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Abstract

This paper investigates the first exit time and the ruin time of a risk reserve process with reserve-dependent income under the assumption that the claims arrive as a Poisson process. We show that the Laplace transform of the distribution of the first exit time from an interval satisfies an integro-differential equation. The exact solution for the classical model and for the Embrechts-Schmidli model are derived.

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1. Introduction

Consider the risk reserve process $(X_t)_{t\geq 0}$ described by

$$X_t = x + \int_0^t c(X_s)\mathrm{d}s - S_t, \quad t \geq 0,$$

where x is the non-negative initial capital, $c(\cdot)$ a continuously differentiable Lipschitz function which represents the positive reserve-dependent income rate, and $(S_t)_{t\geq 0}$ the aggregate claim process defined by $S_t = \sum_{i=1}^{N_t} Y_i$ such that $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ and $(Y_i)_{i\geq 1}$ a sequence of independent claim amounts. Assume that $(N_t)_{t\geq 0}$ and Y_i's are independent and Y_i's are identically distributed having an absolutely continuous distribution function P with $P(0) = 0$ and a finite mean. An important particular case is the classical risk process obtained by taking $c(\cdot)$ as a constant.

From Dassios and Embrechts (1989) or Embrechts and Schmidli (1994) we know that $(X_t)_{t\geq 0}$ is a piecewise deterministic Markov process (PDMP) taking values in $\mathbb{R}$ with extended generator $\mathcal{A}$ that satisfies

$$\mathcal{A}f(x) = \chi f(x) + \lambda \int_0^\infty (f(x-y) - f(x))\mathrm{d}P(y),$$

where f belongs to the domain $\mathcal{D}(\mathcal{A})$ of the generator $\mathcal{A}$ of $(X_t)_{t\geq 0}$, $\chi = c(x)\frac{\mathrm{d}}{\mathrm{d}x}$ is the vector field of the integral curves of the PDMP.

Assume that all processes and random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F})_{t\geq 0})$, where $\mathcal{F}_t$ is right-continuous. Denoted by $\mathbf{P}_x$ and $\mathbf{E}_x$ the conditional distribution of X_t and its expectation operator, given that $X_0 = x$. For two constants $b < a$, define the first exit time from (b, a) by $\tau_{a,b} = \inf\{t > 0 : X_t \leq b \text{ or } X_t \geq a\}$ and so $\tau_{\infty,0}$ is the time of ruin. Note that $\tau_{a,b}$ is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$. In order to avoid $\mathbf{P}_x(\tau_{\infty,0} < \infty) = 1$, we assume the net profit condition $\mathbf{E}_0(X_t) > 0$ for $t \geq 0$.

We conclude this section by showing that the Laplace transform of $\tau_{a,b}$ satisfies the integro-differential equation given in (1.1).

Lemma 1.1. *For $\alpha > 0$, consider the Laplace transform $V(x) := \mathbf{E}_x \mathrm{e}^{-\alpha\tau_{a,b}}, b < x < a$. Then V is the unique solution in $\mathcal{D}(\mathcal{A})$ of*

$$\chi V(x) - \lambda \int_b^x V(t)\mathrm{d}P(x-t) + \lambda(1 - P(x-b)) = (\lambda+\alpha)V(x),\ x \in (b,a), \qquad (1.1)$$

with the boundary condition $V(a) = 1$.

Proof. The Markov property implies that for $s < \tau_{a,b}$

$$\mathbf{E}_x \left(V(X_{\tau_{a,b}})\mathrm{e}^{-\alpha\tau_{a,b}} | \mathcal{F}_s \right) = V(X_s)\mathrm{e}^{-\alpha s},$$

where $\mathcal{F}_s$ is the σ-algebra generated by $\{X_t, t \le s\}$. The left-hand side is a local martingale on $[0, \tau_{a,b})$, and so is $\mathrm{e}^{-\alpha s}V(X_s)$. Note that

$$V(x) = 1 - \alpha \mathbf{E}_x \int_0^\infty \mathrm{e}^{-\alpha t}\ \mathbf{1}(\tau_{a,b} > t)\mathrm{d}t.$$

