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THE TIME OF COMPLETION OF A LINEAR

BIRTH-GROWTH MODEL

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Abstract

Consider the following birth-growth model in $\mathbb{R}$. Seeds are born randomly according to an inhomogeneous space-time Poisson process. A newly formed point immediately initiates a bi-directional coverage by sending out a growing branch. Each frontier of a branch moves at a constant speed until it meets an opposing one. New seeds continue to form on the uncovered parts on the line. We are interested in the time until a bounded interval is completely covered. The exact and limiting distributions as the length of interval tends to infinity are obtained for this completion time by considering a related Markov process. Moreover, some strong limit results are also established.

Keywords: Completion time; coverage; inhomogeneous Poisson process; Johnson-Mehl model; linear birth-growth model; Markov process; strong limit theorem

AMS 1991 Subject Classification: Primary 60G55, 60J25

Secondary 60F05, 60F15, 60D05

1. Introduction

Consider the following linear random birth-growth model. Points arrive indepen-
dently on a line at random positions and times according to a space-time Poisson pro-
cess \( \Phi \equiv \{(x_i, t_i) \in \mathbb{R} \times [0, \infty)\} \) with intensity measure \( dx \lambda(t) dt \). The first arrived
point \((x_1, t_1)\) immediately initiates a bi-directional coverage by sending out a growing
branch centered at \( x_1 \). Each frontier of the branch moves with a constant speed \( v \)
until it meets an opposing one. Other points continue to arrive according to \( \Phi \). If a
point arrives at a position that has already been covered by a branch, it will be deleted
(or thinned); otherwise, another bi-directional coverage will be initiated by the same
mechanism as that initiated by the first point. Applications of such processes can be
found in cell biology (Wolk [13]), molecular biology (Vanderbei and Shepp [12]; Cowan
et al. [4]) and neurobiology (Quine and Robinson [10, 11]) as well as other more ob-
vious areas such as crystal growth (Kolmogorov [8]; Johnson and Mehl [7]; Meijering
[9]). The distributions of random variables such as the number of unthinned points and
the time required to cover an interval of a given length have been studied under various
assumptions on the arrival regime. Quine and Robinson [10], Holst et al. [6], Chiu [2]
and Chiu and Quine [3] showed the asymptotic normality of the number of unthinned
points. Chiu [1] and Erhardsson [5] proved that the number of uncovered components
has an asymptotic Poisson distribution. Vanderbei and Shepp [12] and Cowan et al.
[4] studied the limiting distributions, by different means, of the completion time of the
birth-growth model with \( \lambda(x) = \lambda \) and \( \lambda(x) = \frac{\gamma}{\mu} e^{-\frac{x}{\mu}} \), respectively, where \( \lambda, \gamma \) and \( \mu \)
are positive finite constants. Weak limit theorems have also been proved. The general
model considered by Holst et al. [6] incorporates both these models as special cases.
For limit theorems of the completion time in higher dimensional cases see Chiu [1]. The
current paper deals with the linear birth-growth model. We use the Markov process approach suggested by Vanderbei and Shepp [12] (see also Erhardsson [5] and Holst et al. [6]) to establish, under more general conditions than Holst et al. [6], the exact and limiting distributions and strong limit theorems for the time of complete coverage of a sufficiently long interval.

2. Laplace transform of the completion time

Assume that the space-time Poisson process \( \Phi \) with intensity measure \( dx \lambda(t)dt \) is defined in the probability space \( (\Omega, \mathcal{F}, P) \), and \( \lambda(\cdot) \) is integrable and such that for all \( t > 0 \),

\[
0 < \Lambda(t) := \int_0^t \lambda(y)dy < \infty.
\]

Because the process is homogeneous in space, the growth velocity \( v \) of the seed can be taken as \( \frac{1}{2} \) by a change of scale. After a shear transformation \( (x, t) \rightarrow (x + \frac{t}{2}, t) \), a stationary Markov process \( \{\xi_x, -\infty < x < \infty\} \) with the filtration \( \{\mathcal{F}_x, -\infty < x < \infty\} \) is obtained, where \( \mathcal{F}_x \) is the \( \sigma \)-algebra generated by the points \( \{(x_i, t_i) \in \Phi : -\infty < x_i \leq x\} \) (for details see Holst et al. [6]). Denote by \( T_L \) the lowest time level at which the interval \( (0, L) \) is completely covered and by \( \tau_z \) the value of \( x \) at which the process \( \{\xi_x\} \) first hits the level \( z \) (note that the level of \( \{\xi_x\} \) is the time and the parameter space is the positions). Then for \( t < z \),

\[
P_t(T_L < z) = P_t(\tau_z > L),
\]

where \( P_t \) denotes the conditional probability given that the initial level is \( \xi_0 = t \).

