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A regularity condition and strong limit theorems for linear birth–growth processes

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A regularity condition and strong limit theorems for linear birth – growth processes

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Abstract. A linear birth-growth process is generated by an inhomogeneous Poisson process on $\mathbb{R} \times [0, \infty)$. Seeds are born randomly according to the Poisson process. Once a seed is born, it commences immediately to grow bidirectionally with a constant speed. The positions occupied by growing intervals are regarded as covered. New seeds continue to form on the uncovered part of $\mathbb{R}$. This paper shows that the total number of seeds born on a very long interval satisfies the strong invariance principle and some other strong limit theorems. Also, a gap (an unproved regularity condition) in the proof of the central limit theory in [5] is filled in.

1. Introduction

Let $\Psi \equiv \{(y_i, t_i) \in \mathbb{R} \times [0, \infty)\}$ denote a Poisson point process which is homogeneous in space ($\mathbb{R}$) but can be inhomogeneous in time ($[0, \infty)$). That is, the intensity measure of the process is $l \times \Lambda$, where l is the Lebesgue measure on $\mathbb{R}$ and Λ an arbitrary locally finite nonzero measure on $[0, \infty)$. Consider a linear birth-growth process generated by Ψ as follows. *Seeds* are *born* according to Ψ; once a seed is born, it immediately initiates a bidirectional movement commencing at the location where the seed is born. Each moving frontier progresses with a constant speed $v > 0$. The interval passed over by a moving frontier is regarded as covered. New seeds continue to form according to Ψ only on uncovered intervals in $\mathbb{R}$ and each seed initiates the same bidirectional grow process. Eventually, the whole line will be completely covered. The resultant structure is known as a one-dimensional Johnson-Mehl tessellation [12], [13] and [16]. Such a linear birth-growth process was studied in details by [4], [5], [6], [8], [9], [11],

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[15] and [17]. Applications can be found in various biological context [1], [7], [18] and [19].

The locations and times of seeds born form a space-time point process Φ on $\mathbb{R} \times [0, \infty)$, which is a dependently thinned version of the Poisson process Ψ. The locations of seeds born form a stationary point process on $\mathbb{R}$ with density

$$\mu \equiv \int_0^\infty \exp\left\{-\int_0^t 2v(t-u)\Lambda(du)\right\}\Lambda(dt)$$

(see [16]). Throughout the paper we assume that $\mu < \infty$.

Chiu [4], Holst *et al.* [11] and Quine and Robinson [17] established the asymptotic normality of the total number of seeds born on a very long interval. Chiu and Quine [5] considered a d-dimensional birth-growth process, $d \geq 1$, and obtained the asymptotic normality of the number of seeds born in a very large cube. Moreover, for the case $d = 1$, they also generalised the central limit theorem for the total number of seeds formed to a functional central limit theorem (weak invariance principle), which means that the number of seeds born, after suitable normalisation and interpolation, behaves asymptotically like a Brownian motion. Their argument for establishing these weak limit theorems requires the regularity condition that $\sigma^2 > 0$, where the parameter σ^2 will be explained in Section 2. However, they have only shown numerically that $\sigma^2 > 0$ for $\Lambda(dt) = \lambda\, dt$, $0 < \lambda < \infty$ and $d = 1$, 2, 3 and 4, but have not considered other Λ. We will fill in this gap for the case $d = 1$ and generalise the weak invariance principle to the strong invariance principle for the total number of seeds born. Some other strong limit theorems are also shown.

2. Regularity condition

Chiu and Quine [5] established the weak invariance principle for the total number of seeds by considering the total number as the sum of a mixing sequence of random variables. We follow their notation. Denote by ξ_z the number of seeds born in $[z, z+1)$, $z \in \mathbb{Z}$. Chiu and Quine [5] showed that $\{\xi_z : z \in \mathbb{Z}\}$ is a stationary α-mixing sequence. Let

$$S_n \equiv \sum_{0 \leq z \leq n} (\xi_z - \mu)$$

for $n \in \mathbb{N}$. Many limit theorems for a mixing sequence require the regularity condition that

$$\sigma^2 \equiv \lim_{n \to \infty} \frac{\mathrm{var}(S_n)}{n} > 0,$$

which is not necessarily true [2].

