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Goodness-of-fit test for complete spatial randomness against mixtures of regular and clustered spatial point processes

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Summary

A goodness-of-fit test statistic for spatial point processes is proposed and shown to have an asymptotic chi-squared distribution if the underlying point process is Poisson. Simulations demonstrate that the test, when testing for complete spatial randomness, is more sensitive to mixtures of regular and clustered point processes than the tests using the nearest neighbour distance distribution, the second- or third-order characteristics.

Some key words: Clustered point pattern; Forest stand; Goodness-of-fit; Poisson process; Regular point pattern.

1. Introduction

Point patterns are often assessed against the hypothesis of complete spatial randomness, under which the patterns are realisations of a Poisson or binomial point process. The hypothesis is usually tested through some summary characteristics of the observed pattern, such as nearest-neighbour and inter-point distances and quadrat counts (Ripley, 1981, Ch. 8; Diggle, 1983, Ch. 5; Cressie, 1993, Ch. 8; Stoyan et al., 1995, Ch. 2).

Several comparative studies have investigated the power of different statistics for testing complete spatial randomness (Diggle, 1979; Ripley, 1979; Thönnes & van Lieshout, 1999). Although the set of alternatives was limited, some general conclusions could be drawn: nearest-neighbour methods seem to be especially useful in cases of small sample sizes or departures from complete spatial randomness on small scales; methods based on inter-point distances are effective in discriminating complete spatial randomness from both regular and clustered alternatives; and methods using quadrat counts may be preferable in cases of clustered patterns with large clusters, forming a particular type of heterogeneity (Ripley, 1981, p. 169). Therefore, no test is uniformly 'best'.

Although either regular or clustered point processes are used as alternative models to complete spatial randomness, real systems have often a hierarchical structure and their spatial patterns may exhibit different characteristics at different scales. For instance, ecological systems can have a local regularity within clumps of points, or specific patterns of small clumps. In principle, existing methods can detect departures from complete spatial randomness over a range of scales.

A more difficult situation arises when a pattern is a mixture of regular and clustered

subpatterns at the same scale. For example, in mixed-age forests (Grabarnik & Särkkä, 1999) locations of old trees form a regular pattern as a result of a self-thinning mortality process, whereas young trees grow in clusters associated with canopy gaps created by old trees that have fallen down.

Another, somewhat artificial, example is the cell point process constructed by Baddeley & Silverman (1984) to show that the second-order characteristics of the process and a Poisson process coincide although their realisations are clearly different, so that second-order methods may fail to distinguish between the models. Schladitz & Baddeley (2000) observed however that a third-order characteristic is able to detect the clustered property of the cell process and hence can be used in testing complete spatial randomness. Nevertheless, the statistic does not indicate the regularity of the process, the presence of which is obvious by definition and a visual inspection of realisations.

The aim of this paper is to propose a test statistic that is sensitive to both regular and clustered properties of a point pattern. The new statistic combines information provided by a set of nth nearest-neighbour distance functions. It is designed as a quadratic form and has an asymptotic chi-squared distribution. Thus, critical regions can be approximated without simulations, whereas most of the competing statistics can be applied correctly only by means of a Monte Carlo test. Moreover, the exact significance test is readily constructed for detecting departures from complete spatial randomness on several scales simultaneously.

2. Test statistic

Let Φ and N, respectively, denote the random set and the random counting measure representing a stationary and isotropic point process in $\mathbb{R}^d$ with intensity λ; that is, $N(B)$ is the number of points of Φ in $B \subset \mathbb{R}^d$. Denote volume by $|\cdot|$, Euclidean distance by $\|\cdot\|$, and the ball of radius r centred at x in $\mathbb{R}^d$ by $b(x,r)$. Points x and y are r-close neighbours if $\|x-y\| \le r$.

Define a random measure $M_k(\cdot, r)$ by letting $M_k(B,r)$ be the number of points in $\Phi \cap B$ such that their numbers of r-close neighbours are exactly k, where $k = 0,\ 1,\ 2,\ \ldots$, for $B \subset \mathbb{R}^d$ and $r \ge 0$, i.e.

$$M_k(B,r) = \sum_{x \in \Phi \cap B} 1[N\{b(x,r) \setminus \{x\}\} = k], \qquad k = 0, 1, \ldots,$$

where $1(\cdot)$ is the indicator function.

The refined Campbell theorem and the stationarity lead to

$$EM_k(B,r) = \lambda_k(r)|B| = \lambda |B| P_0^!\{\varphi : |\varphi \cap b(0,r)| = k\},$$

where $P_0^!$ is the reduced Palm distribution of the process Φ, and $\lambda_k(r)$ is the intensity of points having k r-close neighbours. The functions $EM_k(\cdot,\cdot)$ for all k provide a detailed description of the underlying process. Furthermore, the functions λ_k can be used to obtain the nearest-neighbour distribution function $G(r) = 1 - \lambda^{-1}\lambda_0(r)$, the reduced second moment measure $K(r) = \lambda^{-2}\sum_{k=1}^{\infty} k\,\lambda_k(r)$ and the reduced third moment measure (Hanish, 1983) $K^{(3)}(r) = \lambda^{-3}\sum_{k=2}^{\infty} k(k-1)\,\lambda_k(r)$ for $r \ge 0$.

