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Asymptotic Behavior of the Surplus Prior to and at Ruin in the Classical Risk Model Perturbed by Diffusion

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Abstract: We consider the classical risk model that is perturbed by diffusion. Asymptotic formulae for the surplus prior to and at ruin, ruin caused by oscillation and ruin caused by a claim, as the initial capital tends to infinity are obtained for light-tailed and heavy-tailed claim size distributions.

Key Words: Classical risk model, Ruin theory, Surplus at ruin, Surplus prior to ruin.
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1 Introduction

Consider the following model

$$R_t := u + ct + \sigma B_t - S_t, \quad t \geq 0,$$

where R_t is the surplus at time t, $u \geq 0$ the initial surplus, σ a positive constant, c the rate at which premiums are received, $\{B_t, t \geq 0\}$ the standard Brownian motion, $\{S_t, t \geq 0\}$ the aggregate claims process that is a compound Poisson process with Poisson parameter $\lambda > 0$ and the individual nonnegative claim amount distribution function is P, which is assumed to be absolutely continuous. Denote by p and μ the density and the mean, respectively, of P. Furthermore, we assume that $\{B_t, t \geq 0\}$ and $\{S_t, t \geq 0\}$ are independent, and that the safety loading $c - \lambda\mu$ is positive. Thus, the process $\{R_t, t \geq 0\}$ is a homogeneous strong Markov process.

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This model was first introduced and studied by Gerber (1970), see also Dufresne and Gerber (1991), Veraverbeke (1993), Furrer and Schmidli (1994), Gerber and Landry (1998), Wang and Wu (2000) and Wang (2001). For a more general setting see the survey paper Paulsen (1998) and references therein.

Let $T = \inf\{t \geq 0 : R_t < 0\}$ denote the time of ruin with the usual convention that $\inf \emptyset = \infty$. Let $w(\cdot, \cdot)$ be a nonnegative bounded function. We are interested in the quantities

$$W(u,w) = \mathrm{E}[\mathbf{1}(T < \infty) w(R_{T-}, |R_T|) | R_0 = u],$$

$$W_d(u,w) = \mathrm{E}[\mathbf{1}(T < \infty, R_T = 0) w(R_{T-}, |R_T|) | R_0 = u],$$

$$W_s(u,w) = \mathrm{E}[\mathbf{1}(T < \infty, R_T < 0) w(R_{T-}, |R_T|) | R_0 = u],$$

where R_{T-} is the surplus prior to ruin and $|R_T|$ the deficit at ruin. Actually, $W_d(u,w) = w(0,0)\Pr(T < \infty, R_T = 0 | R_0 = u)$. If $w \equiv 1$, then $W(u,w)$, $W_d(u,w)$ and $W_s(u,w)$ are, respectively, the probabilities of ultimate ruin, ultimate ruin caused by oscillation, and ultimate ruin caused by a claim; for simplicity, these three ruin probabilities are denoted by $\psi(u)$, $\psi_d(u)$ and $\psi_s(u)$. Moreover, when $w(x_1, y_1) = \mathbf{1}(x_1 > x, y_1 > y)$, x, $y > 0$, denote $W(u,w)$, $W_d(u,w)$ and $W_s(u,w)$ by $W(u,(x,y))$, $W_d(u,(x,y))$ and $W_s(u,(x,y))$, respectively. Obviously, $W(0,w) = w(0,0)$, $W_d(u,w) = w(0,0)$ if $u = 0$ and zero if $u < 0$, and $W_s(0,w) = 0$.

For the classical model, i.e. $\sigma = 0$, the distributions of $R_{T-}, |R_T|$ and T have been investigated by many authors. Gerber and Shiu (1998) considered the light-tailed claim size distribution and derived a renewal equation for the expected discounted value of a penalty at ruin. Schmidli (1999) considered that the safety loading can be positive, negative or zero. Explicit expressions for the distributions of the surplus prior to and at ruin were given in terms of the ruin probability. Moreover, the limiting distributions as the initial capital tends to infinity were obtained. In particular, exponential, subexponential and some intermediate claim size distributions were discussed for the positive safety loading case. Further studies on $W(u,w)$ in the classical model can be found in Lin and Willmot (1999, 2000).

