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# ON OCCUPATION TIMES FOR A RISK PROCESS WITH RESERVE-DEPENDENT PREMIUM

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## ABSTRACT

Consider a risk reserve process under which the reserve can generate interest. For constants $a$ and $b$ such that $a < b$, we study the occupation time $T_{a,b}(t)$, which is the total length of the time intervals up to time $t$ during which the reserve is between $a$ and $b$. We first present a general formula for piecewise deterministic Markov processes, which will be used for the computation of the Laplace transform of $T_{a,b}(t)$. Explicit results are then given for the special case that claim sizes are exponentially distributed. The classical model is discussed in detail.

*Keywords:* Occupation time; Piecewise deterministic Markov process; risk theory; duration of negative surplus; ruin

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## 1. INTRODUCTION

As a generalization of the classical risk model, we assume that a company can receive interest on its positive reserve at a constant rate $\rho > 0$. More precisely, we consider the following risk reserve process

$$X_t := x + \int_0^t C(X_s)\mathrm{d}s - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0, \tag{1.1}$$

where $x$ is the initial capital, $C(z) = c + \rho z \mathbf{1}(z > 0)$ with $\mathbf{1}(\cdot)$ being the indicator function and $(N_t)_{t\geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$. The positive constants $c$ and $\rho$ are the premium income rate and the interest rate, respectively. The claim sizes $\{Y_i\}_{i\geq 1}$ form a sequence of independent positive random variables having a common general distribution function $G$ and being independent of $(N_t)_{t\geq 0}$.

The probability of ultimate ruin of this nonstationary model has been considered by Dassios and Embrechts [3] and Embrechts and Schmidli [9]. Sundt and Teugels [12] and Küppelberg and Stadtmüller [11] studied the asymptotic behaviour of the ruin probabilities of this model as $c \to \infty$, where the tail of the claim size distribution is exponentially decreasing and heavy, respectively. For a more general setting with subexponential claims, see Asmussen [2]. Dassios and Embrechts [3] described how to apply the piecewise deterministic Markov process set-up of Davis [4] to risk theory. We follow their approach and allow the risk reserve process to continue if it falls below zero. For $a < b$ and $t > 0$, define the occupation time $T_{a,b}(t)$ of $\{X_s : 0 \leq s \leq t\}$ in $[a, b]$ by

$$T_{a,b}(t) = \int_0^t \mathbf{1}(a \leq X_s \leq b)\mathrm{d}s. \tag{1.2}$$

Our goal is to compute the Laplace transform for the distribution of $T_{a,b}(t)$. Problems of this kind have received many attentions over recent years, see for instance Egídio dos Reis [8] and Dickson and Egídio dos Reis [6], who studied the moments and the distribution of the duration of negative surplus (i.e. $T_{a,b}(t)$ for $b = 0$) for the special case that $\rho = 0$. Dickson and Egídio dos Reis [7] considered the density function, the mean and the variance of the duration of the first period of negative surplus under the model assumption that when the surplus is negative, the insurer is allowed to borrow money at a certain interest rate to pay claims.

In the next section we formulate the risk reserve model $X_t$ given in (1.1) as a special piecewise deterministic Markov process (PDMP) introduced by Davis [4] and present a general result. We then in Section 3 apply the result to the special case that the claim size distribution $G$ is exponential to yield explicit expressions for the double Laplace transform of the distribution of the occupation time $T_{a,b}(t)$. The last section discusses the classical model in detail.

## 2. PRELIMINARIES

Roughly speaking, a PDMP taking values on an open subset $E$ of $\mathbb{R}$ is a stochastic process $(Z_t)_{t\geq 0}$ involving a deterministic motion punctuated by random jumps. The motion depends on three local characteristics, namely an integral curve $\phi$ that describes the deterministic motion, a jump rate $\gamma$ that governs how often the jumps are and a transition measure $Q$ that determines the post-jump locations. Imagine that $(Z_t)_{t\geq 0}$ is the path of the movement of a particle starting from $Z_0 = z$. It travels along the curve $\phi(t, z)$ until the first jump time $T_1$ that occurs either spontaneously in a Poisson-like fashion with rate $\gamma(\phi(t, z))$ or the curve $\phi(t, z)$ reaches the boundary of the state space. In either case the location of the process at the jump time $T_1$ is specified by the transition measure $Q(\cdot; \phi(T_1, z))$ and the particle restarts from the post-jump location moving along another curve in the family until the next jump, if any, and so on. For details see Davis [4, 5].