We propose to use a statistic which is based on deviations of M_k from their expected

values under the null hypothesis. A natural candidate is the quadratic form

$$Q^2(r) = (m-\mu)'\Sigma^{-1}(m-\mu),$$

where $m = (m_1, m_2, ..., m_q)'$ consists of $M_k(B,r)$ or their sums, i.e. $m_i = \sum_{k\in I_i} M_k(B,r)$ where B is the observation window and I_i is a nonempty index set containing nonnegative integers and such that $I_{i_1} \cap I_{i_2} = \emptyset$ whenever $i_1 \neq i_2$, $\mu = (\mu_1, \mu_2, \ldots, \mu_q)'$, where $\mu_i = Em_i$ and Σ^{-1} is the inverse of the covariance matrix of the vector m. The choice of the index sets I_i can be arbitrary, provided that the corresponding covariance matrix Σ is invertible.

When Φ is a stationary Poisson point process,

$$P_0^!\{\varphi : |\varphi \cap b(0,r)| \in I\} = \sum_{k\in I} (\lambda\omega_d r^d)^k e^{-\lambda\omega_d r^d}/k!, \tag{1}$$

where $\omega_d = \sqrt(\pi^d)/\Gamma(1+d/2)$ is the volume of the ball of unit radius. Thus, for $m_i = \sum_{k\in I_i} M_k(B,r)$, $i = 1, \ldots, q$,

$$Em_i = \lambda|B| \sum_{k\in I_i} \frac{(\lambda\omega_d r^d)^k e^{-\lambda\omega_d r^d}}{k!}. \tag{2}$$

To find $\mathrm{var}(m_i)$ and $\mathrm{cov}(m_{i_1}, m_{i_2})$ we construct a marked point process Ψ associated with the original Poisson point process. Assign to each point x a mark $s(x)$ which is the number of its r-close neighbours. The first and second factorial moment measures of Ψ are

$$\begin{aligned} \alpha_s^{(1)}(B\times I) &= \lambda \int_B g_1(x)dx, \\ \alpha_s^{(2)}(B_1\times B_2\times I_1\times I_2) &= \lambda^2 \int_{B_1}\int_{B_2} g_{1,2}(x,y)\,dxdy, \end{aligned}$$

where $g_1(x)$ and $g_{1,2}(x,y)$ are the so-called one- and two-point mark distributions (Stoyan et al., 1995, p.108, p.114). Heuristically speaking, the n-point mark distribution $g_{1,\ldots,n}(x_1, \ldots, x_n)$

is the conditional probability that the points x_i have the marks in I_i, provided that $x_i \in \Phi$, $i = 1, \ldots, n$. The variances and covariances can be obtained as second moments of Ψ:

$$\begin{aligned} \mathrm{var}(m_i) &= \lambda \int_B g_i(x)dx + \lambda^2 \int_B \int_B \{g_{i,i}(x,y) - g_i(x)g_i(y)\}dxdy, \quad (3) \\ \mathrm{cov}(m_{i_1}, m_{i_2}) &= \lambda^2 \int_B \int_B \{g_{i_1,i_2}(x,y) - g_{i_1}(x)g_{i_2}(y)\}dxdy. \quad (4) \end{aligned}$$

The n-point mark distribution can be found by using the n-point refined Campbell theorem (Stoyan, 1984). The one-point mark distribution $g_i(x)$ is simply given by the reduced Palm distribution, whereas the two-point mark distribution $g_{i_1,i_2}(x,y)$ is the reduced two-point Palm distribution $P^!_{x,y}\{\varphi : |\varphi \cap b(x,r)| \in I_{i_1} \text{ and } |\varphi \cap b(y,r)| \in I_{i_2}\}$. Formulae for computing these functions can be found in the Appendix.

It is possible to generalise this construction to build a test statistic capable of detecting departures from complete spatial randomness at several scales. Assume that the interval $(0, r_{\max}]$ is partitioned into $(0, r_1]$, $(r_1, r_2]$, ..., $(r_{p-1}, r_p]$, where $0 < r_{\min} = r_1 < \cdots < r_p = r_{\max}$. Compute the statistics $m_i(r_j) = \sum_{k \in I_i(r_j)} M_k(B, r_j)$, where the index sets $I_i(r_j)$, $i = 1, \ldots, q_{r_j}$ may depend on r_j. Finally, calculate the multiscale test statistic $Q^2(r_1, \ldots, r_p) = (m - \mu)'\Sigma^{-1}(m - \mu)$, where the vector m is the concatenation of vectors $m(r_1), \ldots, m(r_p)$ and μ is the vector of corresponding means. The entries of the covariance matrix Σ can still be calculated as second moments of a marked point process associated with the original Poisson process; the marks are vectors the jth components of which are the numbers of r_j-close neighbours, $j = 1, \ldots, p$. However, unlike in the previous case, the covariances have an additional term $\alpha_s^{(1,2)}\{B \times I_{i_1}(r_{j_1}) \times I_{i_2}(r_{j_2})\}$ as a result of the dependence between the