For the model perturbed by diffusion, i.e. $\sigma \neq 0$, Gerber (1970), under the assumption that the claim size distribution has an exponentially decaying tail, found that the probability of ruin satisfies an extended defective renewal equation and derived an asymptotic formula of the exponential type, as the initial surplus becomes large. Veraverbeke (1993) showed that the probability of ruin is asymptotically equivalent to the integrated tail of the claim size distribution for subexponentially distributed claim sizes and to the tail of the claim distribution itself for intermediate cases. These results are an analogue of those in the classical model, see Embrechts and Veraverbeke (1982). Chiu and Yin (2003) proved that the mean discounted penalty satisfies an integro-differential equation and asymptotic results were obtained. The studies of ψ, ψ_d, ψ_s, and the expected discounted value of a penalty at ruin can be found in e.g. Dufresne and Gerber (1991), Gerber and Landry (1998) and Wang (2001).

In this paper, asymptotic formulae for W, W_d and W_s, as the initial surplus becomes large, are obtained for light-tailed, heavy-tailed and intermediate claim size distributions.

2 The exact expressions

Assume that the nonnegative bounded function $w(x, y)$ is continuous except for $(x_i, y_i), i = 1, 2, \cdots$. Put $N_0 = \{x_i, i \geq 1\} \cup \{x_i, i \geq 1\} \cup N_1$. Obviously, the Lebesgue measure of N_0 is zero. Using the same argument as in Wang (2001), we can prove that $W(\cdot) = W(\cdot, w)$ is twice continuously differentiable on $[0, \infty) \setminus N_0$ with a bounded first derivative $W'(\cdot)$, where $W'(0)$ refers to the right-hand derivative of W at 0. Denote by T_1 the first epoch of claim. For $\epsilon, m > 0$ such that $\epsilon < u < m$, define $T^{\epsilon,m} = \inf\{s > 0 : u + cs + \sigma B_s \notin (\epsilon, m)\}$. Note that $W'(\cdot)$ and $W''(\cdot)$ are bounded on $[\epsilon, m] \setminus N_0$ since $W(\cdot)$ is twice continuously differentiable. For $t > 0$, let $T_t^{\epsilon,m} = T^{\epsilon,m} \wedge t$ and $\tau = T_t^{\epsilon,m} \wedge T_1$. By the strong Markov property of R_t, we get $W(u) = \mathrm{E}W(R_\tau)$. Hence, using Itô's formula, we can prove that

$$\frac{1}{2}\sigma^2 W''(u) + cW'(u) + \lambda \int_0^u W(u - z)p(z)\mathrm{d}z + \lambda \int_u^\infty w(u, z - u)p(z)\mathrm{d}z = \lambda W(u), \; u \in [0, \infty) \setminus N_0. \tag{2.1}$$

Similarly, we have

$$\frac{1}{2}\sigma^2 W_d''(u) + cW_d'(u) + \lambda \int_0^u W_d(u - z)p(z)\mathrm{d}z = \lambda W_d(u), u \in [0, \infty) \setminus N_0,$$

$$\frac{1}{2}\sigma^2 W_s''(u) + cW_s'(u) + \lambda \int_0^u W_s(u - z)p(z)\mathrm{d}z + \lambda \int_u^\infty w(u, z - u)p(z)\mathrm{d}z = \lambda W_s(u), u \in [0, \infty) \setminus N_0.$$

Remark 2.1. The proofs of the above formulae follow closely to those in Gerber and Landry (1998), Wang and Wu (2000) and Wang (2001); See also Chiu and Yin (2003). Another way to prove $W(u) = \mathrm{E}W(R_\tau)$ is to show that $W(R_t)$ is a martingale by the Markov property of R_t and then apply Doob's Optional Stopping Theorem.

Throughout the paper, the Laplace transform of a function is denoted by putting a hat on the function and the tail of a distribution by putting a bar on the distribution function. For example, $\hat{W}(\beta) = \int_0^\infty \mathrm{e}^{-\beta u} W(u)\mathrm{d}u$ and $\bar{P}(x) = 1 - P(x)$. For two integrable functions g_1 and g_2 defined on $[0,\infty)$, let $(g_1 * g_2)(u) := \int_0^u g_1(u-z)g_2(z)\mathrm{d}z$ and $(g_1 \star g_2)(u) := \int_0^u g_1(u-z)\mathrm{d}g_2(z)$. In particular, we use notation $g^{*2}(u) = (g_1 * g_2)(u)$ and $g^{\star 2}(u) = (g_1 \star g_2)(u)$ when $g_1 = g_2 = g$. For any positive integer k, $g^{*k}(u)$ and $g^{\star k}(u)$ can be defined similarly. For any probability distribution function F with mean ν, denote $\overline{F}(u) = 1 - F(u)$ the tail of F and $F_I(u) = \nu^{-1}\int_{-\infty}^u \overline{F}(z)\mathrm{d}z$ the integrated tail distribution function.