Denote by $\mathbf{P}_z$ and $\mathbf{E}_z$ the conditional distribution of $Z_t$ and its expectation operator, respectively, given that $Z_0 = z$. Let $\{T_i\}_{i\geq 1}$ be the jump times of $(Z_t)_{t\geq 0}$ and such that

$$\mathbf{E}_z \sum_i \mathbf{1}(T_i \geq t) < \infty, \quad \text{for all } t > 0.$$

Let $\mathcal{A}_1$ denote the extended generator of $(Z_t)_{t\geq 0}$ and $\mathcal{D}(\mathcal{A}_1)$ the domain of $\mathcal{A}_1$. Davis [4] gave necessary and sufficient conditions for a function $f$ to be an element of $\mathcal{D}(\mathcal{A}_1)$. For applications in risk theory, we will use the following simple sufficient condition (see Dassios and Embrechts [3]).

Let $f : E \cup \partial E \to \mathbb{R}$ be a measurable function satisfying

*(i)* $f(u) = \int_E f(y)Q(\mathrm{d}y; u)$ for all $u \in \partial^* E := \{\xi \in \partial E : \xi = \phi(t, z)$ for some $t > 0$ and $z \in E\}$ where $\partial E$ being the boundary of $E$,

*(ii)* $\mathbf{E}_z \sum_{T_i \leq t} |f(Z_{T_i}) - f(Z_{T_i-})| < \infty$, for all $t \geq 0$, and

*(iii)* the mapping $t \mapsto f(\phi(t, z))$ is absolutely continuous for $t < t^*(z)$ and for each $z \in E$, where $t^*(z)$ is defined to be $\inf\{t > 0 : \phi(t, z) \in \partial^* E\}$, with the convention that $\inf \emptyset = \infty$.

Then $f \in \mathcal{D}(\mathcal{A}_1)$ and

$$\mathcal{A}_1 f(z) = \chi_1 f(z) + \gamma(z) \int_E \{f(y) - f(z)\}Q(\mathrm{d}y; z), \quad \text{for } z \in E.$$

The operator $\chi_1 := g(z)\frac{\partial}{\partial z}$ is the vector field of the integral curve $\phi$, i.e.

$$\frac{\partial \phi(t, z)}{\partial t} = g(\phi(t, z)), \quad \phi(0, z) = z, \quad z \in E, \ t \geq 0,$$

where $g : E \to \mathbb{R}$ being a continuously differentiable Lipschitz function.

Note that condition *(ii)* is stronger than the one used in Davis [4]. Hence the set of measurable functions satisfying *(i)* – *(iii)* is a subset of $\mathcal{D}(\mathcal{A}_1)$.

The process $(X_t)_{t\geq 0}$ can be regarded as a PDMP with initial value $x$, state space $\mathbb{R}$, transition measure $Q(\mathrm{d}y;x) = \mathrm{d}G(x-y)$ and associated vector field

$$\chi = \begin{cases} (c+\rho x)\dfrac{\partial}{\partial x}, & x > 0, \\ c\dfrac{\partial}{\partial x}, & x \leq 0. \end{cases}$$

Thus $(X_t)_{t\geq 0}$ is a special PDMP with extended generator $\mathcal{A}$, where

$$\mathcal{A}f(x) = \chi f(x) + \lambda \int_0^\infty \{f(x-y) - f(x)\}\mathrm{d}G(y).$$

For a piecewise continuous function $f : E \to \mathbb{R}$, denote by $D_f$ the set of the discontinuity points of $f$. The following theorem holds for all PDMPs $(Z_t)_{t\geq 0}$ satisfying conditions $(i)-(iii)$.

