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A DIFFUSION PERTURBED RISK PROCESS WITH STOCHASTIC RETURN ON INVESTMENTS

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ABSTRACT

A general risk model that allows for stochastic return on investments as well as perturbation by diffusion is studied. Integro-differential equations for the distributions of the time of ruin, the surplus prior to ruin and the deficit at ruin of this model are established. In particular, we consider a diffusion perturbed risk model with interest force in details.

Keywords: Risk process; Time of ruin; Ruin probability; Stochastic return

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1. INTRODUCTION

Paulsen (1) introduced the following risk process U_t that allows for a stochastic rate of return on investments as well as a stochastic rate of inflation:

$$U_t = \frac{\mathcal{E}(R)_t}{\mathcal{E}(I)_t}\left(u + \int_0^t \frac{\mathcal{E}(I)_{t-}}{\mathcal{E}(R)_{t-}} \mathrm{d}P_s\right).$$

The notation $\mathcal{E}(A)$ denotes the Doléans-Dade exponential of A given as the solution of the stochastic growth equation $\mathcal{E}(A)_t = 1 + \int_0^t \mathcal{E}(A)_{s-}\mathrm{d}A_s$, and P_t, I_t and R_t are all semi-martingales representing the surplus generating process, the inflation generating process and the return on investment generating process, respectively. The initial values are $P_0 = u, I_0 = 0$ and $R_0 = 0$. Paulsen and Gjessing (2) simplified the model above by assuming that there is no inflation and both the surplus generating process P_t and the return on investment generating process R_t are the classical risk processes perturbed by Brownian motions. Paulsen (3) considered a risk process U_t given by

$$U_t = u + P_t + \int_0^t U_{s-}\mathrm{d}R_s, \text{ with } P_0 = R_0 = 0, \tag{1.1}$$

where P_t and R_t are independent Lévy processes. Using the above notation (1.1) can be written as $U_t = \mathcal{E}(R)_t(u + \int_0^t \mathcal{E}(R)_{s-}^{-1}\mathrm{d}P_s)$. For further references see (4).

The present paper generalizes the model given in (1.1) by considering that P_t is a general jump-diffusion process, whilst R_t is still a Lévy process. The organization of the paper is as follows. Section 2 presents a precise formulation of our model. Integro-differential equations for the distributions of time of ruin, surplus prior to ruin and deficit at ruin are established in Section 3. The last section studies the model with interest force.

2. THE MODEL

Assume that all processes and random variables are defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P})$, where $\mathcal{F}_t$ is right-continuous and P-complete. Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the real line and the non-negative half line endowed with the Borel σ-algebra $\mathcal{B}$ and $\mathcal{B}^+$, respectively. A risk process can be described as a marked point process $\{T_n, Z_n\}_{n\geq 1}$, where T_i's are non-negative and represent the times when claims occur, and the marks Z_i's are the corresponding claim amounts, which are assumed to be i.i.d. and positive. In risk theory (e.g. (5)), one often defines the aggregate claim amount at time t as $S_t = \sum_{i=1}^{\infty} \mathbf{1}(T_i \leq t) Z_i$. Møller (6, 7) considered a more general form of the aggregate claim amount, and introduced the risk reserve model of the form

$$V_t = V_0 + \int_0^t b(s, V_s)\mathrm{d}s - \int_0^t \int_0^{\infty} f(s,z) N(\mathrm{d}s, \mathrm{d}z), \tag{2.1}$$

where $b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a piecewise continuous function representing the premium income, $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ a Borel measurable function, $N(t, D) := \sum_{i=1}^{\infty} \mathbf{1}(T_i \leq t, Z_i \in D)$ a locally finite (not necessarily Poisson) counting measure that counts the number of the claims over $(0,t]$ with claim amounts taking values in $D \in \mathcal{B}^+$. The last term in the right-hand side of (2.1) represents the total amount of claims in the time interval $(0,t]$, which can be written as $\sum_{i=1}^{N(t,\mathbb{R}^+)} f(T_i, Z_i)$, where each claim amount may be multiplied by a discount factor.