Taking the Laplace transform of both sides of (2.1) results in

$$\hat{W}(\beta) = \frac{\frac{1}{2}\sigma^2 W'(0) + \frac{1}{2}\sigma^2\beta W(0) + cW(0) - \lambda\Delta(\beta)}{\frac{1}{2}\sigma^2\beta^2 + c\beta - \lambda + \lambda\hat{p}(\beta)}, \tag{2.2}$$

where

$$\Delta(\beta) = \int_0^\infty \mathrm{e}^{-\beta u}\left\{\int_u^\infty w(u, z-u)p(z)\mathrm{d}z\right\}\mathrm{d}u. \tag{2.3}$$

It is clear that $\lim_{u\to\infty} W(u)$ exists and equals 0 since w is bounded. By using the final value theorem of Laplace transforms, we have

$$0 = \lim_{u\to\infty} W(u) = \lim_{\beta\to 0}\beta\hat{W}(\beta) = \frac{\frac{1}{2}\sigma^2 W'(0) + cW(0) - \lambda\Delta(0)}{c - \lambda\mu},$$

which leads to

$$\frac{1}{2}\sigma^2 W'(0) = \lambda\int_0^\infty\int_u^\infty w(u, z-u)p(z)\mathrm{d}z\mathrm{d}u - cW(0). \tag{2.4}$$

Because $W(0) = w(0,0)$, equation (2.2) can be written as

$$\hat{W}(\beta) = \frac{\lambda\{\Delta(0) - \Delta(\beta)\} + \frac{1}{2}\sigma^2\beta w(0,0)}{\frac{1}{2}\sigma^2\beta^2 + c\beta - \lambda + \lambda\hat{p}(\beta)}. \tag{2.5}$$

According to Dufresne and Gerber (1991, equation (3.4)) we have

$$1-\psi(u)=\left(1-\frac{\lambda\mu}{c}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda\mu}{c}\right)^{n}(G^{\star(n+1)}\star L^{\star n})(u), \tag{2.6}$$

where $F^{\star 0}$ is the distribution function corresponding to the Dirac measure at zero, and

$$G(u)=1-\mathrm{e}^{-2cu/\sigma^2},\quad L(u)=\frac{1}{\mu}\int_0^u\{1-P(y)\}\mathrm{d}y,\quad u>0. \tag{2.7}$$

From (2.6) we get

$$\int_0^\infty \mathrm{e}^{-\beta u}\{1-\psi(u)\}\mathrm{d}u=\frac{c-\lambda\mu}{\frac{1}{2}\sigma^2\beta^2+c\beta-\lambda+\lambda\hat{p}(\beta)}.$$

Therefore, we can rewrite (2.5) as

$$\hat{W}(\beta)=\frac{\lambda\{\Delta(0)-\Delta(\beta)\}+\frac{1}{2}\sigma^2\beta w(0,0)}{c-\lambda\mu}\int_0^\infty \mathrm{e}^{-\beta u}\{1-\psi(u)\}\mathrm{d}u. \tag{2.8}$$

Inverting this Laplace transform yields

$$\begin{aligned}W(u,w)=&\frac{\lambda}{c-\lambda\mu}\left[\{1-\psi(u)\}\Delta(0)-\{1-\psi(u)\}*\int_u^\infty w(u,z-u)p(z)\mathrm{d}z\right]\\&-\frac{1}{2(c-\lambda\mu)}\sigma^2 w(0,0)\psi'(u),\quad u>0.\end{aligned}$$

In particular, when $w(x_1,\,y_1)=\mathbf{1}(x_1>x,\,y_1>y)$, for $x,\,y>0$, then

$$\begin{aligned}W(u,(x,y))=\frac{\lambda}{c-\lambda\mu}\Bigg\{&\int_{u+y}^\infty \bar{P}(z)\mathrm{d}z-\psi(u)\int_{x+y}^\infty \bar{P}(z)\mathrm{d}z\\&+\mathbf{1}(u>x)\int_{x+y}^{u+y}\bar{P}(z)\psi(u+y-z)\mathrm{d}z\Bigg\}.\end{aligned} \tag{2.9}$$