By using the same method as in the proof of the first part of Theorem 32.2 in Davis (1993), we have $V \in \mathcal{D}(\mathcal{A})$. Applying the PDMP Differential Formula (Davis, 1993, Theorem 31.3) for $s \in [0, \tau_{a,b})$ leads to

$$V(X_s)\mathrm{e}^{-\alpha s} - V(x) = \int_0^s \mathrm{e}^{-\alpha t}(\mathcal{A} - \alpha)V(X_t)\mathrm{d}t + \text{a local martingale}. \qquad (1.2)$$

It follows that the first integral on the right-hand side of (1.2) is a local martingale. Hence, the smoothness of V implies $\mathcal{A}V(x) - \alpha V(x) = 0$ in (b,a). Since $V(x) = 1$ for $x < b$, equation (1.1) follows. The boundary condition $V(a) = 1$ is obvious.

To show the uniqueness, suppose V is a function that satisfies (1.1), and so $\mathcal{A}V(x) = \alpha V(x)$. We now prove that $V(x) = \mathbf{E}_x \mathrm{e}^{-\alpha\tau_{a,b}}$. If $V \in \mathcal{D}(\mathcal{A})$, then $V\mathrm{e}^{-\alpha s} \in \mathcal{D}(\mathcal{A})$. Applying the PDMP Differential Formula for $s \in [0, \tau_{a,b})$ leads to (1.2) again. Now in view of (1.1), we observe that $M_s := V(X_s)\mathrm{e}^{-\alpha s}$ is a local martingale. Since V is bounded, M_s is a bounded martingale. Note that $\tau_{a,b}$ is a stopping time and $\mathbf{E}_x\tau_{a,b} < \infty$. Thus, by the Optional-Stopping Theorem and the dominated convergence theorem, the result follows.

Remark 1.1. Equation (1.1) remains true when $b \to -\infty$ or $a \to \infty$, but the boundary condition becomes $\lim_{a\to\infty} V(a) = 0$.

2. Application to the classical model

We apply the results in the last section to the classical model, i.e. $c(x) = c$, a constant. The Laplace transform of a function will be denoted by putting a hat on the function. For example, $\hat{g}(\beta) = \int_0^\infty \mathrm{e}^{-\beta x} g(x)\mathrm{d}x$.

Assumption 2.1. We assume that there exists a $\beta_\infty \in [-\infty, 0)$ such that $\hat{p}(\beta) \to \infty$ when $\beta \downarrow \beta_\infty$.

The following theorem is a classical result, see Asmussen (2000, p. 109, Corollary 3.5) for a proof using renewal theory.

Theorem 2.1. *Let $h(x) = \mathbf{E}_x e^{-\alpha\tau_{\infty,0}}$ for $x > 0$. Its Laplace transform is given by*

$$\hat{h}(\beta) = \begin{cases} \dfrac{\frac{\lambda}{\beta}\hat{p}(\beta) - \frac{\lambda}{\beta} + ch(0)}{\lambda\hat{p}(\beta) + \beta c - \lambda - \alpha}, & \hat{p}(\beta) < \infty, \\ \dfrac{1}{\beta}, & \hat{p}(\beta) = \infty. \end{cases} \tag{2.1}$$

Proof. By Lemma 1.1, the function $h(x)$ is the unique solution in $\mathcal{D}(\mathcal{A})$ of the equation

$$ch'(x) - \lambda \int_0^x h(t)\mathrm{d}P(x-t) + \lambda(1 - P(x)) = (\lambda + \alpha)h(x), \ x \in (0, \infty), \tag{2.2}$$

with the boundary condition $h(\infty) = 0$. If $\hat{p}(\beta) < \infty$, equation (2.1) follows from taking the Laplace transforms of the both sides of (2.2). If $\hat{p}(\beta) = \infty$, $\hat{h}(\beta)$ takes the limiting value as $\hat{p}(\beta) \to \infty$.