We define the risk process U_t as the solution of the stochastic differential equation

$$\begin{aligned} U_t = u &+ \int_0^t b(s, U_{s-})\mathrm{d}s + \int_0^t a(s, U_{s-})\mathrm{d}B_s \\ &- \int_0^t \int_0^{\infty} f(s, U_{s-}, z) N(\mathrm{d}s, \mathrm{d}z) + \int_0^t U_{s-}\mathrm{d}R_s, \quad t \geq 0, \end{aligned} \tag{2.2}$$

where u is the initial assets, $\{B\}_{t\geq 0}$ is a standard Brownian motion independent of the inhomogeneous Poisson counting measure $N(\cdot,\cdot)$ and $a, b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^+ \times \mathbb{R} \times$

$\mathbb{R}^+ \to \mathbb{R}^+$ are measurable functions satisfying the condition that for positive m and M, there exists a positive constant $K(m, M)$ such that for $t \le m$ and $|x|, |y| \le M$,

$$\begin{cases} |a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \\ \qquad\qquad + \int_0^\infty |f(t,x,u) - f(t,y,u)|\lambda(t,\mathrm{d}u) \le K(m,M)|x-y|, \\ |a(t,x)| + |b(t,x)| + \int_0^\infty |f(t,x,u)|\lambda(t,\mathrm{d}u) \le K(m,M)(1+|x|), \end{cases} \tag{2.3}$$

where $\lambda(t, D)$ denotes the compensator of the point process $\{N(t, D)\}$. The process $\{R_t\}_{t\ge0}$ is a Lévy process independent of $\{P_t\}_{t\ge0}$ and having the Lévy–Itô decomposition (see (3))

$$R_t = rt + \sigma_R B_{R,t} + \int_0^t \int_{-\infty}^\infty x(\mu_R(\mathrm{d}s, \mathrm{d}x) - \mathbf{1}(|x| \le 1)K_R(\mathrm{d}x)\mathrm{d}s),$$

where r and σ_R are positive constants, $\{B_{R,t}\}_{t\ge0}$ is another standard Brownian motion independent of μ_R which is the Poisson counting measure associated with the jumps of R_t, $K_R(\mathrm{d}x)\mathrm{d}s$ is the compensator of $\mu_R(\mathrm{d}s, \mathrm{d}x)$ and such that $K_R(\{0\}) = 0$ and $\int_{-\infty}^\infty \min(1, x^2)K_R(\mathrm{d}x) < \infty$. For simplicity, assume that $K_R((-\infty, -1]) = 0$.

Because the quadratic variational processes of $\int_0^t a(s, U_{s-})\mathrm{d}B_s + \int_0^t \sigma_R U_{s-}\mathrm{d}B_{R,s}$ and $\int_0^t \sqrt{a^2(s, U_{s-}) + \sigma_R^2 U_{s-}^2}\mathrm{d}W_s$ are the same, where $\{W\}_{t\ge0}$ is a standard Brownian motion independent of the compound Poisson processes involved, by Ikeda and Watanabe [(8), p.85], they have the same distributions. Thus, in distribution, we have

$$\begin{aligned} U_t = u &+ \int_0^t (b(s, U_s) + rU_s)\mathrm{d}s \\ &+ \int_0^t \sqrt{a^2(s, U_{s-}) + \sigma_R^2 U_{s-}^2}\mathrm{d}W_s - \int_0^t \int_0^\infty f(s, U_{s-}, z)N(\mathrm{d}s, \mathrm{d}z) \\ &+ \int_0^t \int_{-\infty}^\infty xU_{s-}(\mu_R(\mathrm{d}s, \mathrm{d}x) - \mathbf{1}(|x| \le 1)K_R(\mathrm{d}x)\mathrm{d}s). \end{aligned} \tag{2.4}$$

If $a(t, x) = a(t), b(t, x) = b(t)$ and $f(t, x, z) = f(t, z)$, then the solution of (2.2) is

$$U_t = \mathcal{E}(R)_t \left(u + \int_0^t \mathcal{E}(R)_{s-}^{-1}\mathrm{d}Q_s\right),$$

where

$$Q_t = u + \int_0^t b(s)\mathrm{d}s + \int_0^t a(s)\mathrm{d}B_s - \int_0^t \int_0^\infty f(s,z)N(\mathrm{d}s,\mathrm{d}z),$$
$$\mathcal{E}(R)_t = \mathrm{e}^{R_t - \frac{1}{2}\sigma_R^2 t} \prod_{0\le s\le t} (1 + R_s - R_{s-})\mathrm{e}^{-(R_s - R_{s-})}.$$