components of mark vectors, and thus

$$\begin{aligned}\mathrm{cov}\{m_{i_1}(r_{j_1}), m_{i_2}(r_{j_2})\} &= \lambda \int_B g_{i_1,i_2}^{(j_1,j_2)}(x,x)dx \\ &\quad +\lambda^2 \int_B \int_B \{g_{i_1,i_2}^{(j_1,j_2)}(x,y) - g_{i_1}^{(j_1)}(x) g_{i_2}^{(j_2)}(y)\} dxdy, \qquad (5)\end{aligned}$$

where $g_i^{(j)}(x)$ and $g_{i_1,i_2}^{(j_1,j_2)}(x,y)$ are the one- and two-point mark distributions similar to those in the single scale case; see the Appendix for their computation.

Remark. The problem of edge-effects intervenes in the calculation of Q^2, because some r-close neighbours may lie outside the observation window B. One simple solution is the border or minus-sampling method (Ripley, 1988, p.25) which is used in §5 for analysing a real dataset.

3. Asymptotic theory

In spatial statistics many asymptotic regimes are possible. We consider the case that a fixed intensity Poisson point process in the whole space $\mathbb{R}^d$ is observed via a sampling window B that expands to cover the whole space and assume that the numbers of r-close neighbours of points in B are known for all $r \leq r_{\max}$.

Let $Y = (m - \mu)/\sqrt{|B|}$, where m and μ correspond to the multiscale Q^2 statistic. As a result of stationarity, $\Sigma/|B|$ does not depend on B; we denote this ratio by Σ^*. We shall show that $Y \to Z$ in distribution where $Z \sim N(0, \Sigma^*)$, as $B \uparrow \mathbb{R}^d$. This asymptotic normality implies that Q^2 asymptotically equals $Z'\Sigma^{*-1}Z$ and hence follows a χ^2-distribution with degrees of freedom that equals the rank of Σ^* and coincides with the dimension of Σ.

Note that $Y \to Z$ in distribution if and only if $\eta'Y \to \eta'Z$ in distribution for all $\eta \in \mathbb{R}^q$, where $q = q_{r_1} + \cdots + q_{r_p}$. Suppose that $\eta = (\eta_{11}, \ldots, \eta_{1q_{r_1}}, \ldots, \eta_{p1}, \ldots, \eta_{pq_{r_p}})'$. We use a fine

mesh to replace the observation window. Without loss of generality we rescale the sizes of the window such that the mesh sizes are 1, and we assume that $B = \{z + a : z \in \Gamma, a \in (0,1]^d\}$, where Γ is a subset of the collection of integers $\mathbb{Z}^d$, so that $|B|$ is equal to the number of integers in Γ. Consider the random field $\{\xi_z : z \in \Gamma\}$, where

$$\xi_z = \sum_{j=1}^{p} \sum_{i=1}^{q_{r_j}} \sum_{x \in \Phi \cap z + (0,1]^d} \eta_{ji} 1[N\{b(x, r_j)\} - 1 \in I_i(r_j)],$$

and $z + (0,1]^d = \{z + a : a \in (0,1]^d\}$. The sum $\sum_{z\in\Gamma} \xi_z = \eta' m$, so that $\sum_{z\in\Gamma} E\xi_z = \eta'\mu$. Therefore, $\eta' Y$ is equal to $\{\sum_z (\xi_z - E\xi_z)\}/\sqrt{|B|}$. Two variables ξ_{z_1} and ξ_{z_2} are independent if $\|z_1 - z_2\| \geq 2r_p + \sqrt[d]{2}$. Thus, $\{\xi_z : z \in \Gamma\}$ is a $(2r_p + \sqrt[d]{2})$-dependent random field. As a result, $\{\sum_z (\xi_z - E\xi_z)\}/\sqrt{|B|}$ converges in distribution to a normal distribution if $\mathrm{var}(\sum_z \xi_z)/|B| = \eta'\Sigma\eta/|B| = \eta'\Sigma^*\eta$ converges to a positive constant as $B \uparrow \mathbb{R}^d$. Since Σ is a nonsingular covariance matrix, it is positive definite, as is Σ^*, and the result follows.

The rate of convergence of the above asymptotic normality is $\sup_x |F_B(x) - F(x)| = O(\log^d |B|/\sqrt{|B|})$ (Takahata, 1983), where F_B and F are, respectively, the distribution functions of $\eta' Y(\eta'\Sigma\eta/|B|)^{-1/2}$ and the standard normal distribution. Hence, the rate of convergence of Q^2 to the χ^2-distribution is $O(2^q \log^d |B|/\sqrt{|B|})$.