Before discussing the inversion formula of (2.1), we need the following lemma.

Lemma 2.1. *For positive λ and c and nonnegative α, consider the function $k_\alpha(\beta) = \lambda\hat{p}(\beta) + \beta c - \lambda - \alpha$ on the complex plane, where $\hat{p}$ is the Laplace transform of p.*

(1) If $\alpha > 0$*, then* $k_\alpha(\cdot)$ *has a unique positive zero and a negative zero; the remaining zeros are either negative or having negative real parts.*

(2) If $\alpha = 0$*, then* $k_\alpha(\cdot)$ *has at least two real zeros, one is a negative and the other is* $\beta = 0$*; the remaining zeros are either negative or having negative real parts.*

Proof. If $\beta \in \mathbb{R}$, $k_\alpha(\beta)$ is a continuous convex function that tends to infinity as $\beta \downarrow \beta_\infty$ or $\beta \uparrow \infty$ and takes the value $-\alpha$ at $\beta = 0$. Hence, if $\alpha > 0$, $k_\alpha(\cdot)$ has a unique positive zero and a negative zero. Moreover, since the positive zero is an increasing function of α, it becomes 0 when $\alpha = 0$. By the Argument Principle (e.g. Ablowits and Fokas, 1997, p. 259), it can be proved that the remaining zeros are all located on the negative half-plane.

For simplicity, we assume that all zeros $\beta_1(\alpha), \beta_2(\alpha), \dots$ of $k_\alpha(\cdot)$ are simple and that $\mathrm{Re}(\beta_j(\alpha)) < 0$ for $j \geq 2$ and $\beta_1(\alpha) > 0$ for $\alpha > 0$ and $\beta_1(0) = 0$.

Theorem 2.2. *If the function* $\hat{h}(\cdot)$ *in (2.1) is analytic on the complex plane except for* $\beta_1(\alpha), \beta_2(\alpha), \dots$*, then for* $\alpha > 0$*,*

$$\mathbf{E}_x e^{-\alpha\tau_{\infty,0}} = \begin{cases} \displaystyle\sum_{j\geq 2} \frac{\frac{\lambda}{\beta_j(\alpha)}\left(\hat{p}(\beta_j(\alpha)) - 1\right) - \frac{\lambda}{\beta_1(\alpha)}\left(\hat{p}(\beta_1(\alpha)) - 1\right)}{\lambda\hat{p}'(\beta_j(\alpha)) + c} e^{\beta_j(\alpha)x}, & x > 0, \\ 1 - \dfrac{\alpha}{c\beta_1(\alpha)}, & x = 0, \end{cases} \tag{2.3}$$

and

$$\mathbf{P}_x(\tau_{\infty,0} < \infty) = \sum_{j\geq 2} \frac{\frac{\lambda}{\beta_j(0)}\left(\hat{p}(\beta_j(0)) - 1\right) + \lambda EY_1}{\lambda\hat{p}'(\beta_j(0)) + c} e^{\beta_j(0)x}, \quad x > 0. \tag{2.4}$$

Proof. The inversion formula is

$$h(x) = \frac{1}{2\pi i} \lim_{w\to\infty} \int_{\mu-iw}^{\mu+iw} \hat{h}(\beta)\mathrm{e}^{\beta x}\mathrm{d}\beta,$$

where $\mu\,(> \beta_1(\alpha))$ is a constant. Since $\beta\hat{h}(\beta)\mathrm{e}^{\beta x} \to 0$ as $|\beta| \to \infty$, it follows from Cauchy's Residue Theorem and Jordan's Lemma that

$$h(x) = \sum_{j\geq 1} \mathrm{Res}(\hat{h}(\beta_j(\alpha))\mathrm{e}^{\beta_j(\alpha)x}) = \sum_{j\geq 1} \frac{\frac{\lambda}{\beta_j(\alpha)}\left(\hat{p}(\beta_j(\alpha)) - 1\right) + ch(0)}{\lambda\hat{p}'(\beta_j(\alpha)) + c} \mathrm{e}^{\beta_j(\alpha)x}.$$