It is known that the Laplace transform of \( \tau_z \) can be obtained by considering the
transition semigroup of operators \( \{ T_x \} \) defined by
\[
T_x f(t) := \mathbb{E}_t f(\xi_x) = \left(1 - \int_0^x \Lambda(t + u) \, du \right) f(t + x) + \left(\int_0^x \Lambda(t + u) \, du \right) \int_0^t f(u) \frac{\lambda(u)}{\Lambda(t)} \, du + o(x)
\]
\[
= (1 - x\Lambda(t + \delta_x)) f(t + x) + x\Lambda(t + \delta_x) \int_0^t f(u) \frac{\lambda(u)}{\Lambda(t)} \, du + o(x),
\]
for some \( \delta_x \) in \((0, x)\), where \( \mathbb{E}_t \) denotes the conditional expectation given \( \xi_0 = t \) and \( f \) is a bounded measurable real-valued function on \([0, \infty)\). Thus, the infinitesimal generator \( \mathcal{A} \) is given by
\[
\mathcal{A} f(t) = f'(t) - \Lambda(t) f(t) + \int_0^t f(u) \frac{\lambda(u)}{\Lambda(t)} \, du.
\]
The Laplace transform \( f(t) = \mathbb{E}_t e^{-\alpha \tau_x} \) is the solution of
\[
\begin{cases}
\mathcal{A} f(t) = \alpha f(t), & 0 < t < z, \\
f(z) = 1.
\end{cases}
\] (2.1)

Holst et al. [6, p. 908] derived the same system of equations by a regenerative argument and obtained explicitly the Laplace transform
\[
\mathbb{E}_t e^{-\alpha \tau_x} = \frac{1 + \alpha \int_0^t e^{\alpha u + \Delta(u)} \, du}{1 + \alpha \int_0^z e^{\alpha u + \Delta(u)} \, du},
\]
where \( \Delta(u) = \int_0^u \Lambda(t) \, dt \).

In principle the inverse Laplace transform can always be found, but it is in the form of a Bromwich integral. Even for the simplest case in which \( \lambda(t) = \lambda \), the Bromwich integral is difficult to calculate (see Vanderbei and Shepp [12, p. 308]). Only limit theorems have been derived in Holst et al. [6]. In the next section we obtain the exact distribution for \( \tau_x \).
3. Exact distribution

Since the Laplace transform
\[
E_t \exp(-\alpha \tau_z) = 1 - \alpha \int_0^\infty e^{-\alpha L} P_t(\tau_z > L) dL
\]
satisfies system (2.1), \( P_t(\tau_z > L) = q(t, L) \) is the unique solution of the following initial-boundary value problem:

\[
\begin{align*}
\frac{\partial q(t, L)}{\partial L} &= \mathcal{A} q(t), \quad L > 0, 0 < t < z, \\
\lim_{t \to z} q(t, L) &= 0, \quad L > 0, \\
\lim_{L \to 0} q(t, L) &= 1, \quad 0 < t < z,
\end{align*}
\]

(3.1)

where \( q(t) = q(t, L) \).

**Theorem 3.1.** Let \( T_L \) denote the earliest time that the interval \((0, L)\) is completely covered. For \( 0 \leq t < z \) and \( L > 0 \),

\[
P_t(T_L < z) = \sum_k C_k \left( 1 + a_k(z) \int_0^t \exp(a_k(z)u + \Delta(u)) du \right) e^{a_k(z)L},
\]

where \( a_1(z) > a_2(z) > a_3(z) > \cdots \) are all negative zeros of \( g(a) := 1 + a \int_0^z \exp(au + \Delta(u)) du \), and

\[
C_k = -\frac{1}{a_k^2(z) \int_0^z u e^{a_k(z)u + \Delta(u)} du - 1}.
\]

**Proof.** Setting \( q(t, L) = U(t)V(L) \) yields two equations:

\[
V'(L) = a V(L), \quad L > 0,
\]