By the same token, we have for $u>0$,

$$\begin{aligned}W_d(u,w)&=-\frac{1}{2(c-\lambda\mu)}\sigma^2 w(0,0)\psi'(u),\\W_s(u,w)&=\frac{\lambda}{c-\lambda\mu}\left[\{1-\psi(u)\}*\left(\Delta(0)-\int_u^\infty w(u,z-u)p(z)\mathrm{d}z\right)\right].\end{aligned}$$

In particular, $W_d(u,(x,y)) = 0$ and $W_s(u,(x,y)) = W(u,(x,y))$, for x, y, $u > 0$.

Let $-\gamma$ denote the left abscissa of convergence of the Laplace transform $\hat{p}$ of the claim size density; it is obvious that $-\gamma \le 0$. In view of (2.5), the behaviour of W depends on $-\gamma$. The case $\gamma = 0$ will be considered in Section 4. For $-\gamma < 0$, Sections 3 and 5 will discuss the cases that the adjustment coefficient does and does not exist, respectively.

3 Light-tailed case

In this section, we assume that $-\gamma < 0$ and there exists an $R > 0$, which is called the Lundberg exponent or the adjustment coefficient, such that $-R$ is the unique negative root of $\frac{1}{2}\sigma^2\beta^2 + c\beta - \lambda + \lambda\hat{p}(\beta) = 0$, i.e.

$$\frac{1}{2}\sigma^2 R^2 - cR - \lambda + \lambda\hat{p}(-R) = 0.$$

From the proof of Chiu and Yin (2003, Theorem 2.1) we know that the zero set N_0 in equation (2.1) does not effect the value of W. Consequently, if $\lim_{u\to\infty} W'(u,w) = 0$, using above Theorem 2.1 again we have

$$W(u,w) := W(u) = (W * g_1 * g_1)(u) + \mathrm{e}^{-\frac{2c}{\sigma^2}u} w(0,0) + (h_1 * h_2)(u), \qquad (3.1)$$

where $g_1(x) = \frac{2\lambda}{\sigma^2}\mathrm{e}^{-\frac{2c}{\sigma^2}x}$, $g_2(x) = \overline{P}(x)$ and

$$h_1(x) = \frac{2}{\sigma^2}\mathrm{e}^{-\frac{2c}{\sigma^2}x},\ h_2(x) = \int_x^\infty g(x)dy,\ g(x) = \lambda\int_x^\infty w(x, z-x)p(z)dz.$$

If $\int_0^\infty \mathrm{e}^{Ry} g(y)dy < \infty$, $\lim_{y\to\infty} \mathrm{e}^{Ry}g(y) = 0$, $\hat{p}'(-R) < \infty$ and $c - \frac{1}{2}\sigma^2 R > 0$, then the conditions of Chiu and Yin (2003, Theorem 2.3) are satisfied, together with (3.1) we have

$$\begin{aligned}&\lim_{u\to\infty} \mathrm{e}^{Ru} W(u,w)\\ &\qquad = \frac{-\frac{1}{2}\sigma^2 R w(0,0) - \lambda\displaystyle\int_0^\infty\int_0^\infty (\mathrm{e}^{Rx}-1)w(x,y)p(x+y)\mathrm{d}x\mathrm{d}y}{\lambda\hat{p}'(-R) + c - \sigma^2 R}.\end{aligned}$$

We can apply the same technique to $W_d(\cdot)$ and $W_s(\cdot)$, and the results are summarized as the following theorem.

Theorem 3.1. *Assume* $\int_0^\infty e^{Ry}g(y)dy < \infty$, $\lim_{y\to\infty} e^{Ry}g(y) = 0$, $\hat{p}'(-R) < \infty$ *and* $c - \frac{1}{2}\sigma^2 R > 0$, *then*

$$(i) \lim_{u\to\infty} \mathrm{e}^{Ru}W(u,w)$$
$$= \frac{-\frac{1}{2}\sigma^2 Rw(0,0) - \lambda \int_0^\infty \int_0^\infty (\mathrm{e}^{Rx}-1)w(x,y)p(x+y)\mathrm{d}x\mathrm{d}y}{\lambda\hat{p}'(-R) + c - \sigma^2 R},$$

$$(ii) \lim_{u\to\infty} \mathrm{e}^{Ru}W_d(u,w) = \frac{-\frac{1}{2}\sigma^2 Rw(0,0)}{\lambda\hat{p}'(-R) + c - \sigma^2 R},$$

$$(iii) \lim_{u\to\infty} \mathrm{e}^{Ru}W_s(u,w) = \frac{-\lambda \int_0^\infty \int_0^\infty (\mathrm{e}^{Rx}-1)w(x,y)p(x+y)\mathrm{d}x\mathrm{d}y}{\lambda\hat{p}'(-R) + c - \sigma^2 R}.$$