The boundedness of $h(\cdot)$ implies

$$\frac{\lambda}{\beta_1(\alpha)}\left(\hat{p}(\beta_1(\alpha))-1\right)+ch(0)=0,$$

as $\beta_1(\alpha)>0$. Equation (2.3) follows. By letting $\alpha\to 0$, we get (2.4).

Remark 2.1. In particular, when $x=0$, (2.3) coincides with Asmussen (2000, p. 109, Corollary 3.4). From (2.4) we can get the well-known Cramér-Lundberg approximation (Rolski *et al.*, 1999, p. 172).

3. Application to the Embrechts-Schmidli model

In this section, instead of the classical model, we consider that a company earns interest at an interest rate $\rho_1>0$ when the reserve is positive, and borrows money at an interest rate $\rho_2>0$ when the reserve is negative. This model was considered by Embrechts and Schmidli (1994). The vector field of the integral curves in the model is

$$\chi := \begin{cases} (c+\rho_1 x)\dfrac{\partial}{\partial x}, & x\ge 0,\\ (c+\rho_2 x)\dfrac{\partial}{\partial x}, & x<0,\end{cases}$$

where c is the constant premium income rate. For the sake of simplicity we restrict our attention to exponentially distributed claim sizes, i.e. $P(x)=1-\mathrm{e}^{-rx}$ for $x>0$. Denote by $M(a;c;x)$ the confluent hypergeometric function, and $U(a;c;x)$ the confluent hypergeometric function of the second kind; see e.g. Magnus *et al*. (1966). We will also use the following notation to simplify the expressions:

$$\omega(\rho)=1-\frac{\lambda}{\rho},\qquad \delta(\rho)=1-\frac{\lambda}{\rho}-\frac{\alpha}{\rho},\qquad \mu_x(\rho)=rx+\frac{cr}{\rho},$$

$$M^*(\rho,x)=M(\omega(\rho);\delta(\rho);\mu_x(\rho)),\qquad M_1^*(\rho,x)=M(\omega(\rho);1+\delta(\rho);\mu_x(\rho)),$$

$$U^*(\rho,x)=U(\omega(\rho);\delta(\rho);\mu_x(\rho)),\qquad U_1^*(\rho,x)=U(\omega(\rho);1+\delta(\rho);\mu_x(\rho)).$$

Theorem 3.1. *Let $\alpha > 0$ and $-c/\rho_2 \le b < 0 < a < \infty$. Define*

$$V(x) = \mathbf{E}_x e^{-\alpha\tau_{a,b}},\ b < x < a.$$

(1) If $\lambda = \rho_1 = \rho_2 \equiv \rho$, then

$$V(x) = \begin{cases} A_1 e^{-rx} \displaystyle\int_{\mu_0(\rho)}^{\mu_x(\rho)} e^y y^{\alpha/\rho} dy + A_2 e^{-rx}, & 0 < x < a, \\ A_3 e^{-rx} \displaystyle\int_{\mu_x(\rho)}^{\mu_0(\rho)} e^y y^{\alpha/\rho} dy + A_4 e^{-rx}, & b < x \le 0, \end{cases}$$

where

$$A_1 = -A_3 = \frac{(\rho + \alpha + cr + br)e^{ra} - \rho r e^{rb}}{(\rho + \alpha + cr + br)\int_{\mu_0(\rho)}^{\mu_a(\rho)} e^y y^{\alpha/\rho} dy},$$

$$A_2 = A_4 = \frac{\rho r e^{br}}{\rho + \alpha + cr + br}.$$

(2) If $\lambda = \rho_1 \neq \rho_2$, and $\delta(\rho_2) \neq -n$, for $n = 0, 1, 2, \dots$, then