A natural generalization of (2.4) is the time inhomogeneous risk process obtained by replacing the integrands of the first two integrals in the right-hand side of (2.4) by two general measurable functions, say, q and σ, i.e.

$$\begin{aligned} U_t = u &+ \int_0^t q(s,U_s)\mathrm{d}s + \int_0^t \sigma(s,U_s)\mathrm{d}W_s - \int_0^t \int_0^\infty f(s,U_{s-},z)N(\mathrm{d}s,\mathrm{d}z) \\ &+ \int_0^t \int_{-\infty}^\infty xU_{s-}(\mu_R(\mathrm{d}s,\mathrm{d}x) - \mathbf{1}(|x|\le 1)K_R(\mathrm{d}x)\mathrm{d}s). \end{aligned} \tag{2.5}$$

If q, σ and f satisfy condition (2.3), then U_t defined by (2.5) is a strong Markov process (see e.g. (9), Theorem 32).

Using Itô's formula, we can prove that the infinitesimal generator $\mathcal{L}$ of $\{U_t\}_{t\ge 0}$ in (2.5) is given by

$$\begin{aligned} \mathcal{L}_{t,y}g(t,y) = \frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2(t,y)\frac{\partial^2 g}{\partial y^2} + q(t,y)\frac{\partial g}{\partial y} + \int_0^\infty [g(t,y-f(t,y,z)) - g(t,y)]\lambda(t,\mathrm{d}z) \\ + \int_{-\infty}^\infty [g(t,y+yx) - g(t,x) - \mathbf{1}(|x|\le 1)xy\frac{\partial g}{\partial y}]K_R(\mathrm{d}x). \end{aligned} \tag{2.6}$$

3. INTEGRO-DIFFERENTIAL EQUATIONS

Let $T(s) = \inf\{t \ge s : U_t < 0\}$ with the usual convention that $\inf\emptyset = \infty$. For $\alpha \ge 0$, denote $\psi_1(s,u) = \mathrm{P}(T(s) < \infty | U_s = u)$, $\psi_2(s,u,t) = \mathrm{P}(T(s) < t | U_s = u)$, $\psi_3(s,\alpha,u) = \mathrm{E}[\mathrm{e}^{-\alpha T(s)} | U_s = u]$, $H_1(s,u;x,y) = \mathrm{P}(T(s) < \infty, U_{T(s)-} \le x, -U_{T(s)} \le y | U_s = u)$, $H_2(s,u;x,y,t) = \mathrm{P}(T(s) < t, U_{T(s)-} \le x, -U_{T(s)} \le y | U_s = u)$, and $H_3(s,u;x,y) = \mathrm{E}(\mathrm{e}^{-\alpha T(s)}\mathbf{1}(T(s) < \infty, U_{T(s)-} \le x, -U_{T(s)} \le y) | U_s = u)$. We will derive integro-differential equations for H_i's.

Theorem 3.1. *Assume that H_i's have bounded continuous derivatives with respect to t and s and bounded continuous first and second derivatives with respect to $u \geq 0$. We have*

(i) *$H_1(s,u) := H_1(s,u;x,y)$ satisfies $\mathcal{L}_{s,u}H_1(s,u) = 0$ in $(0,\infty)\times(0,\infty)$, together with the boundary conditions, for $s,x,y>0$,*

$$\begin{aligned} H_1(s,u;x,y) &= \mathbf{1}(-y \leq u), \quad && u \leq 0, \\ H_1(s,u;x,y) &\to 0, && u \to \infty. \end{aligned}$$