Chiu and Quine [5] derived an expression for σ^2 of a d-dimensional birth-growth process:

$$\sigma^2 = \mu - \int_0^\infty \exp\{-\Delta(t_1)\} \int_0^\infty \omega_d v^d (t_1 + t_2)^d \exp\{-\Delta(t_2)\}\Lambda(dt_2)\Lambda(dt_1)$$

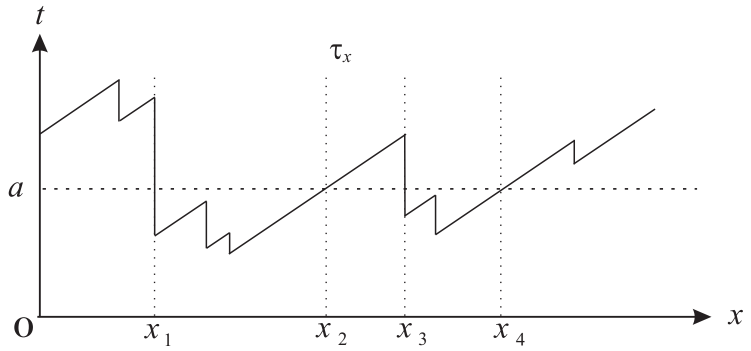

Figure 1: A realisation of the unidirectional process τ_x. The path of τ_x in $[x_1, x_2)$, $[x_2, x_3)$ and $[x_3, x_4)$ are, respectively, in a downblock, upblock and downblock. The upblock and the downblock in $[x_2, x_4)$ form a complete block. However, the path in $[0, x_1)$ is neither in a upblock nor in a downblock because $\tau_0 \neq a$. In this realisation, the number Y_1 of drops in $[0, x_2)$ is 4 and the number B_2 of drops in $[x_2, x_4)$ is 2.

$$+\int_0^\infty \exp\{-\Delta(t_1)\} \int_0^{t_1} \exp\{\Delta(y)\}$$
$$\times \int_y^\infty 2d\omega_d v^d (t_1 + t_2 - 2y)^{d-1} \exp\{-\Delta(t_2)\}\Lambda(dt_2) dy \Lambda(dt_1),$$

where $\Delta(t) = \int_0^{t\vee 0} \omega_d v^d (t-u)^d \Lambda(du)$, $\omega_d = \sqrt{\pi^d}/\Gamma(1 + d/2)$ being the volume of a d-dimensional sphere of unit radius, and $x \vee y = \max(x, y)$. They showed numerically that $\sigma^2 > 0$ for $\Lambda(dt) = \lambda\, dt$, $0 < \lambda < \infty$, in the cases $d = 1, 2, 3$ and 4. However, when they discussed the following two models,

$$\Lambda(dt) = Kt^{j-1}\, dt, \qquad K > 0,\ j \geq 1, \tag{2.1}$$

$$\Lambda(dt) = \frac{\lambda t^{\alpha-1} e^{-t}}{\Gamma(\alpha)}\, dt, \qquad \lambda > 0,\ \alpha > 1, \tag{2.2}$$

they did not show $\sigma^2 > 0$. A brute force proof for $\sigma^2 > 0$, even if possible, would be extremely laborious. In this paper we present a simple proof for a general Λ in the case $d = 1$.

We use the approach of [11] and [18]. Assuming, without loss of generality, that $v = 1/2$, we take a shear transformation $(x, t) \to (x + t/2, t)$ for the spatio-temporal path of the moving frontiers of the bidirectional growth process. This transformed process can be regarded as the path of the moving frontier of a unidirectional growth process that a seed born initiates not a bidirectional but only a unidirectional movement to the right-hand side with unit speed; the movement stops whenever the frontier meets another seed.

Denote by $\{\tau_x : x \in \mathbb{R}\}$ the transformed spatial (x-axis) temporal (y-axis) stochastic process. See Figure 1. It consists of uniform upward motion at unit speed, representing the unidirectional movement, with occasional drops that occur wherever the moving frontier meets a seed; the post-drop level is the birth-time of the seed. Let $x_1 < x_2 < x_3 < \dots$ be the locations such that $\tau_{x_i} = a$ and $\tau_x \neq a$ for all $x \neq x_i$, $i = 1, 2, 3, \dots$, where $a > 0$ is an arbitrary fixed time level. Thus, x_i's are the location points where $\{\tau_x\}$ crosses the time level a. Following the terminology used in [11], we say that the path of $\{\tau_x\}$ has been separated by $\{x_1, x_2, \dots\}$ into *upblocks* where $\tau_x > a$ and *downblocks* with $\tau_x < a$. That is to say, the path of $\{\tau_x\}$ in $[x_i, x_{i+1})$, $i = 2, 3 \dots$, is in an upblock if $\tau_x > a$ for (x_i, x_{i+1}) or in a downblock if $\tau_x < a$ for (x_i, x_{i+1}). Note that the path in $[0, x_1)$ is neither in an upblock nor in a downblock because $\tau_0 \neq a$ almost surely. The upblocks and downblocks form an alternating sequence. If the block on $[x_2, x_3)$ is an upblock (such as in Figure 1), then define a complete block as an upblock and the succeeding downblock; otherwise define it as a downblock and the succeeding upblock. Denote by B_i the total numbers of drops in a complete block from x_{2i-2} to x_{2i}, $i = 2, 3, \dots$. The process $\{\tau_x : x \in \mathbb{R}\}$ is a stationary Markov process, and so the behaviour of the process within each block is independent of its behaviour in other blocks. Therefore, the B_i's are independent and identically distributed for all $i \geq 2$.