(3.2)

\[
\mathcal{A} U(t) = a U(t), \quad 0 < t < z,
\]

(3.3)
where $a$ is a separation constant. The boundary condition of $q(t, L)$ leads to

$$
\lim_{t \to z} U(t) = 0.
$$  \hfill (3.4)

For $a \geq 0$ the only solution of (3.3) with boundary condition (3.4) is zero. Next, assume $a < 0$. The solution of equation (3.3) is of the form

$$
U(t) = B_1 \left( 1 + a \int_0^t \exp(au + \Delta(u)) du \right),
$$

where $B_1$ is a constant and $a$ can be determined by (3.4), that is,

$$
1 + a \int_0^z \exp(au + \Delta(u)) du = 0.
$$

Let $a_1(z), a_2(z), a_3(z), \cdots$ denote its all negative roots, which are all simple, and so without loss of generality assume that $0 > a_1(z) > a_2(z) > a_3(z) > \cdots$. Then the solutions to equations (3.2) and (3.3) are, respectively,

$$
V_k(L) = B_{2k} e^{a_k(z)L},
$$

$$
U_k(t) = B_{1k} \left( 1 + a_k(z) \int_0^t \exp(a_k(z)u + \Delta(u)) du \right),
$$

for $k = 1, 2, \cdots$, where $B_{1k}$ and $B_{2k}$ are constants. Hence the general solution to problems (3.1) is of the form

$$
q(t, L) = \sum_k C_k \left( 1 + a_k(z) \int_0^t \exp(a_k(z)u + \Delta(u)) du \right) e^{a_k(z)L},
$$  \hfill (3.5)

where $C_k$’s are constants. The initial condition given in (3.1) yields

$$
C_k = -\frac{1}{a_k^2(z) \int_0^z u e^{a_k(z)u + \Delta(u)} du - 1},
$$

and the result follows.

4. Limiting distributions
Theorem 4.1. Let $T_L$ denote the earliest time that the interval $(0, L)$ is completely covered. For $z > 0$ and $0 \leq t < z$,

$$
\lim_{L \to \infty} \frac{1}{L} \log P_t(T_L < z) = a_1(z),
$$

(4.1)

where $a_1(z)$ is the principal zero of

$$
g(a) = 1 + a \int_0^z e^{au + \Delta(u)} du.
$$

Proof. From Theorem 3.1

$$
P_t(T_L < z) = e^{a_1(z) L} C_1 \left(1 + a_1(z) \int_0^t \exp(a_1(z)u + \Delta(u)) du \right)
+ e^{a_1(z) L} \sum_{k \geq 2} C_k \left(1 + a_k(z) \int_0^t \exp(a_k(z)u + \Delta(u)) du \right) e^{(a_k(z) - a_1(z))L}.
$$

(4.2)

The Abel test suggests that the series above is uniformly convergent with respect to $L$ on $[L_0, \infty)$, where $L_0 > 0$. Since $C_1 \left(1 + a_1(z) \int_0^t \exp(a_1(z)u + \Delta(u)) du \right) > 0$, and $a_k(z)$’s are decreasing, the result follows.

Theorem 4.2. For each real $u$, let $L = L(z)$ be a function of $z$. If $L \to \infty$ as $z \to \infty$ in such a manner that $a_1(z)L \to -e^{-u}$ as $z \to \infty$, then

$$
\lim_{z \to \infty} P_t(T_L < z) = \exp(-e^{-u}).
$$

(4.3)

Proof. For each $a < 0$, $g(a) = 1 + a \int_0^z e^{au + \Delta(u)} du < 0$ as $z$ is large enough. This implies that $\lim_{z \to \infty} a_k(z) = -\infty, k \geq 2$ and $\lim_{z \to \infty} a_1(z) = 0$. Moreover, the initial condition given in (3.1) leads to $\lim_{z \to \infty} C_1 = 1$. The result follows.