4. Simulations and comparisons

4·1. *Preamble*

We shall provide evidence that the statistic Q^2 is sensitive to alternatives that exhibit regular and clustered properties on a single scale simultaneously. To make the results practically relevant we considered models, null and alternative, with a fixed number of points, because typically the true intensity of a point process is unknown and inferences are made condition-

ally on the number of points observed in the sampling window B.

4·2. *Alternative model*

The alternative model used in the simulation is a point process that is a mixture of the Matérn cluster process (Matérn, 1960), generating a wide class of clustered patterns, and the Strauss process (Strauss, 1975), producing a spectrum of regular patterns ranging from random to strongly regular. Some other models capable of simulating regular-clustered patterns can be found in Grabarnik & Särkkä (2001).

To avoid edge effects we simulated the point process on a torus obtained by identifying opposite sides of a square. We believe that the conclusions obtained can also be applied to point processes on an arbitrary domain with or without edge-correction.

The simulation procedure consists of two stages. The first stage is to simulate a version of the Matérn cluster process with a fixed number n_1 of points: we create a cluster centre uniformly distributed within the torus and then place a $\mathrm{Po}(\rho)$ number of points, which form the offspring process, one by one uniformly and independently within the disc of radius R centred at the cluster centre; this procedure is repeated until a prescribed number n_1 of offspring $\phi_1 = \{x_1, x_2, ..., x_{n_1}\}$ are generated. The second stage is to draw a sample $\phi_2 = \{y_1, y_2, ..., y_{n_2}\}$ from a Strauss process with a fixed number n_2 of points, provided that n_1 points have already been placed at positions ϕ_1. The density of the conditional Strauss process with respect to a binomial process is $f(\phi_2|\phi_1) = Z(\beta, \phi_1)^{-1} \exp\{-\beta S(\phi_2, \phi_1)\}$, where $S(\phi_2, \phi_1) = \sum_{y_i \in \phi_2} \{\sum_{y_j \in \phi_2, y_j > y_i} 1(\|y_i - y_j\| \leq R) + \sum_{x_j \in \phi_1} 1(\|y_i - x_j\| \leq R)\}$. The parameter $\beta \geq 0$ controls the strength of interaction: $\beta = 0$ corresponds to the case of no interaction and the interaction becomes stronger as β increases. The radius of interaction $R > 0$ is the

same as for the clustered part of the process. The normalising constant Z is not computable in an explicit form, and so direct simulation methods do not work. Markov Chain Monte Carlo algorithms can be used instead because they are based on the likelihood ratio, in which case the normalising constant disappears. Note that the limiting case, $\beta \to \infty$, of the Strauss process is a hard-core process, and therefore the density $n_2/|B \setminus \cup_{x_i \in \phi_1} b(x_i, R)|$ cannot exceed the packing density for a given radius R. Also, if R or n_2 is large and the interaction is strong, the Strauss process produces specific clustered patterns (Møller, 1999). Thus, care should be taken in choosing n_2, β and R so as to simulate the desired type of mixture pattern.

Figure 1 presents a realisation of such a mixture simulated on a unit square mapped on to a torus with parameters $n_1 = 100$, $\rho = 3$, $R = 0.06$, $n_2 = 100$ and $\beta = 0.7$.

*** Figure 1 about here ***

4·3. *Test statistics*

The formulae for the mean and the covariance matrix of m in the Q^2 statistic for a binomial point process in the observation window B are slightly different from those for a Poisson process; see the Appendix.

As competitors of the Q^2 statistic we considered the empirical version of Ripley's K-function,

$$\widehat{K}(r) = \frac{|B|}{n(n-1)} \sum_{i \neq j} 1(\|x_i - x_j\| \leq r),$$

a statistic related to the variance-area curve estimator (Ripley, 1981, pp. 160-1),

$$\widehat{V}(r) = \sum_{i \neq j} |b(x_i, r) \cap b(x_j, r)|,$$

the empirical reduced third moment function,

$$\widehat{K}^{(3)}(r) = \frac{|B|^2}{n(n-1)(n-2)} \sum_{i \neq j \neq k} 1(\|x_i - x_j\| \leq r) 1(\|x_i - x_k\| \leq r),$$

and the empirical nearest-neighbour distance distribution,

$$\widehat{G}(r) = n^{-1} \sum_i 1(\min_{j \neq i} \{\|x_i - x_j\|\} \leq r).$$

This choice was made on the basis of known comparative studies. In particular, we did not include the popular empty space function or J-function (Van Lieshout & Baddeley, 1996) in the comparison, because tests based on the empty space function are less powerful than tests based on the nearest-neighbour distance distribution against alternatives producing regular patterns, and the J-function performs similarly to the nearest-neighbour distance distribution (Thönnes & van Lieshout, 1999).