(ii) *$H_2(s,u,t) := H_2(s,u;x,y,t)$ satisfies $-\frac{\partial H_2(s,u,t)}{\partial t} + \mathcal{L}_{s,u}H_2(s,u,t) = 0$ in $(0,\infty)\times(0,\infty)\times(s,\infty)$, together with the boundary and initial conditions, for $s,x,y>0$,*

$$\begin{aligned} H_2(s,u;x,y,t) &= \mathbf{1}(-y \leq u), \quad && t > s, u \leq 0, \\ H_2(s,u;x,y,t) &\to 0, && t > s, u \to \infty, \\ H_2(s,u;x,y,t) &\to 0, && t \to s, u < 0. \end{aligned}$$

(iii) *$H_3(s,u) := H_3(s,u;x,y)$ satisfies $\mathcal{L}_{s,u}H_3(s,u) = \alpha H_3(s,u)$ in $(0,\infty)\times(0,\infty)$, together with the boundary conditions, for $s,x,y>0$,*

$$\begin{aligned} H_3(s,u;x,y) &= \mathbf{1}(-y \leq u), \quad && u \leq 0, \\ H_3(s,u;x,y) &\to 0, && u \to \infty. \end{aligned}$$

Proof. (i) By the strong Markov property of U_t, it can be proved that $H_1(t\wedge T(s), U_{t\wedge T(s)})$ is an $\mathcal{F}_t$- martingale on $t \in [s,\infty)$. Applying Itô's formula gives

$$H_1(t\wedge T(s), U_{t\wedge T(s)}) - H_1(s,u) = \int_0^{t\wedge T(s)} \mathcal{L}_{r,u}H_1(r,U_r)\mathrm{d}r + \text{martingale}.$$

It follows that $\int_0^{t\wedge T(s)} \mathcal{L}_{r,u}H_1(r,U_r)\mathrm{d}r$ is a zero-mean $\mathcal{F}_t$-martingale. Hence, $\int_0^{t\wedge T(s)} \mathcal{L}_{r,u}H_1(r,U_r)\mathrm{d}r = 0$, and thus $\mathcal{L}_{s,u}H_1(s,u) = 0$, since t is arbitrary and $\mathcal{L}_{s,u}H_1(s,u)$ is continuous. The boundary conditions are obvious.

(ii) and (iii) can be proved by using the same argument as in (i).

Remark 3.1. Notice that $\psi_1(s,u) = H_1(s,u;\infty,\infty)$, $\psi_2(s,u,t) = H_2(s,u;\infty,\infty,t)$ and $\psi_3(s,u) = H_3(s,u;\infty,\infty)$. Thus, from Theorem 3.1 we can obtain the integro-differential equations for ψ_i's. In particular, for the homogeneous Markov process $\{U_t\}_{t\geq 0}$ in (2.5) with $f(t,y,z) = z$, $q(s,\cdot) = q(\cdot)$, $\sigma(s,\cdot) = \sigma(\cdot)$ and $\lambda(t,\mathrm{d}z) = \lambda \mathrm{d}G(z)$, where G is the common distribution function of the claim sizes and λ is the constant intensity of the homogeneous Poisson process N, we obtain the integro-differential equations for $\psi_1(0,u)$,$\psi_2(0,u,t)$ and $\psi_3(0,u)$ that have been considered in (2).

4. RISK MODEL WITH INTEREST FORCE

We have shown in the last section that the probability of ultimate ruin and the distributions of the time of ruin, the surplus immediately prior to ruin and the deficit at ruin satisfy certain integro-differential equations. However, the equations involved are so complicated that explicit solutions are generally hard to get even for particular cases. Examples of the solutions for some non-trivial special cases can be found in (2). In this section we give the Laplace–Stieltjes transform of H_1 as well as its estimation for the compound Poisson risk model with a constant interest force.

Consider the risk process U_t in (2.5) with $q(s,x) = c + \rho x$, $\sigma(s,x) = \sigma$, $\mu_R \equiv 0$ and $K_R \equiv 0$, i.e. $\mathrm{d}U_t = c\mathrm{d}t + \rho U_t \mathrm{d}t + \sigma \mathrm{d}W_t - \mathrm{d}X_t$, its solution is given by

$$U_t = u\mathrm{e}^{\rho t} + c\int_0^t \mathrm{e}^{\rho y}\mathrm{d}y + \sigma\int_0^t \mathrm{e}^{\rho(t-s)}\mathrm{d}W_s - \int_0^t \mathrm{e}^{\rho(t-v)}\mathrm{d}X_v,$$