**Theorem 2.1.** *Let $f : E \to \mathbb{R}$ and $h : E \to [0,\infty)$ be piecewise-continuous functions and $\alpha > 0$ a fixed constant. If*

$$\int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_z|f(Z_t)|\mathrm{d}t < \infty, \text{ for all } z \in E,$$

*then the function*

$$V(z) = \int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_z\left\{f(Z_t)\mathrm{e}^{-\int_0^t h(Z_s)\mathrm{d}s}\right\}\mathrm{d}t, \quad z \in E,$$

*is absolutely continuous on $E$ and satisfies*

$$\mathcal{A}_1 V(z) + f(z) = \{\alpha + h(z)\}V(z) \text{ on } E \setminus (D_f \cup D_h). \tag{2.1}$$

*Remark 2.1.* Theorem 2.1 strengthens Proposition 32.25 in Davis [5], in which it is only shown that (2.1) holds almost everywhere on $E$. The set on which (2.1) does not hold has not been discussed in Davis [5] but is essential in solving equation (3.3).

*Proof.* Define the resolvent operator $G_\alpha$ by

$$(G_\alpha g)(z) = \int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_z g(Z_t)\mathrm{d}t,$$

where $g$ is a piecewise-continuous function that satisfies

$$\int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_z|g(Z_t)|\mathrm{d}t < \infty, \text{ for any } z \in E.$$

Fubini's theorem and the Markov property lead to

$$\begin{aligned}
\frac{1}{\alpha} - V(z) &= \int_0^\infty \mathrm{e}^{-\alpha t}\mathrm{d}t - \int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_z \exp\left\{-\int_0^t h(Z_s)\mathrm{d}s\right\}\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_z \int_0^t h(Z_s)\exp\left\{-\int_s^t h(Z_u)\mathrm{d}u\right\}\mathrm{d}s\mathrm{d}t \\
&= \mathbf{E}_z \int_0^\infty \left\{\mathrm{e}^{-\alpha s}h(Z_s)\mathbf{E}_{Z_s}\int_0^\infty \mathrm{e}^{-\alpha t - \int_0^t h(Z_u)\mathrm{d}u}\mathrm{d}t\right\}\mathrm{d}s \\
&= \mathbf{E}_z \int_0^\infty \mathrm{e}^{-\alpha s}h(Z_s)V(Z_s)\mathrm{d}s = G_\alpha(hV)(z), \quad z \in E,
\end{aligned} \tag{2.2}$$

which implies $G_\alpha(|hV|)(z) < \infty$, for all $z \in E$. Moreover, if $P(t,z,y)$ denotes the transition probability function of $(Z_t)_{t\geq 0}$ from the initial position $z$ to $y$ at time $t$, then by the Kolmogorov equation

$$\begin{aligned}
\mathcal{A}_1(G_\alpha hV)(z) &= \int_{-\infty}^\infty (hV)(y)\int_0^\infty \mathrm{e}^{-\alpha t}\mathcal{A}_1 P(t,z,\mathrm{d}y)\mathrm{d}t \\
&= \int_{-\infty}^\infty (hV)(y)\int_0^\infty \mathrm{e}^{-\alpha t}\frac{\partial P(t,z,\mathrm{d}y)}{\partial t}\mathrm{d}t \\
&= -(hV)(z) + \alpha(G_\alpha hV)(z), \quad z \in E \setminus D_{hV}.
\end{aligned} \tag{2.3}$$

Similarly,

$$\mathcal{A}_1(G_\alpha f)(z) = -f(z) + \alpha(G_\alpha f)(z), \quad z \in E \setminus D_f. \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) yields (2.1). The proof of the absolute continuity of $V$ is similar to that of Theorem 32.2 in Davis [5].

## 3. OCCUPATION TIMES

This section presents expressions for the double Laplace transform of the distribution of $T_{a,b}(t)$ of the risk process $(X_t)_{t\geq 0}$ given in (1.1) for the case that claim sizes have the exponential distribution with mean $1/r$. Note that the results also hold when the initial capital $x$ is negative.

For simplicity, let $v_s^+$ and $v_s^-$ denote the positive and negative root, respectively, of the quadratic equation $cv^2+(rc-s-\lambda)v-sr=0$. Moreover, we use $\omega = 1-\lambda/\rho$ and $\mu(x) = cr/\rho + rx$ to simplify the expressions. Furthermore, denote by $M$ the confluent hypergeometric function, and by $U$ the confluent hypergeometric function of the second kind, see Andrews [1].

