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Link to published article: https://dx.doi.org/10.1080/10920277.2005.10596233

**APA Citation**

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DISCUSSIONS to
“The Time Value of Ruin in a Sparre Andersen Model,”
Hans U. Gerber and Elias S. W. Shiu, April 2005

Chuancun Yin* and Sung Nok Chiu**

The ruin problem for a renewal risk model is often difficult to solve; Professors Gerber and Shiu have elegantly extended the results for the classical model in Gerber and Shiu (1998) to a Sparre Andersen risk model with Generalized Erlang interclaim times. The purpose of this discussion is to present explicit formulas for $\phi(u)$ and $f(x, y|u)$ for the special case in which $\hat{p}(\xi)$ is a rational function.

Gerber and Shiu express $\hat{\phi}$ by (eq. 7.2)

$$\hat{\phi}(\xi) = \frac{\hat{\omega}(\xi) - q(\xi)}{\gamma(\xi) - p(\xi)}$$

It is showed by Gerber and Shiu in Section 4 that, in the right half of the complex plane, the function in the denominator of (1) has $n$ zeros $\rho_1, \rho_2, \cdots, \rho_n$. For simplicity, we assume that they are distinct. By (9.2) and (11.1), we can write

$$\hat{\phi}(\xi) = \frac{\hat{S}\omega(\xi)}{1 - \hat{S}p(\xi)}, \quad (1)$$

where the operator $S$ is defined by

$$S = \frac{\lambda_1 \cdots \lambda_n}{c^n} \prod_{j=1}^{n} T_{\rho_j},$$

Moreover, by (9.10), $(Sp)(y)$ can be written as a linear function of $T_{\rho_j}p(y)$'s:

$$(Sp)(y) = g(y) = \frac{\prod_{i=1}^{n} \lambda_i}{c^n} \left[ \sum_{j=1}^{n} \left( \prod_{k=1, k \neq j}^{n} \frac{1}{\rho_k - \rho_j} \right) T_{\rho_j}p(y) \right].$$

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It follows that the denominator of (1) is a rational function if and only if \( \hat{\rho}(\xi) \) is a rational function. Let \( \theta_1, \ldots, \theta_k \) be the distinct roots with negative real parts of the equation \( \hat{g}(\xi) = 1 \) and \( n_1, \ldots, n_k \) be their multiplicities, respectively. From equation (11.4) we can see that the \( \theta_k \)'s are exactly the zeros with negative real parts of equation (4.2). By the principle of partial fractions we may write

\[
\frac{1}{1 - \hat{g}(\xi)} = P_1 \left( \frac{1}{\xi - \theta_1} \right) + \cdots + P_k \left( \frac{1}{\xi - \theta_k} \right) + 1,
\]

where

\[
P_j(\alpha) = c_{j,n_j}\alpha^{n_j} + c_{j,n_j-1}\alpha^{n_j-1} + \cdots + c_{j,1}\alpha, \quad j = 1, 2, \ldots, k.
\]

Denote by

\[
Q_j(u) = c_{j,n_j} \frac{1}{(n_j - 1)!} u^{n_j-1} e^{\theta_j(\delta)u} + c_{j,n_j-1} \frac{1}{(n_j - 2)!} u^{n_j-2} e^{\theta_j(\delta)u} + \cdots + c_{j,1} e^{\theta_j(\delta)u},
\]

we have

\[
\hat{Q}_j(\xi) = P_j \left( \frac{1}{\xi - \theta_j} \right), \quad j = 1, 2, \ldots, k.
\]

Hence (1) can be written as

\[
\hat{\phi}(\xi) = \sum_{j=1}^{k} (\hat{Q}_j(\xi) \cdot \hat{S}\omega(\xi)) + \hat{S}\omega(\xi).
\]

Inverting the Laplace transform yields

\[
\phi(u) = \sum_{j=1}^{k} \int_{0}^{u} Q_j(u - x) \cdot (S\omega)(x) dx + (S\omega)(u). \tag{2}
\]

In particular, when \( n_1 = \cdots = n_k = 1 \), we have

\[
\phi(u) = \sum_{j=1}^{k} A_j \int_{0}^{u} e^{\theta_j(u-x)} \cdot (S\omega)(x) dx + (S\omega)(u), \tag{3}
\]

where

\[
A_j = -\frac{1}{\hat{g}'(\theta_j)}, \quad j = 1, 2, \ldots, k.
\]
Moreover, the discounted joint probability density function \( f(x, y|u) \) defined by (2.1) can be obtained from (2) or (3). For simplicity, we assume that all roots with negative real parts of \( \hat{g}(\xi) = 1 \) are simple, that is, \( n_1 = \cdots = n_k = 1 \). It follows from (9.11) that
\[
(S\omega)(u) = \int_0^\infty \int_0^\infty w(x, y) 1(x \geq u) f(x - u, y + u|0) dx dy,
\]
from which we obtain
\[
\int_0^u e^{\theta_j(u-z)}(S\omega)(z) dz = \int_0^\infty \int_0^\infty w(x, y) \left( \int_0^{u\wedge x} e^{\theta_j(u-z)} f(x - z, y + z|0) dz \right) dx dy.
\]
Consequently,
\[
\phi(u) = \int_0^\infty \int_0^\infty w(x, y) \sum_{j=1}^k A_j \left( \int_0^{u\wedge x} e^{\theta_j(u-z)} f(x - z, y + z|0) dz \right) dx dy
\]
\[
= \int_0^\infty \int_0^\infty w(x, y) 1(x \geq u) f(x - u, y + u|0) dx dy.
\]
(4)

Recall that \( \phi(u) = \int_0^\infty \int_0^\infty w(x, y) f(x, y|u) dx dy; \) by comparing this with (4) we find that
\[
f(x, y|u) = \sum_{j=1}^k A_j \left( \int_0^{u\wedge x} e^{\theta_j(u-z)} f(x - z, y + z|0) dz \right)
\]
\[
+ 1(x \geq u)f(x - u, y + u|0)
\]
\[
= \frac{\prod_{l=1}^n \lambda_l p(x + y) \sum_{j=1}^k A_j}{c^n} \left[ e^{\theta_j u} \sum_{i=1}^n e^{-\rho_i x} \frac{1 - e^{-(u\wedge x)(\theta_j - \rho_i)}}{\theta_j - \rho_i} \prod_{k=1, k \neq i}^n \frac{1}{\rho_k - \rho_i} \right]
\]
\[
+ 1(x \geq u)f(x - u, y + u|0),
\]
with (see eq. 8.3)
\[
f(x, y|0) = \frac{\prod_{l=1}^n \lambda_l p(x + y) \sum_{j=1}^k A_j}{c^n} \left[ e^{\theta_j x} \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right].
\]

In particular, we get the marginal discounted probability density function
\[
f_{U(T^-)}(x|u) = \frac{\prod_{l=1}^n \lambda_l p(x) \sum_{j=1}^k A_j}{c^n} \left[ e^{\theta_j u} \sum_{i=1}^n e^{-\rho_i x} \frac{1 - e^{-(u\wedge x)(\theta_j - \rho_i)}}{\theta_j - \rho_i} \prod_{k=1, k \neq i}^n \frac{1}{\rho_k - \rho_i} \right]
\]
\[
+ 1(x \geq u)\frac{\prod_{l=1}^n \lambda_l p(x) \sum_{j=1}^k A_j}{c^n} \left[ e^{-\rho_j (x-u)} \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right].
\]
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