Remark 4.1. This proof fills in the gap mentioned in Vanderbei and Shepp [12, p. 311].
In the Holst et al. [6] they assumed that $\lambda(\cdot)$ satisfies $(\Lambda_1)$ $\lim_{t \to \infty} \Lambda(t) < \infty$,

$(\Lambda_2)$ $\Lambda(t) \to \infty$ and $\frac{\lambda(t)}{\Lambda(t)} \to \rho$ for some $0 \leq \rho < \infty$ as $t \to \infty$, or $(\Lambda_3)$ $\Lambda(t) \to \infty$

and $\frac{\lambda(t)}{\Lambda(t)} \to c$ with $0 < c < \infty$ as $t \to \infty$. For these three classes of $\lambda(\cdot)$, the condition $a_1(z)L \to -e^{-u}$ is equivalent to

$$\Delta(z) - \log \Lambda(z) = \log L + u + o(1).$$  \hspace{1cm} (4.4)

(see Holst et al. [6, p. 902 equation (2.2)]), which is very useful in finding an explicit expression for $L(z)$ in Theorem 4.2. However, the equivalence between (4.4) and $a_1(z)L \to -e^{-u}$ does not hold for general $\lambda(\cdot)$. The following theorem gives a sufficient condition, which includes $(\Lambda_1) - (\Lambda_3)$, for this equivalence being true.

**Theorem 4.3.** Suppose that $\lim_{z \to \infty} \frac{\lambda(z)}{\Lambda(z)} = 0$. Then the condition $\lim_{z \to \infty} a_1(z)L = -e^{-u}$ is equivalent to

$$\Delta(z) - \log \Lambda(z) = \log L + u + o(1),$$  \hspace{1cm} (4.5)

where $u$ is a real number.

**Proof.** From $\lim_{z \to \infty} \frac{\lambda(z)}{\Lambda(z)} = 0$, one can obtain

$$g'(0) = \int_{z} e^{\Delta(u)} du \sim \frac{1}{\Lambda(z)} e^{\Delta(z)}, \text{ as } z \to \infty.$$

Using one step of Newton’s method yields

$$a_1(z) \sim -\frac{1}{g'(0)} \sim -\Lambda(z) e^{-\Delta(z)}, \text{ as } z \to \infty,$$

and the equivalence follows.
The following example shows that \((\Lambda_1) - (\Lambda_3)\) do not include all cases.

**Example.** Suppose \(\lambda(z) = \frac{1}{2}(2z^2 + 1)e^{z^2}\), so that \(\Lambda(z) = \frac{1}{2}z e^{z^2}\) and \(\Delta(z) = e^{z^2} - 1\). Moreover, \(\lim_{z \to \infty} \frac{\lambda(z)}{\Lambda(z)} = \infty\) and \(\lim_{z \to \infty} \frac{\lambda(z)}{\Lambda(z)} = 0\). Hence \(\lambda(\cdot)\) does not satisfy \((\Lambda_1) - (\Lambda_3)\), but satisfies the condition in Theorem 4.3. It follows from (4.5) that

\[
e^{z^2} = 1 + \log \frac{1}{2} z + \log e^{z^2} + \log L + u + o(1)
\]

which is equivalent to

\[
z = \sqrt{\log \log L} + \frac{1 - \log 2 + \frac{1}{2} \log \log L + \log \log L + u}{\sqrt{\log \log L \log L}} + o \left( \frac{1}{\sqrt{\log \log L \log L}} \right),
\]

and hence (4.3) gives

\[
\lim_{L \to \infty} \text{P}_t \left( \log L \sqrt{\log \log L} T_L - G(L) < u \right) = \exp(-e^{-u}),
\]

where \(G(L) = \log L \log \log L + \frac{1}{2} \log \log L + \log \log L + 1 - \log 2\).

**5. Strong Limit Theorems**

For \(\lambda(x) = \lambda\), Vanderbei and Shepp [12] proved that

\[
\textbf{E} T_L^n \sim (\lambda^{-1} \log \lambda L^2)^{\frac{n}{2}}, \text{ as } L \to \infty,
\]

and Cowan et al. [4] showed that

\[
\frac{\sqrt{\lambda} T_L}{\sqrt{\log \lambda L^2}} \rightarrow 1 \text{ in probability as } L \to \infty.
\]

Actually, a stronger version can be obtained.
Theorem 5.1. Let $T_L$ denote the earliest time that the interval $(0, L)$ is completely covered.