**Theorem 3.1.** *For $a < 0 < b$ and $\alpha, \beta > 0$, let $v_1 = v_\alpha^+$, $v_2 = v_\alpha^-$, $v_3 = v_{\alpha+\beta}^+$ and $v_4 = v_{\alpha+\beta}^-$ and $\delta(\beta) = 1-(\lambda+\alpha+\beta)/\rho$. Define*

$$f(x) = \int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_x \mathrm{e}^{-\beta T_{a,b}(t)}\mathrm{d}t.$$

*(1) If $\lambda = \rho$, then*

$$f(x) = \begin{cases} \dfrac{1}{\alpha} + B_1\mathrm{e}^{-rx}, & x \geq b, \\ \dfrac{1}{\alpha+\beta} + B_2\mathrm{e}^{-rx}\displaystyle\int_{\frac{rc}{\rho}}^{\mu(x)} \mathrm{e}^y y^{(\alpha+\beta)/\rho}\mathrm{d}y + B_3\mathrm{e}^{-rx}, & 0 < x < b, \\ \dfrac{1}{\alpha+\beta} + B_4\mathrm{e}^{v_3 x} + B_5\mathrm{e}^{v_4 x}, & a < x \leq 0, \\ \dfrac{1}{\alpha} + B_6\mathrm{e}^{v_1 x}, & x \leq a, \end{cases}$$

*where $B_i$'s are constants, which are the solutions of the following system of linear equations:*

$$
\begin{cases}
B_1 - B_2 \displaystyle\int_{\frac{rc}{\rho}}^{\mu(b)} \mathrm{e}^y y^{(\alpha+\beta)/\rho} \mathrm{d}y - B_3 = -\frac{\beta \mathrm{e}^{rx}}{\alpha(\alpha+\beta)}, \\
B_3 - B_4 - B_5 = 0, \\
B_4 \mathrm{e}^{v_3 a} + B_5 \mathrm{e}^{v_4 a} - B_6 \mathrm{e}^{v_1 a} = \dfrac{\beta}{\alpha(\alpha+\beta)}, \\
B_4 v_3 \mathrm{e}^{v_3 a} + B_5 v_4 \mathrm{e}^{v_4 a} - B_6 \mathrm{e}^{v_1 a}(v_1 + \dfrac{\beta}{c}) = \dfrac{\beta}{\alpha c}, \\
B_2 r \mathrm{e}^{cr/\rho} (\dfrac{cr}{\rho})^{(\alpha+\beta)/\rho} - rB_3 - B_4 v_3 - B_5 v_4 = 0, \\
B_1 (r + \dfrac{\beta}{c+\rho b}) + B_3 r + \dfrac{\beta \mathrm{e}^{rb}}{\alpha(c+\rho b)} \\
\qquad\qquad = B_2 \left\{ r\mathrm{e}^{\mu(b)} \mu(b)^{(\alpha+\beta)/\rho} - r \displaystyle\int_{\frac{rc}{\rho}}^{\mu(b)} \mathrm{e}^y y^{(\alpha+\beta)/\rho} \mathrm{d}y \right\}.
\end{cases}
\tag{3.1}
$$

*(2) If $\lambda \neq \rho$ and both $\omega, \delta(\beta) \neq 0, -1, -2, \cdots$, then*

$$
f(x) = \begin{cases}
\dfrac{1}{\alpha+\beta} + \mathrm{e}^{-rx} W(\mu(x)), & 0 < x < b \\
\dfrac{1}{\alpha} + A_3 \mathrm{e}^{-rx} U(\omega; \delta(0); \mu(x)), & x > b, \\
\dfrac{1}{\alpha+\beta} + A_4 \mathrm{e}^{v_3 x} + A_5 \mathrm{e}^{v_4 x}, & a < x \leq 0, \\
\dfrac{1}{\alpha} + A_6 \mathrm{e}^{v_1 x}, & x \leq a,
\end{cases}
$$

*where $W(\mu(x)) = A_1 M(\omega; \delta(\beta); \mu(x)) + A_2 U(\omega; \delta(\beta); \mu(x))$, and $A_i$'s are constants, which are the solutions of the following system of linear equations:*