Diggle (1979) observed that the square-root transformation of the K-function leads to a much more powerful test than using the original statistic, and hence we also used transformations that stabilise the variances of $\widehat{K}$, $\widehat{V}$ and $\widehat{K}^{(3)}$, namely

$$\sqrt{\{\widehat{K}(r)/\pi\}}, \qquad \sqrt[4]{[|B|\widehat{V}(r)/\{\pi^2 n(n-1)\}]}, \qquad \text{and} \qquad \sqrt[4]{\{\widehat{K}^{(3)}(r)/\pi^2\}}.$$

The nearest-neighbour distance statistic took the untransformed empirical form $\widehat{G}$.

Since in our experiments the alternative model has been fully defined, the interaction radius $r = R$, which defines the scale of patterns, is known a priori. However, it is more natural, from a practical point of view, to consider a range of r around a suspected value rather than the single value.

For the statistics taken for the comparative study we adopted the maximum statistic,

$$U = \max_{j=1,\ldots,p} |\widehat{W}(r_j) - W_0(r_j)|, \tag{6}$$

where W_0 is the theoretical summary function under complete spatial randomness and $\widehat{W}$ is the corresponding empirical function for a given point pattern, because the supremum statistic $\sup_{r_{\min}\le r\le r_{\max}} |\widehat{W}(r) - W_0(r)|$ is not exactly appropriate for the comparison with the Q^2 statistic.

The distribution theory of the empirical summary functions used here has not yet been successfully established; only some asymptotic results are known in a few simple models. A way to overcome this difficulty is to use a Monte Carlo test (Diggle, 1983, pp. 7-9).

4·4. *Results*

Binomial point processes with $n = 100$, 200 and 300 points were simulated on squares with areas 1/2, 1 and 3/2, respectively, mapped on to a torus, so that the intensities of points were the same for all cases. For each case 9999 realisations were recorded. For simplicity the index sets consisted of singletons and did not vary with scales r_j. The Q^2 statistics were computed from $m_1(r_j) = M_0(B, r_j), \ldots, m_q(r_j) = M_{q-1}(B, r_j)$, where $q = 4$, 5 and 6 for $n = 100$, 200 and 300, respectively, and scales were fixed at $r_1 = 0.05, r_2 = 0.06$ and $r_3 = 0.07$ for all cases. The empirical distributions of Q^2 were compared with the chi-squared distribution with the corresponding degrees of freedom, and proved to be adequate.

The values of the scales $r_1, \ldots, r_p$ for the competitive statistics were the same as for the Q^2 test. Note that the supremum statistics give similar powers to the maximum statistics, though the former were typically slightly less powerful.

To compare the powers of the tests, 4×16 pairs of the parameters (ρ, β) were used as the mean cluster size ρ in the Matérn clustered process and the interaction parameter β in the conditional Strauss point process. Other parameters were fixed at $n_1 = n_2 = n/2$ and

$R = 0.06$ for all cases. Ten thousand realisations were generated for each pair of (ρ, β). The proportion of rejections were computed and the results are summarised in Fig. 2.

*** Figure 2 about here ***

These results show the advantage of the new test statistic Q^2 for testing complete spatial randomness against mixtures of two strongly pronounced opposite tendencies, namely clustering and regularity. For $\rho \geq 2$ and $\beta > 0.5$ the test based on Q^2 is more powerful than all other comparative tests. The superiority is especially remarkable for moderate clustering, $\rho = 2$ and $\rho = 3$. In the cases of weak regularity, $0 < \beta < 0.3$, tests based on $\widehat{K}$, $\widehat{V}$ and $\widehat{K}^{(3)}$ are more powerful than those based on Q^2 and $\widehat{G}$. The conclusions remain true if we vary the number of points, though the difference in powers is more substantial for $n = 200$ points.

We also found that the $\widehat{V}$-based test is slightly more powerful than the $\widehat{K}$-based test in detecting a clustering deviation from complete spatial randomness, which agrees with Ripley (1979). In contrast, in the cases of strong regularity, $\beta > 1.0$, tests based on Q^2 and $\widehat{G}$ are more powerful than the others. Note that, for a narrow range of values of β and moderate clustering, tests based on $\widehat{K}^{(3)}$ may be more effective than the others. This confirms the empirical findings of Schladitz & Baddeley (2000) for the cell point process.

An important question is how to choose the index sets used in the construction of the Q^2 statistic so as to improve the sensitivity of the test. We explored several index sets for the case $n = 200$ and $\rho = 3$. One of them was that used in the previous experiment, namely (I) $m_1 = M_0$, $m_2 = M_1$, ..., $m_5 = M_4$. Others were various combinations of M_k statistics: (II) $m_1 = M_0$, $m_2 = M_1 + M_2$, $m_3 = M_3 + M_4 + M_5$; (III) $m_1 = M_0 + M_1$, $m_2 = M_2 + M_3$,

$m_3 = M_4 + M_5$; and (IV) $m_1 = M_0 + M_1 + M_2$, $m_2 = M_3 + M_4 + M_5$. The results, given in Fig. 3, show some, though not a significant, advantage of variant (II) of the Q^2 statistic in comparison with variant (I). On the other hand, the test based on variant (IV) loses power with increasing repulsive interaction. Thus, a reasonable strategy for fine tuning of the test is first to try variant (I) and then to combine successive M_k's whose deviations from the expectation are of the same sign.

