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"The Time Value of Ruin in a Sparre Andersen Model," Hans U. Gerber and Elias S. W. Shiu, April 2005

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DISCUSSIONS to

"The Time Value of Ruin in a Sparre Andersen Model,"

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The ruin problem for a renewal risk model is often difficult to solve; Professors Gerber and Shiu have elegantly extended the results for the classical model in Gerber and Shiu (1998) to a Sparre Andersen risk model with Generalized Erlang interclaim times. The purpose of this discussion is to present explicit formulas for $\phi(u)$ and $f(x, y|u)$ for the special case in which $\hat{p}(\xi)$ is a rational function.

Gerber and Shiu express $\hat{\phi}$ by (eq. 7.2)

$$\hat{\phi}(\xi) = \frac{\hat{\omega}(\xi) - q(\xi)}{\gamma(\xi) - p\hat{\ }(\xi)}.$$

It is showed by Gerber and Shiu in Section 4 that, in the right half of the complex plane, the function in the denominator of (1) has n zeros $\rho_1, \rho_2, \cdots, \rho_n$. For simplicity, we assume that they are distinct. By (9.2) and (11.1), we can write

$$\hat{\phi}(\xi) = \frac{\widehat{S\omega}(\xi)}{1 - \widehat{Sp}(\xi)}, \tag{1}$$

where the operator S is defined by

$$S = \frac{\lambda_1 \cdots \lambda_n}{c^n} \prod_{j=1}^{n} T_{\rho_j}.$$

Moreover, by (9.10), $(Sp)(y)$ can be written as a linear function of $T_{\rho_j}p(y)$'s:

$$(Sp)(y) = g(y) = \frac{\prod_{i=1}^{n} \lambda_i}{c^n} \left[\sum_{j=1}^{n} \left(\prod_{k=1, k \neq j}^{n} \frac{1}{\rho_k - \rho_j} \right) T_{\rho_j} p \right] (y).$$

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It follows that the denominator of (1) is a rational function if and only if $\hat{p}(\xi)$ is a rational function. Let $\theta_1, \ldots, \theta_k$ be the distinct roots with negative real parts of the equation $\hat{g}(\xi) = 1$ and $n_1, \ldots, n_k$ be their multiplicities, respectively. From equation (11.4) we can see that the θ_k's are exactly the zeros with negative real parts of equation (4.2). By the principle of partial fractions we may write

$$\frac{1}{1 - \hat{g}(\xi)} = P_1\left(\frac{1}{\xi - \theta_1}\right) + \cdots + P_k\left(\frac{1}{\xi - \theta_k}\right) + 1,$$

where

$$P_j(\alpha) = c_{j,n_j}\alpha^{n_j} + c_{j,n_j-1}\alpha^{n_j-1} + \cdots + c_{j,1}\alpha, \; j = 1, 2, \ldots, k.$$

Denote by

$$Q_j(u) = c_{j,n_j}\frac{1}{(n_j - 1)!}u^{n_j-1}\mathrm{e}^{\theta_j(\delta)u} + c_{j,n_j-1}\frac{1}{(n_j - 2)!}u^{n_j-2}\mathrm{e}^{\theta_j(\delta)u} + \cdots + c_{j,1}\mathrm{e}^{\theta_j(\delta)u},$$

we have

$$\hat{Q}_j(\xi) = P_j\left(\frac{1}{\xi - \theta_j}\right), \; j = 1, 2, \cdots, k.$$

Hence (1) can be written as

$$\hat{\phi}(\xi) = \sum_{j=1}^{k}\left(\hat{Q}_j(\xi) \cdot \hat{S\omega}(\xi)\right) + \widehat{S\omega}(\xi).$$

Inverting the Laplace transform yields

$$\phi(u) = \sum_{j=1}^{k}\int_0^u Q_j(u - x) \cdot (S\omega)(x)dx + (S\omega)(u). \tag{2}$$