$$V(x) = \begin{cases} e^{-rx}\left(A_5 \displaystyle\int_{\mu_0(\rho_1)}^{\mu_x(\rho_1)} e^y y^{\alpha/\rho_1} dy + A_6\right), & 0 < x < a, \quad (3.1) \\ e^{-rx}(A_7 M^*(\rho_2, x) + A_8 U^*(\rho_2, x)), & b < x \le 0, \quad (3.2) \end{cases}$$

where

$$A_5 = \frac{(\rho_1 + \alpha + cr + br)e^{ra} - \rho_1 r e^{rb}}{(\rho_1 + \alpha + cr + br)\int_{\mu_0(\rho_1)}^{\mu_a(\rho_1)} e^y y^{\alpha/\rho_1} dy},$$

$$A_6 = \frac{\rho_1 r e^{br}}{\rho_1 + \alpha + cr + br},$$

$$A_7 = \frac{r\omega(\rho_2)U(\omega(\rho_2) + 1; \delta(\rho_2) + 1; \mu_0(\rho_2))A_6 + re^{\mu_0(\rho_1)}\mu_0(\rho_1)^{\alpha/\rho_1}U^*(\rho_2, 0)A_5}{rM^*(\rho_2, 0)U_1^*(\rho_2, 0) + \frac{\alpha r}{\rho_2\delta(\rho_2)}M_1^*(\rho_2, 0)U^*(\rho_2, 0)},$$

$$A_8 = \frac{A_6 - A_7 M^*(\rho_2, 0)}{U^*(\rho_2, 0)}.$$

(3) If $\lambda = \rho_2 \neq \rho_1$, and $\delta(\rho_1) \neq -n$, for $n = 0, 1, 2, \dots$, then

$$V(x) = \begin{cases} e^{-rx}\left(A_9 M^*(\rho_1, x) + A_{10} U^*(\rho_1, x)\right), & 0 < x < a, \\ e^{-rx}\left(A_{11} \displaystyle\int_{\mu_x(\rho_2)}^{\mu_0(\rho_2)} e^y y^{\alpha/\rho_2} dy + A_{12}\right), & b < x \le 0, \end{cases}$$

where

$$A_{12} = \frac{\rho_2 r e^{br}}{\rho_2 + \alpha + cr + br},$$
$$A_9 = \frac{e^{ra}U^*(\rho_1, 0) - A_{12}U^*(\rho_1, a)}{M^*(\rho_1, a)U^*(\rho_1, 0) - M^*(\rho_1, 0)U^*(\rho_1, a)},$$
$$A_{10} = \frac{e^{ra} - A_9 M^*(\rho_1, a)}{U^*(\rho_1, a)},$$
$$A_{11} = \frac{rA_{10}U_1^*(\rho_1, 0) - A_9 \frac{r\alpha}{\rho_1\delta(\rho_1)} M_1^*(\rho_1, 0) - rA_{12}}{re^{\mu_0(\rho_2)}\mu_0(\rho_2)^{\alpha/\rho_2}}.$$

(4) If $\lambda \neq \rho_1 \neq \rho_2$, and both $\delta(\rho_1), \delta(\rho_2) \neq -n$, for $n = 0$, 1, 2, ..., then

$$V(x) = \begin{cases} e^{-rx}(A_{13}M^*(\rho_1, x) + A_{14}U^*(\rho_1, x)), & 0 < x < a, \\ e^{-rx}(A_{15}M^*(\rho_2, x) + A_{16}U^*(\rho_2, x)), & b < x \leq 0, \end{cases}$$

where A_i's are constants that are the solutions of the following system of linear equations