*** Figure 3 about here ***

5. Example

Our example concerns the analysis of a spatial pattern of a forest stand. The data came from a study of broad-leaved multispecies old-growth forest in the south-east of Central Russia (Smirnova, 1994). We chose a homogeneous part 75m × 75m from the original 500m × 200m plot, and excluded the trees which are fully overshadowed by neighbouring trees.

Figure 4 depicts the locations of 270 trees. A band of width 5m around the central 65m × 65m part was used as 'guard' area to eliminate edge effects. As mentioned in the introduction, such a mixed-age forest may exhibit both regularity and clustering at the same scale.

As the range of scales we took the interval 3.5m - 4.5m because it is related to the zone of influence of individual trees of this type of forest. First, we applied tests based on $\widehat{K}$, $\widehat{K}^{(3)}$, $\widehat{V}$ and $\widehat{G}$ using the maximum statistics as in §4.2. Three scales, 3.5m, 4m and 4.5m, were used to calculate the maximum statistics, and the Monte Carlo p-values estimated on the basis of 10,000 simulations were 0.61, 0.39, 0.45 and 0.20 for each test, respectively. Thus, these tests

give no clear evidence to reject complete spatial randomness. In contrast, the value of the Q^2 statistic computed from $m_k = M_{k-1}(B, r_j)$, $k = 1$, 2, 3, 4, 5 and 6, for the same scales, $r_1 = 3.5$m, $r_2 = 4.0$m and $r_3 = 4.5$m, was 37.8, and with 18 degrees of freedom it gives a p-value of 0.0041, thereby providing strong evidence against complete spatial randomness.

*** Figure 4 about here ***

6. Discussion

Application of the test based on Q^2 is not straightforward because we need to choose some parameters arbitrarily. We recommend combining small values of M_k such that the expectation of their sum will be greater than 5. This improves the chi-squared approximation and is usually recommended for Pearson's chi-squared test. Another problem is the choice and the partition of the interval of the scales $[r_{\min}, r_{\max}]$, which depends very much on the nature of the data under study. One might be able to gather enough information about the data by applying just a single-scale Q^2 test at different scales. An additional benefit of the multiscale Q^2 test is that it allows exact significance testing at several scales simultaneously.

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Appendix

Formulae for the one- and two-point mark distributions

In the case of a Poisson process the one-point mark distribution $g_i^{(j)}(x)$ is given by the right-hand side of (1), because Slivnyak's theorem (Stoyan et al., 1995, p. 121) states that the reduced Palm distribution of a Poisson process is the same as the distribution of the Poisson process. In addition, using the property of a Poisson process that the numbers of points in disjoint sets are independent, we obtain the formula of the two-point mark distribution:

$$g_{i_1,i_2}^{(j_1,j_2)}(x,y) = \sum_{k\in I_{i_1}} \sum_{l\in I_{i_2}} \sum_t \frac{(\lambda b_1)^{k-t-c} e^{-\lambda b_1}}{(k-t-c)!} \frac{(\lambda b_2)^t e^{-\lambda b_2}}{t!} \frac{(\lambda b_3)^{l-t-d} e^{-\lambda b_3}}{(l-t-d)!},$$

where $b_1 = |b(x, r_{j_1}) \setminus b(y, r_{j_2})|$, $b_2 = |b(x, r_{j_1}) \cap b(y, r_{j_2})|$, $b_3 = |b(y, r_{j_2}) \setminus b(x, r_{j_1})|$, $c = 1[y \in b(x, r_{j_1}) \setminus \{x\}]$, $d = 1[x \in b(y, r_{j_2}) \setminus \{y\}]$, and t runs over all nonnegative integers such that the values in powers are nonnegative. We use the convention that $0^0 = 1$.

In the case of a binomial process the formulae are as follows:

$$\begin{aligned} g_i^{(j)}(x) &= \sum_{k\in I_i} \binom{n-1}{k} p_b^k (1-p_b)^{n-1-k}, \\ g_{i_1,i_2}^{(j_1,j_2)}(x,y) &= \sum_{k\in I_{i_1}} \sum_{l\in I_{i_2}} \sum_t \frac{(n-2)!}{(k-t-c)!t!(l-t-d)!\{n-2-(k-t-c)-t-(l-t-d)\}!} \\ &\quad \times p_{b_1}^{k-t-c} p_{b_2}^t p_{b_3}^{l-t-d} p_{b_4}^{n-2-(k-t-c)-t-(l-t-d)}, \end{aligned}$$

where c and d are as above, $p_b = |b(x, r_j)|/|B|$, $p_{b_i} = b_i/|B|$, for $i = 1, \ldots, 4$, where b_1, b_2 and b_3 are as above and $b_4 = 1 - (b_1 + b_2 + b_3)$, and t runs over all nonnegative integers such that the values in powers are all nonnegative. Calculation of $g_{i_1,i_2}^{(j_1,j_2)}(x,x)$ follows the same formula except that 2 has to be replaced by 1. Also, in formulae (2), (3), (4) and (5) λ and λ^2 should be replaced by $n/|B|$ and $n(n-1)/|B|^2$, respectively.