Let $\alpha_i = x_{2i-2} - x_{2i}$, $i = 2, 3, \dots$, denote the length of a complete block. Holst *et al*. [11], p. 911, showed that

$$\mathbf{E}(\alpha_i) = \frac{1}{\Lambda([0, a))} \exp\left\{ \int_0^a (a - u)\Lambda(du) \right\}.$$

The path of τ_x in $[0, x_2)$ is not in a complete block because $\tau_0 \neq a$. Let $\beta_1 = x_2$ denote the length of this incomplete block. Similarly, the path of τ_x in $[x_{2(M_n-1)}, n)$ is not a complete block since $\tau_n \neq a$, where M_n is a random variable such that

$$\{M_n = m\} = \begin{cases} \{\beta_1 + \alpha_2 + \cdots + \alpha_{m-1} < n < \beta_1 + \alpha_2 + \cdots + \alpha_m\} & m \geq 3, \\ \{\beta_1 < n < \beta_1 + \alpha_2\} & m = 2, \\ \{\beta_1 > n\} & m = 1. \end{cases}$$

That is, M_n is the total number of complete and incomplete blocks in $[0, n)$. Let Y_1 and Y_2 be the numbers of drops in the first and the last incomplete blocks in $[0, n)$, respectively. Note that $Y_2 = 0$ if $M_n = 1$. Because of locational stationarity, the distribution is unchanged under the shear transformation (see [11], p. 904 and [18], p. 302) we have

$$\text{distribution of } S_n + n\mu = \text{distribution of } Y_1 + \sum_{i=2}^{M_n - 1} B_i + Y_2. \tag{2.3}$$

Denote by $\mathcal{F}_1 \equiv \sigma\{\beta_1, Y_1\}$ and $\mathcal{F}_m \equiv \sigma\{\beta_1, \alpha_2, \dots, \alpha_m, Y_1, B_2, \dots, B_m\}$, $m \geq 2$, the σ-algebras generated by $\{\beta_1, Y_1\}$ and $\{\beta_1, \alpha_2, \dots, \alpha_m, Y_1, B_2, \dots, B_m\}$, respectively. Then $\{\mathcal{F}_m : m \geq 1\}$ is an increasing sequence of σ-algebras such that B_m is $\mathcal{F}_m$-measurable and independent of $\mathcal{F}_{m-1}$ for all $m \geq 2$, and $\{M_n = m\} \in \mathcal{F}_m$.

Lemma 2.1.(Gut [10], Theorem 5.3) *Let* $\mathbf{E}(B_2) = \mu_B$ *and* $(M_n - 2)_+ = (M_n - 2) \vee 0$. *We have*

$$\mathbf{E}\left(\sum_{i=2}^{M_n - 1} B_i \right) = \mu_B \mathbf{E}(M_n - 2)_+ = \mu \mathbf{E}(\alpha_2) \mathbf{E}(M_n - 2)_+, \tag{2.4}$$

$$\mathbf{E}\left[\sum_{i=2}^{M_n-1} B_i - \mu_B(M_n-2)_+\right]^2 = \mathrm{var}(B_2)\mathbf{E}(M_n-2)_+. \tag{2.5}$$

Since $\mathbf{E}(Y_1)/n$ and $\mathbf{E}(Y_2)/n$ vanish as $n \to \infty$, equations (2.3) and (2.4) lead to the result that $\mathbf{E}(\alpha_2)\mathbf{E}(M_n-2)_+/n$ tends to 1 as $n \to \infty$.

The variance of S_n can be expressed as the sum of the variances of Y_1, $\sum_{i=2}^{M_n-1} B_i$ and Y_2 as these random variables are independent. By (2.5), we have

$$\mathrm{var}(S_n) \geq \mathrm{var}(B_2)\mathbf{E}(M_n-2)_+ + [\mathbf{E}(B_2)]^2\mathrm{var}(M_n-2)_+ \geq \mathrm{var}(B_2)\mathbf{E}(M_n-2)_+.$$

Since $\{\tau_x : x \in \mathbb{R}\}$ is Markov, $\mathrm{var}(B_2) > 0$, and so we obtain:

Theorem 2.2.

$$\sigma^2 \equiv \lim_{n\to\infty} \frac{\mathrm{var}(S_n)}{n} \geq \lim_{n\to\infty} \frac{\mathrm{var}(B_2)\mathbf{E}(M_n-2)_+}{n} = \frac{\mathrm{var}(B_2)}{\mathbf{E}(\alpha_2)} > 0.$$

3. Strong limit theorems

Chiu and Quine [5] showed that for $d = 1$, if Λ is given by either (2.1) or (2.2) and $\sigma^2 > 0$, the α-mixing coefficient for the sequence $\{\xi_z : z \geq 0\}$ is

$$\begin{aligned}\alpha(k) &\equiv \sup_{n\in\mathbb{Z}}\{|\Pr(A_1 \cap A_2) - \Pr(A_1)\Pr(A_2)| : A_1 \in \sigma(\xi_z : z \leq n), \\ &\qquad A_2 \in \sigma(\xi_z : z \geq n+k)\} \\ &= O(e^{-k\rho}) \qquad \text{as} \quad k \to \infty,\end{aligned}$$

for some $\rho > 0$, and S_n satisfies the *weak* invariance principle. Using recently published results, we show in the following that for $d = 1$, S_n satisfies the *strong* invariance principle as well as other strong limit theorems.

Lemma 3.1.[Complete convergence] (Lin and Lu [14], Theorem 8.5.0) *Let* $1/2 < s \leq 1$, $2 < r \leq \infty$, *and* $1/s < p < r$. *If* $\mathbf{E}(|\xi_1 - \mu|^r) < \infty$ *and* $\sum_{k=1}^{\infty} \alpha^{1/\theta}(k) < \infty$ *for some* $\theta > [2 + r/(r-p)]ps/(ps-1)$, *then for every* $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} k^{s-2}\Pr\left\{\max_{1\leq i\leq k} |S_i| \geq \varepsilon k^s\right\} < \infty. \tag{3.1}$$

Lemma 3.2.[Strong law of large numbers] (Chen and Wu [3]; Lin and Lu [14], Remark 8.2.3) *Suppose that* $\varepsilon > 0$, $\sup_z \mathbf{E}(|\xi_z - \mu|^p) < \infty$ *for some* $p > 1$, *and*

$$\alpha(k) = \begin{cases} O(k^{-p/(2p-2)-\varepsilon}) & \textit{if} \quad 1 < p < 2, \\ O(k^{-2/p-\varepsilon}) & \textit{if} \quad p \geq 2. \end{cases}$$

Then

$$\frac{S_n - \mathbf{E}(S_n)}{n} = o(1) \qquad \text{almost surely.} \tag{3.2}$$

Lemma 3.3.[Strong invariance principle] (Lin and Lu [14], Corollary 9.3.1) *If $\sigma^2 > 0$, $\mathbf{E}|\xi_z - \mu|^r < \infty$ for some $r > 4$, and $\sum_{k=1}^{\infty} \alpha(k)^{1/4-1/r} < \infty$, then there exists a standard Wiener process $\{W(x) : x \geq 0\}$ defined on a suitably extended probability space such that*

$$S_n - W(n\sigma^2) = O(n^{1/4}(\log n)^{3/2}) \qquad \text{almost surely.} \tag{3.3}$$

Lemma 3.4.[Laws of the iterated logarithm] (Lin and Lu [14], Corollary 9.3.2) *If $\sigma^2 > 0$ and for some $\delta > 0$, $\sup_z \mathbf{E}(|\xi_z - \mu|^{2+\delta}) < \infty$ and $\alpha(k) = O(k^{-r})$ for $r > 1 + 2/\delta$, then*

$$\limsup_{n\to\infty} \frac{S_n}{\sigma\sqrt{2n \log\log n}} = 1 \qquad \text{almost surely,} \tag{3.4}$$

$$\liminf_{n\to\infty} \frac{\max_{1\leq m\leq n} |S_m|}{\sqrt{\pi^2 n/8 \log\log n}} = 1 \qquad \text{almost surely.} \tag{3.5}$$

Consider the linear birth-growth processes. For every positive ρ and θ, $\sum_{k=1}^{\infty} e^{-k\rho/\theta} < \infty$. If Λ is locally finite, then $\mathbf{E}(|\xi_z - \mu|^r) < \infty$ for every $z \in \mathbb{Z}$ and $r > 0$. Thus, we have

Theorem 3.5. *If Λ satisfies either (2.1) or (2.2), then equations (3.1)-(3.5) hold.*

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