 0, T_3 = 0, U = uL, X = xL\}$ where $0 < u < 1$ and $0 < x < u$,

$$I_{xL}^{(0)} \stackrel{d}{=} 1 + I_Y^{(0)} + I_{xL-Y}^{(0)}, \tag{3.4}$$

where $Y$ is uniform on $(0, xL)$, and conditional further on $\{Y = yL\}$, where $0 < y < x$, the random variables $I_{yL}^{(0)}$ and $I_{xL-yL}^{(0)}$ are independent. Moreover, denote

$$I(x) \equiv I_{xL}^{(0)} + 1$$

for $0 < x < u$. For each fixed $L$,

$$\mathbf{E}\left[|I(x + \delta) - I(x)|\right] \le 2v|\delta|\Lambda([0, uL/(2v)]) + uL\Lambda([0, |\delta|]) \to 0$$

for any $x \in (0, u)$ as $\delta \to 0$, and so $\mathbf{E}[\exp\{-\xi I(\cdot)\}]$ is uniformly continuous on $(0, u)$.

From (3.4), we have

$$\mathcal{L}_{xL}^{(I)}(\xi) = \int_0^x \mathcal{L}_{yL}^{(I)}(\xi)\mathcal{L}_{(x-y)L}^{(I)}(\xi)\frac{dy}{x}. \tag{3.5}$$

For each $x$ and $\xi$ define $g : [-x/2, x/2] \mapsto \mathbb{R}$ by

$$g(z) \equiv \mathcal{L}_{xL}^{(I)}(\xi) - \mathcal{L}_{(2^{-1}x+z)L}^{(I)}(\xi)\mathcal{L}_{(2^{-1}x-z)L}^{(I)}(\xi),$$

and rewrite (3.5) as

$$\int_{-2^{-1}x}^{2^{-1}x} g(z)\frac{dz}{x} = 0. \tag{3.6}$$

Since $g$ is continuous and symmetric about $g(0)$, (3.6) implies that $g(0) = 0$. Thus for all integers $n \in \{k \in \mathbb{Z} : 0 < 2^{-k}x < u\}$, where $\mathbb{Z}$ denotes the set of all integers,

$$\mathcal{L}_{xL}^{(I)}(\xi) = \mathcal{L}_{2^{-n}xL}^{(I)}(\xi)^{2^n}.$$

As $\mathcal{L}_1^{(I)}$ is bounded on $[0, \infty)$, there exists a real-valued function $\alpha$ such that $\mathcal{L}_1^{(I)}(\xi) \equiv \exp\{\alpha(\xi)\}$, which is finite for each nonnegative $\xi$. Define $f(w) \equiv \exp\{-w\alpha(\xi)\}\mathcal{L}_w^{(I)}(\xi)$, for all $0 < w < uL$, and 1 otherwise. Then $f(1) = 1$ and $f(2^{-n}) = 1$ for every positive or negative integer $n$ such that $2^{-n} < uL$. Since $f$ is continuous on $(0, uL)$, $f = 1$. Therefore, for all $0 < x < u$,

$$\mathcal{L}_{xL}^{(I)}(\xi) = \exp\{xL\alpha(\xi)\}.$$

□

**Remark.** As Quine and Robinson (1990) remarked, if instead on an interval, the growth process took place on a circle of perimeter length $L$, the number of points formed would be distributed as $1+I_L^{(T_1)}$. Let $N_L^*$ be a random variable with the same distribution as $1 + I_L^{(T_1)}$. Lemmas 3.2(b) and 3.3 mean that in this case the Laplace transform $\mathbf{E}[\exp\{-\xi N_L^*\}] \sim \exp\{L\alpha(\xi)\}$. By Lemma 3.1, asymptotic normality holds, and by Theorem 3.1 below, $N_L^*$ and $N_L$ have the same asymptotic distribution as $L \to \infty$.