1. If $\lambda(x) = \lambda$, where $\lambda$ is a positive finite constant, then

$$\lim_{L \to \infty} \frac{\sqrt{\lambda T_L}}{\sqrt{\log \lambda L^2}} = 1, \quad P_t\text{-almost surely.}$$

2. If $\Lambda(x) = \gamma F(x)$, where $F$ is a distribution function with support on $(0, \infty)$ and finite mean, and $\gamma$ is a positive finite constant, then

$$\lim_{L \to \infty} \frac{\gamma T_L}{\log \gamma L} = 1, \quad P_t\text{-almost surely.}$$

3. If $\lambda(x) = e^x$, then

$$\lim_{L \to \infty} \frac{T_L}{\log \log L} = 1, \quad P_t\text{-almost surely.}$$

4. If $\lambda(x) = \frac{1}{2}(2x^2 + 1)e^x$, then

$$\lim_{L \to \infty} \frac{T_L}{\sqrt{\log \log L}} = 1, \quad P_t\text{-almost surely.}$$

Proof. We prove (2) only, and the others can be proved in a similar way.

For any $0 < \rho < 1$ and for any $C > 1$,

$$P_t \left( \liminf_{L \to \infty} \frac{\gamma T_L}{\log \gamma L} < \rho \right) \leq P_t \left( \inf_{C^n \leq L \leq C^{n+1}} \frac{\gamma T_L}{\log \gamma L} < \rho, \text{ i.o.} \right)$$

$$\leq P_t (\gamma T_{C^n} \leq \rho \log \gamma C^{n+1}, \text{ i.o.})$$

$$= P_t (\gamma T_{C^n} - \log \gamma C^n - \gamma \mu \leq \rho \log \gamma C^{n+1} - \log \gamma C^n - \gamma \mu, \text{ i.o.}).$$

By Theorems 4.2 and 4.3 (see also Holst et al. [6, p. 909]),

$$\lim_{n \to \infty} P_t (\gamma T_{C^n} - \log \gamma C^n - \gamma \mu \leq u) = \exp(-e^{-u}). \quad (5.1)$$
Thus,

\[ P_t(\gamma T_{C_n} - \log \gamma C^n - \gamma \mu \leq \rho \log \gamma C^{n+1} - \log \gamma C^n - \gamma \mu) \]
\[ \sim \exp(-\exp(\mu \gamma - \log(\gamma^{\rho - 1} C^{(\rho - 1)n + \rho}))). \]

Note that

\[ \sum_n \exp(-\exp(\mu \gamma - \log(\gamma^{\rho - 1} C^{(\rho - 1)n + \rho}))) < \infty. \]

Hence, by the Borel-Cantelli Lemma,

\[ P_t \left( \liminf_{L \to \infty} \frac{\gamma T_L}{\log \gamma L} < \rho \right) = 0, \]

which implies

\[ P_t \left( \liminf_{L \to \infty} \frac{\gamma T_L}{\log \gamma L} \geq 1 \right) = 1. \quad (5.2) \]

On the other hand, for any \( \epsilon > 0 \) and \( C > 1 \),

\[ P_t \left( \limsup_{L \to \infty} \frac{\gamma T_L}{\log \gamma L} > 1 + \epsilon \right) \leq P_t \left( \sup_{\gamma^{C_n} \leq L \leq \gamma^{C_{n+1}}} \frac{\gamma T_L}{\log \gamma L} > 1 + \epsilon, \text{ i.o.} \right) \]
\[ \leq P_t (\gamma T_{C^{n+1}} \geq (1 + \epsilon) \log \gamma C^n, \text{ i.o.}) \]
\[ = P_t (\gamma T_{C^{n+1}} - \log \gamma C^{n+1} - \gamma \mu \geq (1 + \epsilon) \log \gamma C^n - \log \gamma C^{n+1} - \gamma \mu, \text{ i.o.}). \]

Using (5.1) again yields

\[ P_t (\gamma T_{C^{n+1}} - \log \gamma C^{n+1} - \gamma \mu \geq (1 + \epsilon) \log \gamma C^n - \log \gamma C^{n+1} - \gamma \mu) \sim e^{\mu \gamma} C^{1-\epsilon}. \]

The Borel-Cantelli Lemma leads to

\[ P_t \left( \limsup_{L \to \infty} \frac{\gamma T_L}{\log \gamma L} > 1 + \epsilon \right) = 0. \]
By the arbitrariness of $\epsilon > 0$,

$$P_{\epsilon} \left( \limsup_{L \to \infty} \frac{\gamma T_L}{\log \gamma L} \leq 1 \right) = 1,$$

and the result follows.

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**References**