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Figure Captions

Figure 1: A realisation of the mixture of a Matérn clustered process (with $n_1 = 100$ points, mean cluster size $\rho = 3$ and cluster radius $R = 0.06$) and a conditional Strauss point process (with $n_2 = 100$ points, interaction parameter $\beta = 0.7$ and interaction radius $R = 0.06$).

Figure 2: Estimated powers against a mixture of Matérn cluster process with $n/2$ points and mean cluster size ρ and the conditional Strauss process with $n/2$ points and interaction parameter β. The value of R is 0.06. The percentages of rejections correspond to the 5% significance level. The symbols represent tests based on $\widehat{K}$ ($\square$), $\widehat{K}^{(3)}$ ($\triangle$), $\widehat{V}$ ($\times$), $\widehat{G}$ ($\triangledown$) and Q^2 ($\bullet$).

Figure 3: Estimated powers of the Q^2 statistic with four different choices of index sets by using a mixture of Matérn cluster process with 100 points and mean cluster size $\rho = 3$ and the conditional Strauss process with 100 points and interaction parameter β. The value of R is 0.06. The percentages of rejections correspond to the 5% significance level.

Figure 4: Locations of 270 trees of a broad-leaved multispecies old-growth forest in a 75m $\times$ 75m square.

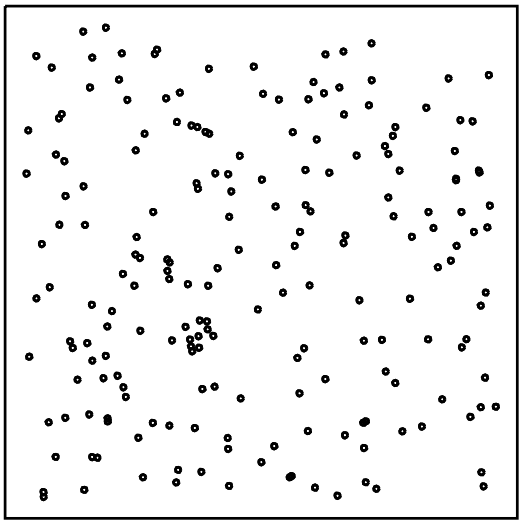
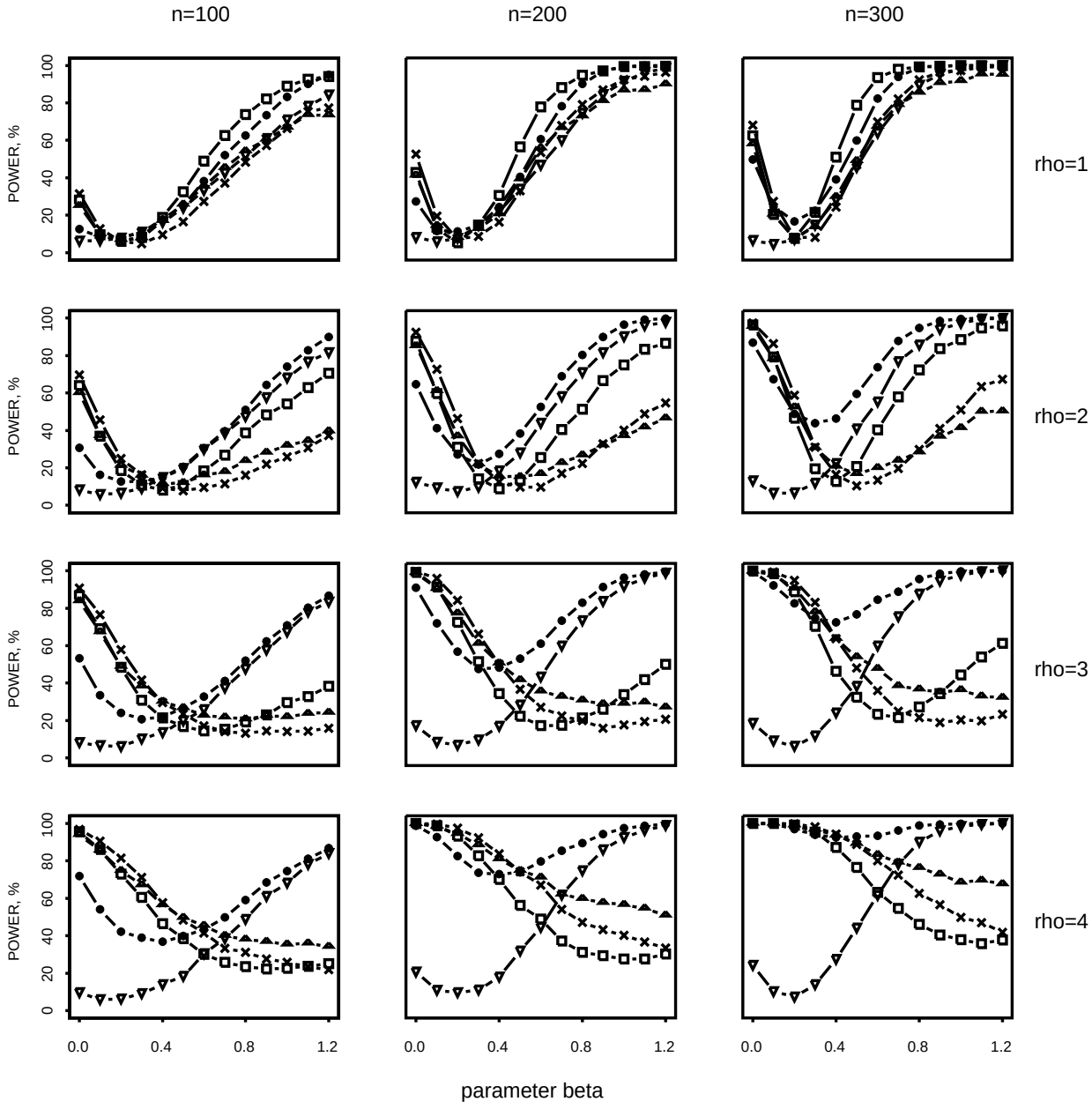