**Lemma 3.4** *For each real $\xi$ and positive $u$ and $L$ denote*

$$\mathcal{L}_{uL}^{(R)}(\xi) \equiv \mathbf{E}[\exp\{-\xi(R_{uL}^{(0)} + 1)\}].$$

*Conditional on $\{T_1 = 0, T_2 = 0, T_3 = 0, U = uL\}$, where $0 < u < 1$,*

$$\mathcal{L}_{uL}^{(R)}(\xi) = \exp\{uL\alpha(\xi) + \beta(\xi)\}$$

*for some real-valued function $\beta$ such that $\beta(\xi)$ is bounded for each $\xi \in [0, \infty)$, where $\alpha$ is the same as in Lemma 3.3.*

**Proof.** Similar to the proof of Lemma 3.3, $\mathcal{L}_{uL}^{(R)}(\cdot)$ exists and is bounded, and conditional on $\{T_1 = 0, T_2 = 0, T_3 = 0, U = uL\}$, where $0 < u < 1$ and $X$ is uniform on $(0, uL)$, we have

$$R_{uL}^{(0)} \stackrel{d}{=} 1 + R_{uL-X}^{(0)} + I_X^{(0)}.$$

Thus,

$$\mathcal{L}_{uL}^{(R)}(\xi) = \int_0^u \mathcal{L}_{(u-x)L}^{(R)}(\xi)\mathcal{L}_{xL}^{(I)}(\xi)\frac{dx}{u}.$$

Conditional on $\{T_1 = 0, T_2 = 0, T_3 = 0, U = uL, X = xL\}$ where $0 < x < u$, $\mathcal{L}_{xL}^{(I)}(\xi) = \exp\{xL\alpha(\xi)\}$ (Lemma 3.3). Hence, dividing both sides by $\exp\{uL\alpha(\xi)\}$ yields

$$h(u) = \int_0^u h(u-x)\frac{dx}{u} = \int_0^u h(x)\frac{dx}{u}, \tag{3.7}$$

where $h(u) \equiv \mathcal{L}_{uL}^{(R)}(\xi)\exp\{-uL\alpha(\xi)\}$. The derivative (with respect to $u$) of the leftmost term of (3.7) exists and is equal to zero. Thus for each $\xi$, $h$ is a constant on $(0, u]$, say $\exp\{\beta(\xi)\}$ where $\beta$ is some real-valued function such that $\beta(\xi)$ is bounded for each $\xi \in [0, \infty)$. The result follows. □

**Theorem 3.1** *The distribution of* $(N_L - \mathbf{E}[N_L])/\sqrt{\mathbf{var}[N_L]}$ *converges weakly to the standard normal distribution as* $L \to \infty$.

**Proof.** It can be seen from (3.1) that conditional on $\{T_1 = 0\}$,

$$N_L \stackrel{d}{=} 1 + R_U^{(0)} + R_{L-U}^{(0)},$$

where $U$ is uniform on $(0, L)$. Moreover, conditional further on $\{U = uL\}$, where $0 < u < 1$, the random variables $R_{uL}^{(0)}$ and $R_{(1-u)L}^{(0)}$ above are independent. Thus, by letting

$$\mathcal{L}_L(\xi) \equiv \mathbf{E}[\exp\{-\xi(N_L + 1)\}],$$

we have

$$\mathcal{L}_L(\xi) = \int_0^1 \mathcal{L}_{uL}^{(R)}(\xi)\mathcal{L}_{(1-u)L}^{(R)}(\xi)du.$$

From Lemma 3.4, conditional further on $\{T_2 = 0, T_3 = 0\}$,

$$\mathcal{L}_L(\xi) = \exp\left\{L\alpha(\xi) + \beta(\xi)\right\}.$$

By Lemma 3.2(b), this conditional distribution of $N_L$ converges weakly to the unconditional distribution of $N_L$ as $L \to \infty$. The asymptotic normality follows from Lemma 3.1. □

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