where $X_t = \int_0^t\int_0^\infty zN(\mathrm{d}s,\mathrm{d}z)$, $u > 0$ is the initial capital, $c > 0$ the premium rate and $\rho > 0$ the interest force. In order to indicate the role of ρ and σ, we let $\psi_{\rho,\sigma}(u) := \psi_1(u) = \mathrm{P}(\inf_{t\geq 0} U_t < 0 | U_0 = u)$ denote the ultimate ruin probability for this risk process. For the claim distribution G with mean μ, denote by $L(x)=\mu^{-1}\int_0^x(1-G(y))\mathrm{d}y$ the integrated tail of the claim size distribution. Moreover, let

$$Q(\alpha,x,y):=\int_0^x \mathrm{e}^{-\alpha u}\mathrm{d}L(u)-\mathrm{e}^{\alpha y}\int_y^{y+x}\mathrm{e}^{-\alpha u}\mathrm{d}L(u),\ \alpha,x,y>0.$$

The Laplace-Stieltjes transform of a function is denoted by putting a hat on the top of the function, for example, $\hat{G}(\beta)=\int_0^\infty \mathrm{e}^{-\beta u}\mathrm{d}G(u)$. Because the process U_t is homogeneous, we consider $H_1(0,u;x,y)$ instead of $H_1(s,u;x,y)$ and denote it by $H_1(u;x,y)$.

Theorem 4.1. *Assume that $H_1(u):=H_1(u;x,y)$ has bounded continuous first and second derivatives. Then*

(i) If $\sigma=0$, then

$$\hat{H}_1(\beta)=\int_\beta^\infty \frac{\lambda\mu}{\rho}\Big(H_1(0)\hat{L}(\alpha)-Q(\alpha,x,y)\Big)\exp\left(-\frac{a_1(\alpha)-a_1(\beta)}{\rho}\right)\mathrm{d}\alpha,$$

where $a_1(\beta)=c\beta-\lambda\mu\int_0^\beta\hat{L}(\alpha)\mathrm{d}\alpha$, and

$$H_1(0)=\frac{\lambda\mu\int_0^\infty Q(\alpha,x,y)\exp\left(-\frac{a_1(\alpha)}{\rho}\right)\mathrm{d}\alpha}{\rho+\lambda\mu\int_0^\infty\hat{L}(\alpha)\exp\left(-\frac{a_1(\alpha)}{\rho}\right)\mathrm{d}\alpha}. \tag{4.1}$$

(ii) If $\sigma\neq 0$, then

$$\hat{H}_1(\beta)=-\frac{1}{\rho}\int_\beta^\infty\left(\frac{1}{2}\sigma^2h_2'(0)-\lambda\mu\hat{L}(\alpha)+\lambda\mu Q(\alpha,x,y)\right)\exp\left(-\frac{a_2(\alpha)-a_2(\beta)}{\rho}\right)\mathrm{d}\alpha,$$

where $a_2(\beta)=(c+\frac{1}{2}\sigma^2)\beta-\lambda\mu\int_0^\beta\hat{L}(\alpha)\mathrm{d}\alpha$, and

$$\frac{1}{2}\sigma^2h_2'(0)=\frac{\rho+\lambda\mu\int_0^\infty\exp\left(-\frac{a_2(\alpha)}{\rho}\right)\Big(\hat{L}(\alpha)-Q(\alpha,x,y)\Big)\mathrm{d}\alpha}{\int_0^\infty\exp\left(-\frac{a_2(\alpha)}{\rho}\right)\mathrm{d}\alpha}.$$

Proof. It follows from Theorem 3.1(i) that $H_1(u)$ satisfies

$$\frac{1}{2}\sigma^2H_1''(u)+(\rho u+c)H_1'(u)+\lambda\int_0^u H_1(u-z)\mathrm{d}G(z)$$
$$+\lambda(G(u+y)-G(u))\mathbf{1}(u\le x)=\lambda H_1(u),\quad u>0. \tag{4.2}$$