In particular, when $n_1 = \cdots = n_k = 1$, we have

$$\phi(u) = \sum_{j=1}^{k} A_j \int_0^u \mathrm{e}^{\theta_j(u-x)} \cdot (S\omega)(x)dx + (S\omega)(u), \tag{3}$$

where

$$A_j = -\frac{1}{\hat{g}'(\theta_j)}, \; j = 1, 2, \ldots, k.$$

Moreover, the discounted joint probability density function $f(x,y|u)$ defined by (2.1) can be obtained from (2) or (3). For simplicity, we assume that all roots with negative real parts of $\hat{g}(\xi)=1$ are simple, that is, $n_1=\cdots=n_k=1$. It follows from (9.11) that

$$(S\omega)(u)=\int_0^\infty\int_0^\infty w(x,y)1(x\geq u)f(x-u,y+u|0)dxdy,$$

from which we obtain

$$\int_0^u \mathrm{e}^{\theta_j(u-z)}(S\omega)(z)dz=\int_0^\infty\int_0^\infty w(x,y)\left(\int_0^{u\wedge x}\mathrm{e}^{\theta_j(u-z)}f(x-z,y+z|0)dz\right)dxdy.$$

Consequently,

$$\begin{aligned}\phi(u)&=\int_0^\infty\int_0^\infty w(x,y)\sum_{j=1}^k A_j\left(\int_0^{u\wedge x}\mathrm{e}^{\theta_j(u-z)}f(x-z,y+z|0)dz\right)dxdy\\&=\int_0^\infty\int_0^\infty w(x,y)1(x\geq u)f(x-u,y+u|0)dxdy.\end{aligned}\tag{4}$$

Recall that $\phi(u)=\int_0^\infty\int_0^\infty w(x,y)f(x,y|u)dxdy$; by comparing this with (4) we find that

$$\begin{aligned}f(x,y|u)=&\sum_{j=1}^k A_j\left(\int_0^{u\wedge x}\mathrm{e}^{\theta_j(u-z)}f(x-z,y+z|0)dz\right)\\&+1(x\geq u)f(x-u,y+u|0)\\=&\frac{\prod_{l=1}^n\lambda_l}{c^n}p(x+y)\sum_{j=1}^k A_j\left[\mathrm{e}^{\theta_j u}\sum_{i=1}^n \mathrm{e}^{-\rho_i x}\frac{(1-\mathrm{e}^{-(u\wedge x)(\theta_j-\rho_i)})}{\theta_j-\rho_i}\prod_{k=1,k\neq i}^n\frac{1}{\rho_k-\rho_i}\right]\\&+1(x\geq u)f(x-u,y+u|0),\end{aligned}$$

with (see eq. 8.3)

$$f(x,y|0)=\frac{\prod_{l=1}^n\lambda_l}{c^n}p(x+y)\sum_{j=1}^n\left[\mathrm{e}^{-\rho_j x}\prod_{k=1,k\neq j}^n\frac{1}{\rho_k-\rho_j}\right].$$

In particular, we get the marginal discounted probability density function

$$\begin{aligned}f_{U(T-)}(x|u)=&\frac{\prod_{l=1}^n\lambda_l}{c^n}\overline{P}(x)\sum_{j=1}^k A_j\left[\mathrm{e}^{\theta_j u}\sum_{i=1}^n \mathrm{e}^{-\rho_i x}\frac{(1-\mathrm{e}^{-(u\wedge x)(\theta_j-\rho_i)})}{\theta_j-\rho_i}\prod_{k=1,k\neq i}^n\frac{1}{\rho_k-\rho_i}\right]\\&+1(x\geq u)\frac{\prod_{l=1}^n\lambda_l}{c^n}\overline{P}(x)\sum_{j=1}^n\left[\mathrm{e}^{-\rho_j(x-u)}\prod_{k=1,k\neq j}^n\frac{1}{\rho_k-\rho_j}\right].\end{aligned}$$

REFERENCE

Gerber, Hans U., and Elias S. W. Shiu. 1998. On the Time Value of Ruin. *North American Actuarial Journal* 2(1): 48-78.