$$
\begin{cases}
W(\mu(b)) - A_3 U(\omega;\delta(0);\mu(b)) = \dfrac{\beta \mathrm{e}^{rb}}{\alpha(\alpha+\beta)}, \\
A_1 M(\omega;\delta(\beta);\mu(0)) + A_2 U(\omega;\delta(\beta);\mu(0)) - A_4 - A_5 = 0, \\
A_4 \mathrm{e}^{v_3 a} + A_5 \mathrm{e}^{v_4 a} - A_6 \mathrm{e}^{v_1 a} = \dfrac{\beta}{\alpha(\alpha+\beta)}, \\
A_4 v_3 \mathrm{e}^{v_3 a} + A_5 v_4 \mathrm{e}^{v_4 a} - A_6 \mathrm{e}^{v_1 a}(v_1 + \dfrac{\beta}{c}) = \dfrac{\beta}{c\alpha}, \\
A_1 \dfrac{r(\alpha+\beta)}{\rho\delta(\beta)} M(\omega;1+\delta(\beta);\mu(b)) - A_2 r U(\omega;1+\delta(\beta);\mu(b)) \\
\qquad\qquad\qquad\qquad - A_4 v_3 - A_5 v_4 = 0, \\
A_1 \dfrac{r(\alpha+\beta)}{\rho\delta(\beta)} M(\omega;1+\delta(\beta);\mu(b)) - A_2 r U(\omega;1+\delta(\beta);\mu(b)) \\
= A_3 \left\{ \dfrac{\beta}{c+\rho b} U(\omega;\delta(0);\mu(b)) - r U(\omega;1+\delta(0);\mu(b)) \right\} + \dfrac{\beta \mathrm{e}^{rb}}{\alpha(c+\rho b)}.
\end{cases}
\tag{3.2}
$$

*Proof.* By Theorem 2.1, $f$ is absolutely continuous on $\mathbb{R}$ and satisfies

$$
\mathcal{A}f(x) + 1 = \begin{cases} (\alpha+\beta) f(x), & a < x < b, \\ \alpha f(x), & x < a, \ \text{or } x > b, \end{cases} \tag{3.3}
$$

where

$$
\mathcal{A}f(x) = C(x) f'(x) + \lambda r \mathrm{e}^{-rx} \int_0^x f(y) \mathrm{e}^{ry} \mathrm{d}y - \lambda f(x).
$$

Equation (3.3) implies that $f'(x)$ is absolutely continuous on $\mathbb{R} \setminus \{a, b\}$ and satisfies the following boundary conditions

$$
\begin{cases} C(b)\{f'(b-0) - f'(b+0)\} = \beta f(b), \\ C(a)\{f'(a+0) - f'(a-0)\} = \beta f(a). \end{cases} \tag{3.4}
$$

Multiplying (3.3) by $\mathrm{e}^{rx}$ and taking derivative with respect to $x$ on both sides lead to the following second order ordinary differential equations:

$$
\begin{cases}
(c+\rho x)f''(x)+\eta(\beta)f'(x)-r(\alpha+\beta)f(x)+r=0, & 0<x<b,\\
(c+\rho x)f''(x)+\eta(0)f'(x)-r\alpha f(x)+r=0, & x>b,\\
cf''(x)+(rc-\lambda-\alpha-\beta)f'(x)-r(\alpha+\beta)f(x)+r=0, & a<x<0,\\
cf''(x)+(rc-\lambda-\alpha)f'(x)-r\alpha f(x)+r=0, & x<a,
\end{cases}
\tag{3.5}
$$

where $\eta(\beta)=rc+\rho rx+\rho-\lambda-\alpha-\beta$.

The last two equations of (3.5) have constant coefficients, and so the general solution can be found easily.

Substituting $y=cr/\rho+rx$ and $f(x)=\mathrm{e}^{cr/\rho-y}g(y)$ into the first two equations of (3.5), we get

$$
\begin{aligned}
&yg''(y)+\{\delta(\beta)-y\}g'(y)-\omega g(y)+\frac{1}{\rho}\mathrm{e}^{y-cr/\rho}=0, \quad \frac{cr}{\rho}<y<\frac{cr}{\rho}+rb,\\
&yg''(y)+\{\delta(0)-y\}g'(y)-\omega g(y)+\frac{1}{\rho}\mathrm{e}^{y-cr/\rho}=0, \quad y>\frac{cr}{\rho}+rb.
\end{aligned}
$$

If $\lambda=\rho$, i.e. $\omega=0$, the bounded solution is of the form

$$
g(y)=\begin{cases}
\dfrac{1}{\alpha}\mathrm{e}^{y-cr/\rho}+B_1, & y>\dfrac{cr}{\rho}+rb,\\
\dfrac{\mathrm{e}^{y-cr/\rho}}{\alpha+\beta}+B_2\displaystyle\int_{\frac{cr}{\rho}}^{y}\mathrm{e}^s s^{(\alpha+\beta)/\rho}\mathrm{d}s+B_3, & \dfrac{cr}{\rho}<y<\dfrac{cr}{\rho}+rb,
\end{cases}
$$

where $B_i$'s are constants.