$$\begin{cases} A_{13}\frac{\alpha r}{\rho_1\delta(\rho_1)}M_1^*(\rho_1, 0) - A_{14}rU_1^*(\rho_1, 0) = A_{15}\frac{\alpha r}{\rho_2\delta(\rho_2)}M_1^*(\rho_2, 0) - A_{16}rU_1^*(\rho_2, 0), \\ A_{13}M^*(\rho_1, 0) + A_{14}U^*(\rho_1, 0) = A_{15}M^*(\rho_2, 0) + A_{16}U^*(\rho_2, 0), \\ A_{13}M^*(\rho_1, a) + A_{14}U^*(\rho_1, a) = e^{ra}, \\ A_{13}G_1 + A_{14}G_2 = A_{15}G_3 + A_{16}r^2U(\omega(\rho_2), 2 + \delta(\rho_2), \mu_0(\rho_2)), \end{cases}$$

where

$$G_1 = \frac{\alpha r^2(\omega(\rho_1) - \delta(\rho_1) - 1)}{\rho_1\delta(\rho_1)(1 + \delta(\rho_1))}M(\omega(\rho_1), 2 + \delta(\rho_1), \frac{cr}{\rho_1}) - \frac{\alpha r(\rho_2 - \rho_1)}{c\rho_1\delta(\rho_1)}M_1^*(\rho_1, 0),$$
$$G_2 = r^2U(\omega(\rho_1); 2 + \delta(\rho_1); \mu_0(\rho_1)) + \frac{r(\rho_2 - \rho_1)}{c}U_1^*(\rho_1, 0),$$
$$G_3 = \frac{\alpha r^2(\omega(\rho_1) - \delta(\rho_1) - 1)}{(\rho_1 - \lambda - \alpha)(1 + \delta(\rho_1))}M(\omega(\rho_2); 2 + \delta(\rho_2); \mu_0(\rho_2)).$$

Proof. By Lemma 1.1, $V(\cdot)$ is absolute continuous in (b, a) and satisfies (1.1) with the boundary condition $V(a) = 1$. Multiplying (1.1) by e^{-rx} and differentiating with respect to x yield

$$(\rho x + c)V''(x) + (\rho rx + rc + \rho - \lambda - \alpha)V'(x) - \alpha rV(x) = 0, \quad (3.3)$$

where $\rho = \rho_1$ if $0 < x < a$ and $\rho = \rho_2$ if $b < x \leq 0$. We only prove case (2) here; the proofs of cases (1), (3) and (4) are similar.

If $\lambda = \rho_1$, then the bounded solution of (3.3) for $0 < x < a$ is of the form of (3.1), where A_5 and A_6 are constants.

If $\lambda \neq \rho_2$, substituting $W(y) = \mathrm{e}^{y-cr/\rho_2} V(\frac{y}{r} - \frac{c}{\rho_2})$ into (3.3) yields

$$yW''(y) + \left(1 - \frac{\alpha}{\rho_2} - \frac{\lambda}{\rho_2} - y\right) W'(y) + \left(\frac{\lambda}{\rho_2} - 1\right) W(y) = 0, \quad \frac{cr}{\rho_2} + br < y < \frac{cr}{\rho_2},$$

which is Kummer's differential equation, see e.g. Magnus *et al.* (1966). If $\delta(\rho_2) \neq -n$, for $n = 0, 1, 2, \ldots$, the complete solution is given by

$$W(y) = A_7 M(\omega(\rho_2); \delta(\rho_2); y) + A_8 U(\omega(\rho_2); \delta(\rho_2); y), \quad \frac{cr}{\rho_2} + br < y < \frac{cr}{\rho_2},$$

where A_7 and A_8 are constants. Thus, equation (3.2) follows. The values of A_5, A_7 and A_8 can be determined by the boundary condition $V(a) = 1$ and the continuity of $V(\cdot)$ and $V'(\cdot)$ at zero. Finally, substituting (3.1) into (1.1) and equating the coefficients of e^{-rx} yield the expression for A_6.

Remark 3.1. Letting $a \to \infty$ and taking $b = -c/\rho_2$, we can obtain the Laplace transform of the distribution of the time of absolute ruin, and from which the probability of absolute ruin in the infinite horizon case can be derived.

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