Integrating (4.2) gives

$$\frac{1}{2}\sigma^2(H_1'(u) - H_1'(0)) + (\rho u + c)H_1(u) - c + \lambda \int_0^{u\wedge x} (G(y+v) - G(v))\mathrm{d}v$$
$$= \rho \int_0^u H_1(v)\mathrm{d}v + \lambda \int_0^u H_1(u-z)(1-G(z))\mathrm{d}z,\ u>0. \tag{4.3}$$

(i) Consider the case $\sigma = 0$. To avoid complications with the jump of $H_1(\cdot)$ at 0, just as Sundt and Teugels (10), we introduce the auxiliary distribution

$$h_1(u) := h_1(u; x, y) := 1 - \frac{H_1(u; x, y)}{H_1(0; x, y)},$$

from which we have $H_1(u; x, y) = H_1(0; x, y)(1 - h_1(u; x, y))$ and $h_1(0; x, y) = 0$. In terms of $h_1(u)$, equation (4.3) can be written as

$$ch_1(u) = -\rho \int_0^u v\mathrm{d}h_1(v) + \lambda\mu(h_1 * L)(u)$$
$$- \lambda\mu L(u) + \frac{\lambda}{H_1(0)} \int_0^{u\wedge x} (G(y+v) - G(v))\mathrm{d}v, \tag{4.4}$$

where $*$ denotes the Stieltjes convolution.

Taking the Laplace-Stieltjes transform of (4.4) gives

$$c\hat{h}_1(\beta) = \rho\hat{h}_1'(\beta) + \lambda\mu\hat{h}_1(\beta)\hat{L}(\beta) - \lambda\mu\hat{L}(\beta) + \frac{\lambda\mu}{H_1(0)} Q(\beta, x, y). \tag{4.5}$$

The solution of (4.5) with the boundary condition $\lim_{\beta\to\infty} \hat{h}_1(\beta) = 0$ is given by

$$\hat{h}_1(\beta) = -\frac{\lambda\mu}{\rho} \int_\beta^\infty \left(\hat{L}(\alpha) - \frac{Q(\alpha, x, y)}{H_1(0)}\right) \exp\left(-\frac{a_1(\alpha) - a_1(\beta)}{\rho}\right) \mathrm{d}\alpha, \tag{4.6}$$

where $a_1(\beta) = c\beta - \lambda\mu \int_0^\beta \hat{L}(\alpha)\mathrm{d}\alpha$.

Letting $\beta \to 0$ in (4.6) yields (4.1), and because $\hat{H}_1(\beta) = -H_1(0)\hat{h}_1(\beta)$, the result follows.

(ii) Consider $\sigma \neq 0$. In this case $H_1(0) = 1$, and so the auxiliary distribution is

$$h_2(u) := h_2(u; x, y) := 1 - H_1(u; x, y).$$

In terms of $h_2(u)$, equation (4.3) can be written as

$$\begin{aligned}\frac{1}{2}\sigma^2(h_2'(u) - h_2'(0)) + ch_2(u) = &- \rho \int_0^u v \mathrm{d}h_2(v) + \lambda\mu(h_2 * L)(u) \\ &- \lambda\mu L(u) + \lambda \int_0^{u \wedge x} (G(y+v) - G(v))\mathrm{d}v.\end{aligned} \tag{4.7}$$

Taking the Laplace-Stieltjes transform of (4.7) gives

$$\frac{1}{2}\sigma^2(-h_2'(0) + \beta\hat{h}_2(\beta)) + c\hat{h}_2(\beta) = \rho\hat{h}_2'(\beta) + \lambda\mu\hat{h}_2(\beta)\hat{L}(\beta) - \lambda\mu\hat{L}(\beta) + \lambda\mu Q(\beta, x, y). \tag{4.8}$$

The solution of (4.8) with the boundary condition $\lim_{\beta\to\infty} \hat{h}_2(\beta) = 0$ is given by

$$\hat{h}_2(\beta) = \frac{1}{\rho}\int_\beta^\infty \left(\frac{1}{2}\sigma^2 h_2'(0) - \lambda\mu\hat{L}(\alpha) + \lambda\mu Q(\alpha, x, y)\right) \exp\left(-\frac{a_2(\alpha) - a_2(\beta)}{\rho}\right) \mathrm{d}\alpha, \tag{4.9}$$

where $a_2(\beta) = (c + \frac{1}{2}\sigma^2)\beta - \lambda\mu \int_0^\beta \hat{L}(\alpha)\mathrm{d}\alpha$.

Letting $\beta \to 0$ in (4.9) yields the result.

Theorem 4.2. *Suppose that the safety loading $c - \lambda\mu$ is positive and $\rho, x, y > 0$. We have*

$$\frac{B_\sigma(u; x, y)}{u} - \int_u^\infty \frac{A_\sigma(z; x, y)}{z^2}\mathrm{d}z \le H_1(u; x, y) \le \frac{A_\sigma(u, x, y)}{u} - \int_u^\infty \frac{B_\sigma(z; x, y)}{z^2}\mathrm{d}z, \tag{4.10}$$

where

$$\begin{aligned}A_\sigma(u,x,y) =&\frac{\lambda}{\rho(1-\psi_{0,\sigma}(u))}\left(\int_u^\infty (G(v+y)-G(v))\mathrm{d}v - \psi_{0,\sigma}(u)\right.\\ &\left.\times\int_0^\infty (G(v+y)-G(v))\mathrm{d}v \,+\, \psi_{0,\sigma}(u) * \int_0^u (G(v+y)-G(v))\mathrm{d}v\right),\\ B_\sigma(u;x,y) =&\frac{\lambda}{\rho+u^{-1}c}\int_u^\infty (G(v+y)-G(v))\mathrm{d}v,\\ 1-\psi_{0,0}(u) =& \left(1-\frac{\lambda\mu}{c}\right)\sum_{n=0}^\infty \left(\frac{\lambda\mu}{c}\right)^n L^{n,*}(u),\\ 1-\psi_{0,\sigma}(u) =& \left(1-\frac{\lambda\mu}{c}\right)\sum_{n=0}^\infty \left(\frac{\lambda\mu}{c}\right)^n (J^{(n+1),*} * L^{n,*})(u),\end{aligned}$$

with $J(u) = \left(1-\exp(-\frac{2c}{\sigma^2}u)\right)\mathbf{1}(u>0)$, $L^{n,*}$, *for* $n \geq 1$, *denotes the* n*-fold convolution of* L *with itself and* $L^{0,*}$ *is the distribution function corresponding to the Dirac measure at zero.*

Proof. Consider the case $\sigma = 0$. By (4.5) we have

$$\hat{h}_1(\beta) = \sum_{n=0}^\infty \left(\frac{\lambda\mu}{c}\hat{L}(\beta)\right)^n \left(\frac{\rho}{c}\hat{h}_1'(\beta) - \frac{\lambda\mu}{c}\hat{L}(\beta) + \frac{\lambda\mu}{cH_1(0)}Q(\beta,x,y)\right).$$

Inverting it gives

$$\begin{aligned}h_1(u) =&\frac{\rho}{c}\sum_{n=0}^\infty \left(\frac{\lambda\mu}{c}\right)^n (L^{n,*} * k_1)(u) - \sum_{n=0}^\infty \left(\frac{\lambda\mu}{c}\right)^{n+1} L^{(n+1),*}(u)\\ &+\frac{\lambda}{cH_1(0)}\sum_{n=0}^\infty \left(\frac{\lambda\mu}{c}\right)^n L^{n,*}(u) * \int_0^u (G(v+y)-G(v))\mathrm{d}v,\end{aligned} \tag{4.11}$$

where $k_1(u) = \int_u^\infty z\mathrm{d}h_1(z)$.

Letting $u \to \infty$ yields

$$H_1(0) = \frac{\lambda\int_0^\infty (G(v+y)-G(v))\mathrm{d}v}{c+\rho\int_0^\infty u\mathrm{d}h_1(u)}. \tag{4.12}$$

We can rewrite (4.11), in terms of $\psi_{0,0}$, as

$$\begin{aligned} h_1(u) =& \frac{\rho}{c-\lambda\mu}((1-\psi_{0,0})*k_1)(u) - \frac{\lambda\mu}{c-\lambda\mu}((1-\psi_{0,0})*L)(u) \\ &+ \frac{\lambda}{H_1(0)(c-\lambda\mu)}((1-\psi_{0,0}(u))*\int_0^u (G(v+y)-G(v))\mathrm{d}v. \end{aligned}$$