Remark 3.1. Result (i) can also be obtained by setting $\delta = 0$ in Chiu and Yin (2003, Theorem 3.3). Theorem 3.1 extends the results in Gerber (1970) and Dufresne and Gerber (1991), in which only $w \equiv 1$ has been considered.

4 Heavy-tailed case

In this section, we consider claim size distributions with $\gamma = 0$. In insurance mathematics the subexponential distribution functions are the most popular candidates for describing heavy-tailed distributions. A distribution function F with support $(0,\infty)$ is subexponential if $\lim_{x\to\infty} \overline{F^{\star 2}}(x)/\bar{F}(x) = 2$. Denoted by $\mathcal{S}$ the class of subexponential distribution functions. Typical examples in $\mathcal{S}$ are the Pareto, the lognormal, the loggamma and the heavy-tailed Weibull distributions.

A distribution function F with support $(0,\infty)$ belongs to the class $\mathcal{L}$ if $\lim_{x\to\infty} \frac{\bar{F}(x-y)}{\bar{F}(x)} = 1 \;\forall y \in (-\infty,\infty)$. For $F \in \mathcal{L}$, we have $\mathrm{e}^{\epsilon x}\bar{F}(x) \to \infty$ as $x \to \infty$ for every $\epsilon > 0$.

For properties of the classes $\mathcal{S}$ and $\mathcal{L}$, see Embrechts *et al.* (1979, 1997), Rolski *et al.* (1999) and references therein.

Definition 4.1. (Klüppelberg (1989)) A function $f : (-\infty,\infty) \to (0,\infty)$ such that $f(x) > 0$ on $[A,\infty)$ for some $A > 0$ belongs to the class $\mathcal{S}d$ if $f \in \mathcal{L}$ and

$$\lim_{x\to\infty} \frac{f^{*2}(x)}{f(x)} = 2.$$

For $f \in \mathcal{S}d$ define a distribution function concentrated on $(0,\infty)$ by

$$F(x) := \left(\int_0^\infty f(y)dy\right)^{-1} \int_0^x f(y)dy.$$

It follows from Klüppelberg (1989, Theorem 1.1) that $F \in \mathcal{S}$.

The objective of this section is to study the asymptotic behaviour of $W(u,(x,y))$ and $W_s(u,(x,y))$ with the tail of the claim size distribution belongs to $\mathcal{S}d$.

Theorem 4.1. *If the claim size distribution function P satisfies that $\overline{P} \in \mathcal{S}d$, then for any $x, y > 0$,*

$$W(u,(x,y)) = W_s(u,(x,y)) \sim \psi(u), \quad u \to \infty.$$

Proof. The integration of (2.1) in the case of $w \equiv 1$, by using $\psi(0) = 1$ and $\psi(\infty) = 0$, together with (2.4), give

$$\int_0^u \bar{P}(u-z)\psi(z)\mathrm{d}z = \frac{\sigma^2}{2\lambda}\psi'(u) + \frac{c}{\lambda}\psi(u) + \int_0^u \bar{P}(z)\mathrm{d}z - \mu. \tag{4.1}$$

Using (4.1), we can rewrite (2.9) as

$$\begin{aligned}\frac{c-\lambda\mu}{\lambda}W(u,(x,y)) = {} & \mathbf{1}(u>x)\left\{\frac{\sigma^2}{2\lambda}\psi'(u+y) + \frac{c}{\lambda}\psi(u+y)\right\} \\ & - \mathbf{1}(u>x)\int_0^{x+y} \bar{P}(z)\psi(u+y-z)\mathrm{d}z \\ & + \mathbf{1}(u \le x)\int_{u+y}^\infty \bar{P}(z)\mathrm{d}z - \psi(u)\int_{x+y}^\infty \bar{P}(z)\mathrm{d}z. \end{aligned} \tag{4.2}$$