If $\lambda\neq\rho$, $g$ satisfies the confluent hypergeometric equation or Kummer's differential equation. When both $\omega,\delta(\beta)\neq 0,-1,-2,\cdots$, the bounded solution is of the form (see Andrews [1]):

$$
g(y)=\begin{cases}
\dfrac{1}{\alpha+\beta}\mathrm{e}^{y-cr/\rho}+W(y), & \dfrac{cr}{\rho}<y<\dfrac{cr}{\rho}+rb,\\
\dfrac{1}{\alpha}\mathrm{e}^{y-cr/\rho}+A_3U(\omega;\delta(0);y), & y>\dfrac{cr}{\rho}+rb,
\end{cases}
$$

where $A_i$'s are constants.

The general solution of equations (3.5) follows from the relation between $f$ and $g$. The continuity of $f$ on $\mathbb{R}$ and $f'$ on $\mathbb{R}\setminus\{a,b\}$, together with condition (3.4) and the recurrence relations for $M$ and $U$, lead to systems (3.1) and (3.2).

## 4. THE CLASSICAL MODEL

Using the same method as in the last section yields the corresponding results for the classical model. In this model we consider the occupation time $T_{a,b}(t)$ for any real numbers $a<b$. In the following we use the same notation as in Theorem 3.1.

**Theorem 4.1.** *Let $a<b$ and $\alpha,\beta>0$. We have*

$$\int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_x \mathrm{e}^{-\beta T_{a,b}(t)}\mathrm{d}t = \begin{cases} \dfrac{1}{\alpha+\beta} + C_1\mathrm{e}^{v_3 x} + C_2\mathrm{e}^{v_4 x}, & a\le x\le b,\\ \dfrac{1}{\alpha} + C_3\mathrm{e}^{v_1 x}, & x<a,\\ \dfrac{1}{\alpha} + C_4\mathrm{e}^{v_2 x}, & x>b, \end{cases}$$

*where $C_1=\Delta_1(v_3,v_4), C_2=\Delta_1(v_4,v_3), C_3=\Delta_2(v_1,a)$ and $C_4=\Delta_2(v_2,b)$, with*

$$\Delta_1(u,v) = \frac{\frac{\beta}{\alpha(\alpha+\beta)}\left\{(v_1-\frac{\alpha}{c})(v_2-v+\frac{\beta}{c})\mathrm{e}^{vb} - (v_2-\frac{\alpha}{c})(v_1-v+\frac{\beta}{c})\mathrm{e}^{va}\right\}}{(v_1-u+\frac{\beta}{c})(v_2-v+\frac{\beta}{c})\mathrm{e}^{ua+vb} - (v_2-u+\frac{\beta}{c})(v_1-v+\frac{\beta}{c})\mathrm{e}^{ub+va}},$$

$$\Delta_2(u,v) = \frac{C_1 v_3 \mathrm{e}^{v_3 v} + C_2 v_4 \mathrm{e}^{v_4 v} - \frac{\beta}{\alpha c}}{u\mathrm{e}^{uv} + \frac{\beta}{c}\mathrm{e}^{uv}}.$$

**Corollary 4.1.** *Let $a,b\in\mathbb{R}$ and $\alpha,\beta>0$, define*

$$\phi(a,b,x) = \int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_x \mathrm{e}^{-\beta T_{a,b}(t)}\mathrm{d}t.$$

*We have*

$$\phi(-\infty,b,x)=\begin{cases}\dfrac{1}{\alpha+\beta}\left\{1+\dfrac{\beta(\frac{1}{c}-\frac{v_2}{\alpha})}{v_3-v_2-\frac{\beta}{c}}\mathrm{e}^{(x-b)v_3}\right\}, & x\le b,\\ \dfrac{1}{\alpha}+\dfrac{\frac{\beta\alpha c}{\alpha+\beta}(\frac{1}{c}-\frac{v_2}{\alpha})v_3-\beta(v_3-v_2-\frac{\beta}{c})}{\alpha c(v_2+\frac{\beta}{c})(v_3-v_2-\frac{\beta}{c})\mathrm{e}^{(b-x)v_2}}, & x>b,\end{cases}$$