Because $1 \geq h_1(u)$ and $((1-\psi_{0,0})*k_1)(u) \geq (1-\psi_{0,0}(u))(k_1(u)-k_1(0))$, it follows that

$$\begin{aligned} \rho(1-\psi_{0,0}(u))k_1(u) \leq& \frac{\lambda}{H_1(0)} \int_u^\infty (G(v+y)-G(v))\mathrm{d}v \\ &- \psi_{0,0}(u)\frac{\lambda\int_0^\infty (G(v+y)-G(v))\mathrm{d}v}{H_1(0)} \\ &+ \frac{\lambda}{H_1(0)}\psi_{0,0}(u)*\int_0^u (G(v+y)-G(v))\mathrm{d}v, \end{aligned} \tag{4.13}$$

where we have used (4.12). By (4.4),

$$\begin{aligned} \frac{1}{c}k_1(u) = \frac{\rho}{c}k_1(0) + h_1(u) + \frac{\lambda\mu}{c}L(u) - \frac{\lambda\mu}{c}(h_1*L)(u) \\ - \frac{\lambda}{cH_1(0)}\int_0^u (G(y+v)-G(v))\mathrm{d}v. \end{aligned} \tag{4.14}$$

Letting $u \to \infty$ gives

$$\frac{\rho}{c}k_1(0) + 1 - \frac{\lambda}{cH_1(0)}\int_0^\infty (G(v+y)-G(v))\mathrm{d}v = 0. \tag{4.15}$$

Because $h_1(u) \geq 1 - u^{-1}k_1(u)$ and $L(u) - (h_1 * L)(u) \geq 0$, it follows from (4.14) and (4.15) that

$$k_1(u) \geq \frac{\lambda}{H_1(0)(\rho + cu^{-1})}\int_u^\infty (G(v+y)-G(v))\mathrm{d}v. \tag{4.16}$$

Note that

$$H_1(u) = H_1(0)\left(\frac{k_1(u)}{u} - \int_u^\infty \frac{k_1(v)}{v^2}\mathrm{d}v\right),$$

the result given in (4.10) follows easily from (4.13) and (4.16).

Next, consider the case $\sigma \neq 0$. Since the proof is similar to the proof of the first case, we only indicate the key steps. Inverting the solution of $\hat{h}_2(\beta)$ obtained from (4.8) yields

$$\begin{aligned} h_2(u) =& \frac{1}{c-\lambda\mu}\left(\frac{\sigma^2}{2}h_2'(0)(1-\psi_{0,\sigma}(u)) + \rho((1-\psi_{0,\sigma}) * k_2)(u)\right. \\ & \left. -\lambda\mu((1-\psi_{0,\sigma}) * L)(u) \ +\lambda(1-\psi_{0,\sigma}(u)) * \int_0^u (G(v+y)-G(v))\mathrm{d}v\right), \end{aligned}$$

where $k_2(u) = \int_u^\infty v\mathrm{d}h_2(v)$.

Letting $\beta \to 0$ in (4.5) gives

$$\frac{\sigma^2}{2c}h_2'(0) = 1 + \frac{\rho}{c}k_2(0) - \frac{\lambda}{c}\int_0^\infty (G(v+y)-G(v))\mathrm{d}v. \tag{4.17}$$

Similar to (4.13) we have

$$\begin{aligned} \rho(1-\psi_{0,\sigma}(u))k_2(u) \leq & \lambda\int_u^\infty (G(v+y)-G(v))\mathrm{d}v - \lambda\psi_{0,\sigma}(u)\int_0^\infty (G(v+y)-G(v))\mathrm{d}v \\ & + \lambda\psi_{0,\sigma}(u) * \int_0^u (G(v+y)-G(v))\mathrm{d}v. \end{aligned} \tag{4.18}$$

Using (4.7) and (4.17), we get

$$k_2(u) \geq \frac{\lambda}{\rho + cu^{-1}}\int_u^\infty (G(v+y)-G(v))\mathrm{d}v. \tag{4.19}$$

The result follows from (4.18) and (4.19) for the case $\sigma \neq 0$.

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