$\overline{P} \in \mathcal{S}d$ implies that $L \in \mathcal{S}$, then from Veraverbeke (1993, Theorem 1), $1-\psi(u) \in \mathcal{S}$, and

$$\psi(u) \sim \frac{\lambda}{c-\lambda\mu}\int_u^\infty \{1-P(z)\}\mathrm{d}z \ \text{ as } u \to \infty. \tag{4.3}$$

Furthermore, we claim that

$$\psi'(u) \sim \frac{-\lambda\mu}{c-\lambda\mu} l(u) \ \text{ as } u \to \infty, \tag{4.4}$$

where $l(u) = L'(u)$. In fact, from (2.6) we see that $-\psi'(u) = (g * R)(u)$, where $g(u) = G'(u)$ and

$$R(u) = \left(1 - \frac{\lambda\mu}{c}\right)\sum_{n=0}^\infty \left(\frac{\lambda\mu}{c}\right)^n (g * l)^{*n}(u).$$

Moreover, $l \in \mathcal{S}d \Rightarrow l \in \mathcal{L} \Rightarrow g(u) = o(l(u))$. By Asmussen *et al.* (2003, Proposition 7, Theorem 3) we have $(g * l)(u) \sim l(u)$ and $R(u) \sim \frac{-\lambda\mu}{c-\lambda\mu} l(u)$ as $u \to \infty$. Thus

$$\psi'(u) = -(g * R)(u) \sim \frac{-\lambda\mu}{c - \lambda\mu} l(u) \text{ as } u \to \infty.$$

It follows from (4.3) and (4.4) that $\lim_{u\to\infty} \frac{\psi'(u)}{\psi(u)} = 0$. Dividing both sides of (4.2) by $\psi(u)$, and letting $u \to \infty$, we prove the theorem.

5 Intermediate case

In this section we study the asymptotic behaviour of $W(u,(x,y))$ in the case that $-\gamma < 0$ but the adjustment coefficient R does not exist. An appropriate class of functions for describing such claim size distributions is the following.

Definition 5.1. A distribution function F on $[0,\infty)$ with unbounded support is said to belong to $\mathcal{S}(\gamma)$ with $\gamma \geq 0$ if

$$(i) \lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\bar{F}(x)} = 2\int_0^\infty \mathrm{e}^{\gamma y} \mathrm{d}F(y) < \infty,$$
$$(ii) \lim_{x\to\infty} \frac{\bar{F}(x-y)}{\bar{F}(x)} = \mathrm{e}^{\gamma y}, \text{ for all real } y.$$

Examples of distributions in $\mathcal{S}(\gamma)$ are those with densities asymptotically equal to $mx^{-\rho}\mathrm{e}^{-\gamma x}$ for large x, where $m > 0$ and $\rho > 1$.

This class of functions was studied in detail in Embrechts and Goldie (1982). If $F \in \mathcal{S}(\gamma)$, then $\lim_{x\to\infty} \mathrm{e}^{(\gamma+\epsilon)x}\bar{F}(x) \to \infty$ and $\int_0^\infty \mathrm{e}^{(\gamma+\epsilon)y}\mathrm{d}F(y) = \infty$, for any $\epsilon > 0$. Note that $\mathcal{S}(0) = \mathcal{S}$, the class of subexponential distribution functions.

Let $\hat{l}(\alpha) = \frac{1-\hat{p}(\alpha)}{\mu\alpha}$ and $\hat{g}(\alpha) = \frac{2c}{2c+\alpha\sigma^2}$. Note that $\hat{l}$ and $\hat{g}$ are, respectively, the Laplace-Stieltjes transforms of the functions L and G, which have been defined in (2.7). According to Veraverbeke (1993), if $\lambda\mu\hat{l}(-\gamma)\hat{g}(-\gamma) < c$, then the adjustment coefficient R does not exist.