Figure 1: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n=200
n=300
POWER, %
0
20
40
60
80
100
rho=1
rho=2
rho=3
rho=4
0.0
0.4
0.8
1.2
parameter beta

Figure 2:

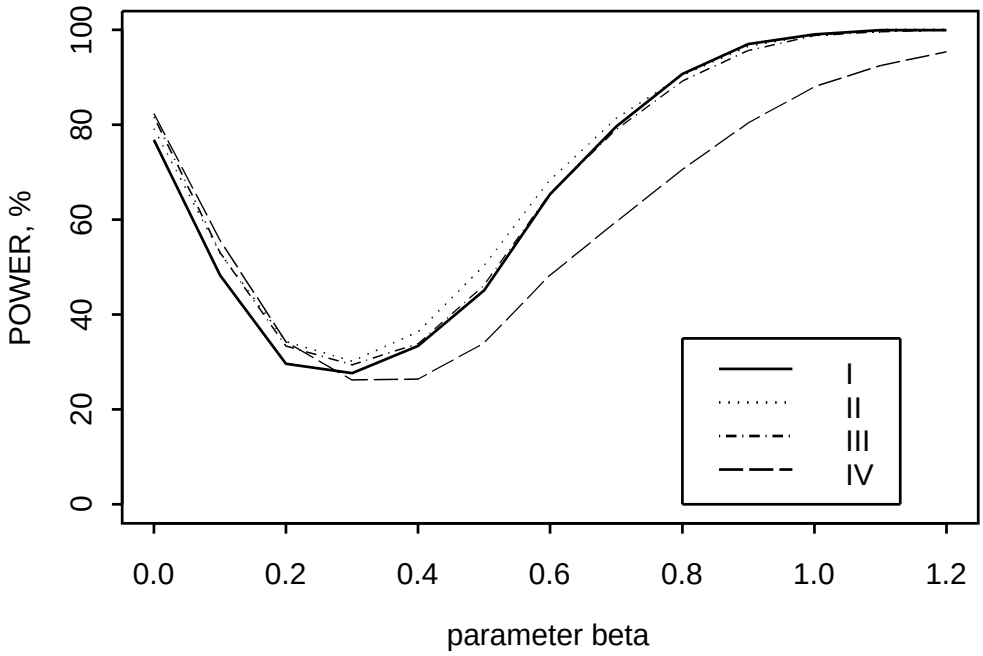
POWER, %
0
20
40
60
80
100
0.0
0.2
0.4
0.6
0.8
1.0
1.2
parameter beta
I
II
III
IV

Figure 3:

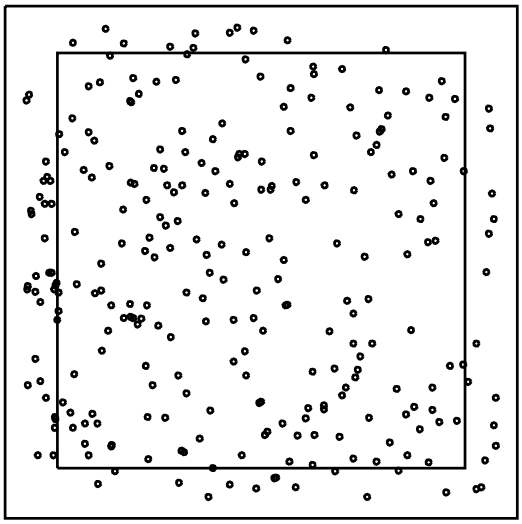

Figure 4: