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A NEW GRAPH RELATED TO THE DIRECTIONS OF NEAREST NEIGHBOURS IN A POINT PROCESS

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Abstract

This paper introduces a new graph constructed from a point process. The idea is to connect a point with its nearest neighbour, then to the second nearest and continue this process until the point belongs to the interior of the convex hull of these nearest neighbours. The number of such neighbours is called the degree of a point. We derive the distribution of the degree of the typical point in a Poisson process, prove a central limit theorem for the sum of degrees, and propose an edge-corrected estimator of the distribution of the degree that is unbiased for a stationary Poisson process. Simulation studies show that this degree is a useful concept that allows the separation of clustering and repulsive behaviour of point processes.

Keywords: Point process; random graph; convex hull; degree

AMS 2000 Subject Classification: Primary 60G55; 60D05
Secondary 60F05; 62H11

1 Introduction

Consider a point process Φ in $\mathbb{R}^d$. We connect each point in Φ to its first nearest neighbour, then to its second nearest neighbours, and so on until the point is first contained in the interior of the convex hull of these nearest neighbours. Such a construction corresponds to inclusion of all nearest neighbours until positive combinations of the directions to them span the whole space. The resultant graph is connected because if there were two disjoint components, a point of Φ on the boundary of the convex hull of a component would not be contained in the interior of points connected to it.

If Φ is a stationary Poisson process on the plane, a point in the Delaunay tessellation generated by Φ has a mean degree six, which is larger than that in communication networks used in reality. The proposed graph provides an alternative model for communication networks that is in a sense better than Delaunay tessellations, because the mean degree of a

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point in such a graph generated by a stationary Poisson process on the plane is only five and the total length of connections from a point is less than that in the corresponding Delaunay tessellation. Figure 1 shows a realisation of such a random graph.

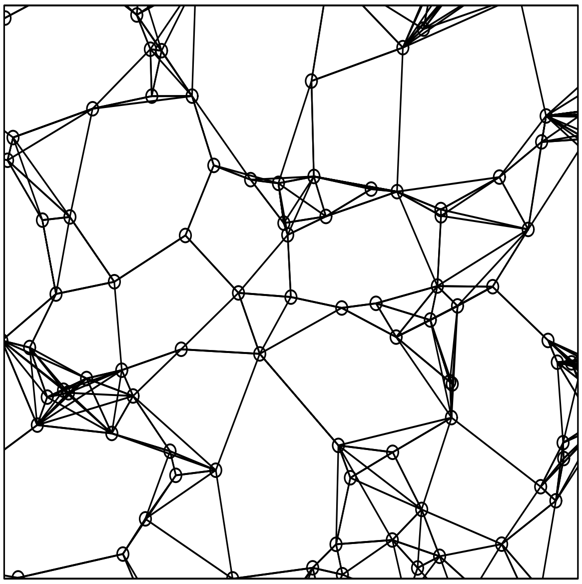

Figure 1: A realisation of the proposed random graph generated by a stationary planar Poisson process.

Let N be the degree of the typical point of a stationary Φ in the corresponding graph, i.e. the smallest number such that the typical point is contained in the interior of the convex hull of the first to the N-th nearest neighbours. In this paper we derive the distribution of N of a stationary Poisson process and prove a central limit theorem for the subprocesses of points of degree n. An edge-corrected estimator for the distribution of N that is unbiased in the case of a stationary Poisson process observed in a window is proposed. We apply this estimator to simulated realisations of various point processes and compare the distributions with that of a stationary Poisson process.

2 Distribution of N

Assume that Φ is a stationary Poisson process. Since the degree of a stationary process is scale invariant and so does not depend on the intensity of the process, we assume without loss of generality that the intensity is 1. Consider a d-dimensional unit ball centred at the typical point o . The direction from o to its k-th nearest neighbour x_k is represented by the point s_k on the surface of the unit ball. The convex hull of $\{x_1, \ldots, x_n\}$ contains o if and only if the unit ball is completely covered by hemispheres with poles $\{s_1, \ldots, s_n\}$.

When Φ is Poisson, s_k are independent and uniformly distributed on the surface of the sphere. Thus, $\{N \leq n\}$ is equivalent to the event that n independent and uniformly distributed hemispheres cover the sphere completely. From Hall [4, Theorem 1.5], we get the following result.

Theorem 1. *Suppose that N is the degree of the typical point of a stationary Poisson process in $\mathbb{R}^d$, we have*

$$\mathbf{P}(N \leq n) = 1 - 2^{-n+1} \sum_{i=0}^{d-1} \binom{n-1}{i}, \quad \text{for } n \geq d.$$

In particular, when $d = 2$, we get

$$\mathbf{P}(N = n) = \frac{n-2}{2^{n-1}}, \quad \text{for } n \geq 2, \tag{1}$$

and so $\mathbf{E}(N) = 5$ and $\mathbf{var}(N) = 4$.

The distribution of the degree follows immediately from the result on coverage of a sphere by hemispheres, while the expectation and variance are obtained from (1) by calculating sums of series.

3 Subprocesses of points of degree n

For z_1 and z_2 in $\mathbb{Z}^d$, let $d(z_1, z_2) = \max_{1 \leq i \leq d} |z_1(i) - z_2(i)|$, where $z(i)$, $1 \leq i \leq d$, are the coordinates of $z \in \mathbb{Z}^d$. For $Z \subset \mathbb{Z}^d$, denote by $\#(Z)$ the number of elements in Z and by ∂Z the boundary of Z, which is defined to be the set of all $z \in Z$ such that there exists $z' \notin Z$ such that $d(z, z') = 1$.

Consider the subprocess $\Phi_n \subset \Phi$ of points with degree n. Denote by $\{Z_m\}$ a fixed increasing sequence of subsets of $\mathbb{Z}^d$ such that $Z_m \uparrow \mathbb{Z}^d$ and $\lim_{m\to\infty} \#(\partial Z_m)/\#(Z_m) = 0$. Let $\xi_z^{(n)} = \Phi_n(\{z + [0,1)^d\})$ be the number of points of degree n in the cube $\{z + [0,1)^d\}$ and $S_m^{(n)} = \sum_{z \in Z_m} \xi_z^{(n)}$ be the sum of the number of points of degree n in $\{Z_m \oplus [0,1)^d\}$, where $\oplus$ denotes the Minkowski addition. We are going to prove a central limit theorem for $S_m^{(n)}$ for Φ being a stationary Poisson process.

Note that the degrees of different points in a Poisson process may still be dependent. However, this dependence is rather weak, which allows to refer to the central limit theorem for α-mixing random fields [2]. Define the mixing coefficient for the stationary random field $\{\xi_z^{(n)} : z \in \mathbb{Z}^d\}$ induced by Φ_n to be

$$\alpha_{a,b}^{(n)}(k) = \sup\{|\mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_1)\mathbf{P}(A_2)| : A_i \in \sigma(\xi_z^{(n)} : z \in Z^{(i)}),\ Z^{(i)} \subset \mathbb{Z}^d,\ i = 1, 2,$$
$$\#(Z^{(1)}) \le a,\ \#(Z^{(2)}) \le b,\ d(Z^{(1)}, Z^{(2)}) \ge k\},$$

where $k \in \mathbb{N}$, $a, b \in \mathbb{N} \cup \{\infty\}$ and $\sigma(\xi_z^{(n)} : z \in Z^{(i)})$ is the σ-algebra generated by $\{\xi_z^{(n)} : z \in Z^{(i)}\}$.

Lemma 1. (Bolthausen [2]) *Suppose $\{\xi_z^{(n)} : z \in \mathbb{Z}^d\}$ is stationary. If $\sum_{k=1}^\infty k^{d-1}\alpha_{a,b}^{(n)}(k) < \infty$ for $a+b \le 4$, $\alpha_{1,\infty}^{(n)}(k) = o(k^{-d})$, and $\mathbf{E}|\xi_z^{(n)}|^{2+\delta} < \infty$ and $\sum_{k=1}^\infty k^{d-1}\alpha_{1,1}^{(n)}(k)^{\delta/(2+\delta)} < \infty$ for some $\delta > 0$, then $\sum_{z\in\mathbb{Z}^d} |\mathbf{cov}(\xi_0^{(n)}, \xi_z^{(n)})| < \infty$, and if $\sigma_n^2 = \sum_{z\in\mathbb{Z}^d} \mathbf{cov}(\xi_0^{(n)}, \xi_z^{(n)}) > 0$, then the distribution of $\{S_m^{(n)} - \mathbf{E}(S_m^{(n)})\}/\sqrt{\#(Z_m)\sigma_n^2}$ converges weakly to the standard normal distribution as $m \to \infty$.*

First, $|\xi_z^{(n)}| = \Phi_n(\{z + [0, 1)^d\}) \le \Phi(\{z + [0, 1)^d\})$, the latter has a bounded $(2 + \delta)$-th moment for, say, $\delta = 1$.

Denote by $E_x = E_x(r)$ the event that there are at least n points in a ball centred at x with radius r. Since points of a Poisson process are independent, for x_1, x_2 in $\mathbb{R}^d$, $\mathbf{P}(E_{x_1}|E_{x_2}) \ge \mathbf{P}(E_{x_1})$ because if $\|x_1 - x_2\| \le 2r$, points in the ball centred at x_2 with radius r may also lie in the ball centred at x_1; if $\|x_1 - x_2\| > 2r$, the equality holds because of the independence. Therefore, for $x_1, \ldots, x_j$ in $\mathbb{R}^d$, $\mathbf{P}(E_{x_1} \cap \cdots \cap E_{x_j}) \ge \mathbf{P}(E_{x_1}) \cdots \mathbf{P}(E_{x_j})$. Thus, for fixed finite a and b, the probability that the first n nearest neighbours of each point, if any, in $Z^{(1)} \oplus [0, 1)^d$ and $Z^{(2)} \oplus [0, 1)^d$ are within a distance r of the point is at least

$$G(r) = \sum_{j=0}^{\infty} \left\{1 - \sum_{i=0}^{n-1} \frac{(\omega_d r^d)^i}{i!} \exp(-\omega_d r^d)\right\}^j \frac{(a+b)^j}{j!} \exp\{-(a+b)\}$$
$$= \exp\left\{-(a+b) \sum_{i=0}^{n-1} \frac{(\omega_d r^d)^i}{i!} \exp(-\omega_d r^d)\right\},$$

where ω_d is the volume of a unit ball.

Suppose that the maximum among all distances from points in $Z^{(i)} \oplus [0, 1)^d$ to their own nth nearest neighbours is R_i and let $R = \max\{R_1, R_2\}$. The above argument implies that $\mathbf{P}(R \le r) \ge G(r)$. Let $B^{(i)} \subset \mathbb{R}^d$ be the set that contains the points of Φ in $Z^{(i)} \oplus [0, 1)^d$ and their first n nearest neighbours. The number $\Phi_n(Z^{(i)} \oplus [0, 1)^d)$ is completely determined by the configuration of points of Φ in $B^{(i)}$. For events A_1 and A_2 as described in the definition of the mixing coefficient, $|\mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_1)\mathbf{P}(A_2)|$ is at most 1 and is zero if $B^{(1)}$ and

$B^{(2)}$ are disjoint, and they are disjoint whenever $R \le k/2$. More precisely, consider

$$\begin{aligned}
&|\mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_1)\mathbf{P}(A_2)| \\
&\quad = |\mathbf{P}(A_1 \cap A_2 | R_1 \le k/2,\ R_2 \le k/2)\mathbf{P}(R_1 \le k/2,\ R_2 \le k/2) \\
&\qquad - \mathbf{P}(A_1|R_1 \le k/2)\mathbf{P}(A_2|R_2 \le k/2)\mathbf{P}(R_1 \le k/2)\mathbf{P}(R_2 \le k/2) \\
&\qquad + \mathbf{P}(A_1 \cap A_2 | R_1 > k/2, R_2 > k/2)\mathbf{P}(R_1 > k/2, R_2 > k/2) \\
&\qquad - \mathbf{P}(A_1|R_1 > k/2)\mathbf{P}(A_2|R_2 > k/2)\mathbf{P}(R_1 > k/2)\mathbf{P}(R_2 > k/2) \\
&\qquad + \mathbf{P}(A_1 \cap A_2 | R_1 > k/2, R_2 \le k/2)\mathbf{P}(R_1 > k/2, R_2 \le k/2) \\
&\qquad - \mathbf{P}(A_1|R_1 > k/2)\mathbf{P}(A_2|R_2 \le k/2)\mathbf{P}(R_1 > k/2)\mathbf{P}(R_2 \le k/2) \\
&\qquad + \mathbf{P}(A_1 \cap A_2 | R_1 \le k/2, R_2 > k/2)\mathbf{P}(R_1 \le k/2, R_2 > k/2) \\
&\qquad - \mathbf{P}(A_1|R_1 \le k/2)\mathbf{P}(A_2|R_2 > k/2)\mathbf{P}(R_1 \le k/2)\mathbf{P}(R_2 > k/2)|. \qquad (2)
\end{aligned}$$

Since $\{R_1 \le k/2\}$ and $\{R_1 \le k/2\}$ are two events concerning the number of points of a Poisson process in two disjoint regions, they are independent. Moreover, given the event $\{R \le k/2\}$, A_1 and A_2 are conditional independent. Hence, the first two terms in equation (2) vanish. Thus, for any fixed finite a and b,

$$\begin{aligned}
|\mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_1)\mathbf{P}(A_2)| &\le \mathbf{P}(R_1 > k/2, R_2 > k/2) + \mathbf{P}(R_1 > k/2, R_2 \le k/2) \\
&\quad + \mathbf{P}(R_1 \le k/2, R_2 > k/2) \\
&= \mathbf{P}(R > k/2) \\
&\le 1 - G(k/2),
\end{aligned}$$

and so

$$\alpha_{a,b}^{(n)}(k) = O\left(k^{dn-d} \exp\{-\omega_d 2^{-d} k^d\}\right),$$

which implies that $\sum_{k=1}^{\infty} k^{d-1} \alpha_{a,b}^{(n)}(k)$ and $\sum_{k=1}^{\infty} k^{d-1} \alpha_{1,1}^{(n)}(k)^{\delta/(2+\delta)}$ are bounded for $a + b \le 4$ and, say, $\delta = 1$.

Finally, consider $\alpha_{1,\infty}^{(n)}(k)$. Without loss of generality, we consider $Z^{(1)} = (0, \dots, 0) \in \mathbb{Z}^d$ and $Z^{(2)}$ be the set of all integers lying in the complement of $[-(k-1), k]^d$. Denote by $z_1 = (k+1, 0, \dots, 0)$ and for $i = 2, 3, \dots, z_i = (2k+i, 0, \dots, 0) \in Z^{(2)}$. For $r < k$, denote by $E_{z_1}^*(r)$ the event that there are at least n points of Φ in the region $b(z_1, r) \oplus [0,1)^d \setminus \{z_1 \oplus [0,1)^d\}$. The event $E_{z_1}^*(r)$ implies that the first n nearest neighbours of each point of Φ in $\bigcup_{i=1}^{\infty} z_i \oplus [0,1)^d$ are at least $k - r$ away from $[0,1)^d$. Thus, the event $\bigcap_z E_z^*(k/2) \bigcap_x E_x(k/2)$, where z are all integers such that $d(z, [0,1)^d) = k$ and x are all points of Φ lying in $[0,1)^d \cup [-2k, 2k+1]^d \setminus [-k, k+1]^d$, implies that $R \le k/2$. By the same argument as in the derivation of the order of $\alpha_{a,b}^{(n)}(k)$, we have

$$\alpha_{1,\infty}^{(n)}(k) \le \left[1 - \mathbf{P}\left\{\bigcap_z E_z^*(k/2) \bigcap_x E_x(k/2)\right\}\right].$$

Because Φ is Poisson,

$$\mathbf{P}\left\{\bigcap_z E_z^*(k/2)\bigcap_x E_x(k/2)\right\}$$
$$\geq \left[1-\sum_{i=0}^{n-1}\frac{O(k^d)^i}{i!}\exp\{-O(k^d)\}\right]^{O(k^{d-1})}\exp\left[-O(k^d)\sum_{i=0}^{n-1}\frac{O(k^d)^i}{i!}\exp\{-O(k^d)\}\right],$$

and hence $\alpha_{1,\infty}^{(n)}(k)=o(k^{-d})$. From Lemma 1 we obtain the following asymptotic normality.

Theorem 2. *Suppose that Φ is a stationary Poisson process. If $\sigma_n^2>0$, then the distribution of $\{S_m^{(n)}-\mathbf{E}(S_m^{(n)})\}/\sqrt{\#(Z_m)\sigma_n^2}$ converges weakly to the standard normal distribution as $m\to\infty$.*

Since the configuration of points of Φ in $Z^{(i)}\oplus[0,1)^d$ and their first n nearest neighbours determines not only $\Phi_n(Z^{(i)}\oplus[0,1)^d)$ but also $\Phi_j(Z^{(i)}\oplus[0,1)^d)$ for $d+1\leq j\leq n$, the above argument immediately leads to a central limit theorem for the sum of degrees.

Theorem 3. *Let $N_m^{(n)}=\sum_{j=d+1}^n jS_m^{(j)}$. If $\sigma_n^{*2}=\sum_{z\in\mathbb{Z}^d}\mathbf{cov}(\sum_{j=d+1}^n j\xi_0^{(j)},\sum_{j=d+1}^n j\xi_z^{(j)})>0$, then $\{N_m^{(n)}-\mathbf{E}(N_m^{(n)})\}/\sqrt{\#(Z_m)\sigma_n^{*2}}$ converges weakly to the standard normal distribution as $m\to\infty$.*

Remark. Similar to many central limit theorems for mixing sequences or mixing random fields, the positivity of the variances σ_n^2 and σ_n^{*2} is given as conditions, because the calculation of the variance term requires not only the vanishing mixing coefficient but a thorough knowledge of the dependence structure, which seems intractable here.

4 Statistical estimation

For statistical application, it is important to be able to estimate the distribution of N from a given realisation observed in a bounded window. However, the existence of the edge-effects is quite pronounced in this case. See Figure 2. Nevertheless, the edge-effects here can be considered as a right censoring and so we suggest an unbiased estimator which is the discrete version of that in Chiu [3].

Denote by W the sampling window. For a point x_i in W, let $r_i^{(j)}$ be the distance from the point to its j-th nearest neighbour. Let $n_i^{\max}$ be the largest j such that a ball centred at the point x_i with radius $r_i^{(j)}$ lies entirely in W. Suppose the point has a degree of n_i and n_i^* with respect to the whole realisation of the point process and the observed points in W, respectively. If x_i does not lie in the interior of the convex hull of all other points in W, let n_i^* be infinity. For a point x_i in W we can observe $(n_i^*,n_i^{\max})$. If $n_i^*\leq n_i^{\max}$, then the degree of x_i is not affected by the edge-effects and so $n_i=n_i^*$. However, if $n_i^*>n_i^{\max}$, then n_i may be equal to, less than or greater than n_i^*; we only know that it happens if and only if $n_i>n_i^{\max}$. Note that if the point process is stationary Poisson, the distribution of the directions from x_i to its nearest points is independent of the distances to these points,

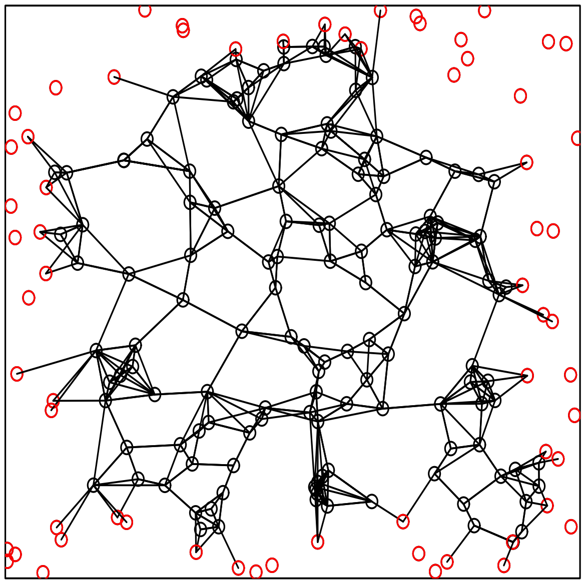

Figure 2: The proposed graph constructed from a realisation of a stationary planar Poisson process observed in a unit square. Grey circles are points subjected to the edge-effects.

i.e. n_i and $n_i^{\max}$ are independent. Thus, the observed pairs $\{(n_i^*, n_i^{\max})\}$ from W lead to the triples $\{(h_i, \delta_i, n_i^{\max})\}$, where $h_i = \min(n_i^*, n_i^{\max}) = \min(n_i, n_i^{\max})$ is the observed degree of x_i subjected to random censorship and $\delta_i = \mathbf{1}(n_i^* \le n_i^{\max}) = \mathbf{1}(n_i \le n_i^{\max})$ is the censoring indicator. Let

$$\widehat{p}_n = \sum_i \frac{\mathbf{1}(h_i = n, \delta_i = 1)}{\sum_j \mathbf{1}(n \le n_j^{\max})}, \tag{3}$$

with the convention that $0/0 = 0$. If n_i are identically distributed and independent of $n_i^{\max}$,

$$\begin{aligned}\mathbf{E}\widehat{p}_n &= \sum_i \mathbf{E}\frac{\mathbf{1}(n_i = n, n_i \le n_i^{\max})}{\sum_j \mathbf{1}(n_i \le n_j^{\max})} = \sum_i \mathbf{E}\frac{\mathbf{1}(n_1 = n, n_1 \le n_i^{\max})}{\sum_j \mathbf{1}(n_1 \le n_j^{\max})} \\ &= \mathbf{E}\mathbf{1}(n_1 = n)\frac{\sum_i \mathbf{1}(n_1 \le n_i^{\max})}{\sum_j \mathbf{1}(n_1 \le n_j^{\max})}.\end{aligned}$$

Thus, unless n is larger than any $n_j^{\max}$, we have $\mathbf{E}\widehat{p}_n = \mathbf{P}(N = n)$. This proves the following statement.

Theorem 4. *If Φ is a stationary Poisson process, then $\widehat{p_n}$ is a conditionally unbiased estimator of $\mathbf{P}(N = n)$, given that $n \leq \max_i\{n_i^{\max}\}$.*

Theorem 4 suggests that it is sensible to use the estimator given in equation (3) even in the non-Poisson case.

5 Distribution of the degree of the typical point of a non-Poisson process

The degrees of points in regular planar square lattices and hexagonal networks are 4 and 3, respectively. Suppose we put discs of unit radius on the plane such that the inter-centre distances are at least, say, 10. If on the circumference of each disc we place n points uniformly, then each point has a degree at least n. Thus, it seems that a low and high mean degree suggest that the point process exhibits regularity, while a high mean degree suggests that it exhibits clustering. However, the point patterns constructed by the above methods are not in general position. In this section we apply the estimator proposed in the previous section to simulated realisations of the Strauss process, the Poisson cluster process and the cell process [1], the last of these generates realisations that exhibit both regularity and clustering but has the same K-function as the stationary Poisson process.

Figure 3 gives the corresponding estimated distributions of the degree of the typical point for 20 independent samples each. We can see that the tail of the distribution from the regular and cluster point patterns are lighter and heavier, respectively, than that from the Poisson process. The tail from the cell process captures the regularity of the pattern. Boxplots shown in Figure 4 confirm that the mean degree can be used as a statistic to discriminate clustering and repulsive behaviour of the processes.

6 Miscellaneous remarks

Such a graph can also be constructed even if the given point process is non-stationary. However, there is no typical point in this case. Nevertheless, we still can study the degree N_o of the origin, at which there may or may not be a point of the process. For a stationary Poisson process, the distribution of N does not depend on the intensity. For a non-stationary Poisson process, such an invariance is still true for N_o in the sense that the distribution of N_o remains the same when the intensity function is multiplied by a finite positive constant. The reason is as follows.

Consider the polar coordinates $(r, \boldsymbol{\theta})$ of points in the process. As discussed in Section 2, the distribution of N_o can be obtained by considering complete coverage of a sphere by independent hemispheres. In case of non-stationarity, the hemisphere corresponding to $x_i = (r_i, \boldsymbol{\theta}_i)$ arrives at time r_i with the pole at the position $\boldsymbol{\theta}_i$. The degree N_o is the number of these hemispheres that have arrived when the sphere is first completely covered. Thus, multiplying the intensity function by a finite positive constant only leads to a rescaling of the time-axis but change neither the order of arrivals nor the positions of the poles.

It should be noted that the construction used in the paper can be applied to all points in $\mathbb{R}^d$ and not necessarily to the points of the process. For $x \in \mathbb{R}^d$ let M_x denote the set of points belonging to the point process that are connected to x. This yields an equivalence relationship, namely $x \sim y$ if and only if $M_x = M_y$, which may be used to construct a tessellation of $\mathbb{R}^d$. Properties of such a tessellation will be left for future endeavour.

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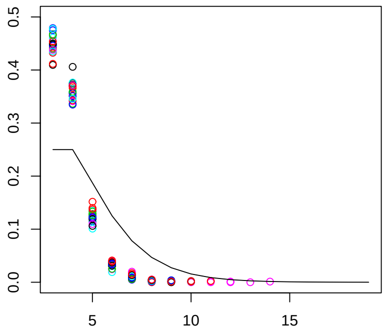
References

[1] Baddeley, A. J. and Silverman, B. W. (1984). A cautionary example on the use of second-order methods for analyzing point patterns. *Biometrics* **40**, 1089–1093.

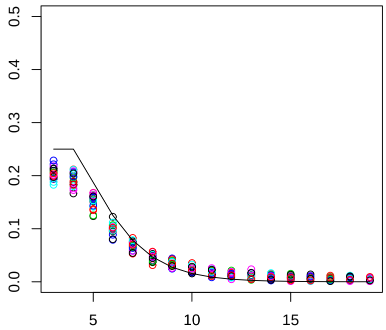
[2] Bolthausen, E. (1982). On the central limit theorem for stationary mixing random fields. *Ann. Probab.* **10**, 1047–1050.

[3] Chiu, S. N. (1999). An unbiased estimator for the survival function of censored data. *Comm. Statist. Theory Methods* **28**, 2249–2260.

[4] Hall, P. (1988). *Introduction to the Theory of Coverage Processes.* Wiley, New York.



(a)



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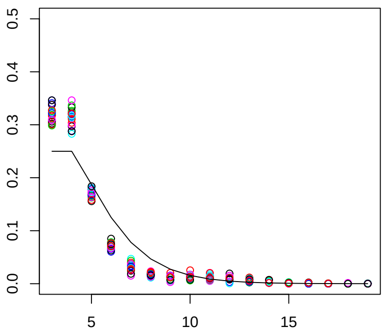

(c)

Figure 3: Estimated distributions of the degree of the typical point of 20 independent realisations in a unit square of (a) the Strauss process (1000 points, hard-core radius 0.02), (b) the Poisson cluster process (50 parent points, 20 points on average in each cluster uniformly in a disk of radius 0.05 centred at the parent), (c) the cell process (unit square split into 1600 equal square cells). The distribution of the degree of the typical point of a stationary planar Poisson process is joined together by a solid line for visual comparison.

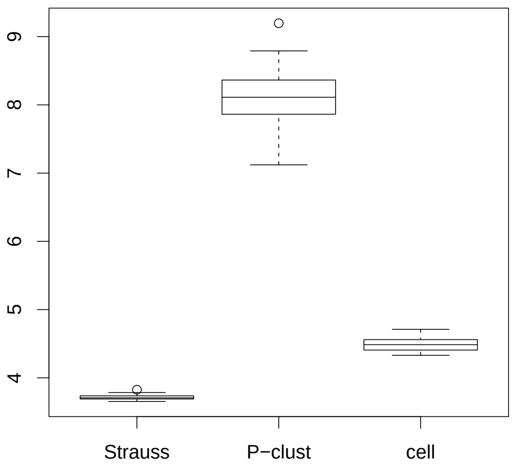

Figure 4: The boxplots of the sample means of the degrees for the typical point of the Strauss process, the Poisson cluster process and the cell process.