*and*

$$\phi(a,\infty,x)=\begin{cases}\dfrac{1}{\alpha+\beta}\left\{1+\dfrac{\beta(\frac{1}{c}-\frac{v_1}{\alpha})}{v_4-v_1-\frac{\beta}{c}}\mathrm{e}^{(x-a)v_4}\right\}, & x\ge a,\\ \dfrac{1}{\alpha}+\dfrac{\frac{\beta\alpha c}{\alpha+\beta}(\frac{1}{c}-\frac{v_1}{\alpha})v_4-\beta(v_4-v_1-\frac{\beta}{c})}{\alpha c(v_1+\frac{\beta}{c})(v_4-v_1-\frac{\beta}{c})\mathrm{e}^{(a-x)v_1}}, & x<a.\end{cases}$$

Applying the Tauberian theorem (Feller [10], p. 445) to Theorem 4.1 leads to

**Theorem 4.2.** *Let $a<b$ and $\alpha>0$. If $\frac{\lambda}{c}<r$, where $1/r$ is the mean claim size, then*

$$\mathbf{E}_x\mathrm{e}^{-\alpha T_{a,b}(\infty)}=\begin{cases}C_5\mathrm{e}^{v_1x}+C_6\mathrm{e}^{v_2x}, & a<x<b,\\ C_7, & x\le a,\\ 1+C_8\mathrm{e}^{(\lambda/c-r)x}, & x\ge b,\end{cases}$$

*where*

$$C_5=-\frac{(\frac{\lambda}{c}-r)(\frac{c}{\alpha}v_2-1)\mathrm{e}^{v_2a}}{H},$$
$$C_6=\frac{(\frac{c}{\alpha}v_1-1)(\frac{\lambda}{c}-r)\mathrm{e}^{v_1a}}{H},$$
$$C_7=\frac{\frac{c}{\alpha}(v_1-v_2)(\frac{\lambda}{c}-r)\mathrm{e}^{(v_2+v_1)a}}{H},$$
$$C_8=\frac{\frac{c}{\alpha}(v_1-\frac{\alpha}{c})(v_2-\frac{\alpha}{c})(\mathrm{e}^{v_1a+v_2b}-\mathrm{e}^{v_1b+v_2a})\mathrm{e}^{-(\lambda/c-r)b}}{H},$$

*with*

$$H=(\frac{c}{\alpha}v_2-1)\mathrm{e}^{v_1b+v_2a}(v_1-\frac{\alpha}{c}-\frac{\lambda}{c}+r)-(\frac{c}{\alpha}v_1-1)\mathrm{e}^{v_1a+v_2b}(v_2-\frac{\alpha}{c}-\frac{\lambda}{c}+r).$$

**Corollary 4.2.** *Let $a, b \in \mathbb{R}$, $\alpha > 0$ and $\frac{\lambda}{c} < r$. Then for all $x \in \mathbb{R}$, we have*

$$\mathbf{E}_x \mathrm{e}^{-\alpha T_{-\infty,b}(\infty)} = \begin{cases} \dfrac{\frac{\lambda}{c} - r}{\frac{\alpha}{c} + \frac{\lambda}{c} - v_1 - r} \mathrm{e}^{(x-b)v_1}, & x < b, \\ 1 + \dfrac{v_1 - \frac{\alpha}{c}}{\frac{\alpha}{c} + \frac{\lambda}{c} - v_1 - r} \mathrm{e}^{(\lambda/c - r)(x-b)}, & x \geq b, \end{cases} \tag{4.1}$$

*and*

$$\mathbf{P}_x \left\{ \int_0^\infty \mathbf{1}(X_s > a) \mathrm{d}s = \infty \right\} = 1.$$

*Remark 4.1.* Letting $b = 0$ in the case $x \geq b$ in (4.1) leads to the corresponding result in Egídio dos Reis [8].

*Remark 4.2.* Our method can also be applied to other risk models, such as the restricted borrowing risk models (Zhang and Wu [13]) or Embrechts-Schmidli models (Embrechts and Schmidli [9]).


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