Theorem 5.1. *For x, $y > 0$, if $P \in \mathcal{S}(\gamma)$ $(\gamma > 0)$ satisfies*

$$\lambda\mu\hat{l}(-\gamma)\hat{g}(-\gamma) < c, \quad \int_0^\infty e^{\gamma y} dL(y) < 1,$$

then

$$W(u,(x,y)) = W_s(u,(x,y)) \sim C\{1-P(u)\}, \quad u \to \infty,$$

where

$$C = \frac{\lambda^2 \left\{ \left(-\frac{\gamma\sigma^2}{2\lambda} + \frac{c}{\lambda} \right) \mathrm{e}^{-\gamma y} - \int_{x+y}^{\infty} \bar{P}(z)\mathrm{d}z - \int_0^{x+y} \bar{P}(z)\mathrm{e}^{-\gamma(y-z)}\mathrm{d}z \right\}}{c^2\gamma \left\{ 1 - \frac{\gamma\sigma^2}{2c} - \frac{\lambda}{c}\int_0^{\infty} \mathrm{e}^{\gamma t}\bar{P}(t)\mathrm{d}t \right\}^2}.$$

Proof. If the conditions are fulfilled, then according to Veraverbeke (1993, Theorem 2), $1-\psi(u) \in \mathcal{S}(\gamma)$ and

$$\psi(u) \sim K\{1-P(u)\}, \quad u \to \infty, \tag{5.1}$$

where

$$K = \frac{\lambda}{c\gamma}\left(1 - \frac{\lambda\mu}{c}\right)\left\{1 - \frac{\gamma\sigma^2}{2c} - \frac{\lambda}{c}\int_0^{\infty} \mathrm{e}^{\gamma t}\bar{P}(t)\mathrm{d}t\right\}^{-2}.$$

From (2.6) we see that $-\psi'(u) = (g * R)(u)$, where $g(u) = G'(u)$ and

$$R(u) = \left(1 - \frac{\lambda\mu}{c}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda\mu}{c}\right)^n (g * l)^{*n}(u).$$

Since $P \in \mathcal{S}(\gamma)$ we have $\lim_{u\to\infty} \frac{l(u-y)}{l(u)} = \lim_{u\to\infty} \frac{\overline{P}(u-y)}{\overline{P}(u)} = \mathrm{e}^{\gamma y}$ for any $y \in \mathcal{R}$. Using dominated convergence gives

$$\lim_{u\to\infty} \frac{(g*l)(u)}{l(u)} = \int_0^{\infty} \mathrm{e}^{\gamma y} g(y)\mathrm{d}y = \hat{g}(-\gamma).$$

By Klüppelberg (1989, Theorem 3.2) we have $R(u) \sim C_1 l(u)$ where

$$\begin{aligned} C_1 &= \sum_{n=1}^{\infty} n\left(1 - \frac{\lambda\mu}{c}\right)\left(\frac{\lambda\mu}{c}\right)^n \hat{g}^n(-\gamma)\left(\int_0^{\infty} \mathrm{e}^{\gamma y} dL(y)\right)^{n-1} \\ &= \left(1 - \frac{\lambda\mu}{c}\right)\left(\frac{\lambda\mu}{c}\right)\hat{g}(-\gamma)\left(1 - \frac{\lambda\mu}{c}\hat{g}(-\gamma)\int_0^{\infty} \mathrm{e}^{\gamma y} dL(y)\right)^{-2}. \end{aligned}$$

It follows that

$$\begin{aligned} -\psi'(u) &= (g * R)(u) \sim C_1\hat{g}(-\gamma)l(u) \\ &= \frac{\lambda}{c}\left(1 - \frac{\lambda\mu}{c}\right)\left\{1 - \frac{\gamma\sigma^2}{2c} - \frac{\lambda}{c}\int_0^{\infty} \mathrm{e}^{\gamma t}\bar{P}(t)\mathrm{d}t\right\}^{-2}(1 - P(u)). \end{aligned} \tag{5.2}$$

By (5.1) and (5.2) we obtain $\lim_{u\to\infty} \frac{\psi'(u+y)}{\psi(u)} = -\gamma e^{-\gamma y}$, together with (2.8) we obtain

$$\lim_{u\to\infty} \frac{W(u,(x,y))}{\psi(u)} = K_1,$$

where

$$K_1 = \frac{\lambda}{c-\lambda\mu}\left\{-\frac{\gamma\sigma^2}{2\lambda}e^{-\gamma y} + \frac{c}{\lambda}e^{-\gamma y} - \int_{x+y}^{\infty} \bar{P}(z)\mathrm{d}z \right.$$
$$\left. - \int_0^{x+y} \bar{P}(z)e^{-\gamma(y-z)}\mathrm{d}z\right\},$$

and the result follows.

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