2013

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On the Complete Monotonicity of the Compound Geometric Convolution with Applications in Risk Theory

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Abstract

We prove that the complete monotonicity is preserved under mixed geometric compounding, and hence show that the ruin probability, the Laplace transform of the ruin time, and the density of the tail of the joint distribution of ruin and the deficit at ruin in the Sparre Andersen model are completely monotone if the claim size distribution has a completely monotone density.

Keywords: Complete monotonicity; Compound geometric convolution; Pollaczeck–Khinchine formula; Ruin probability; Sparre Andersen model

1 Introduction

Let $f \in C^\infty((a,b))$, $0 < a < b \leq \infty$. We say f is completely monotone (Widder, 1941, Chapter IV, Definition 2c) on (a,b) if $(-1)^n f^{(n)} \geq 0$ for all nonnegative integers n. The following result is known as Bernstein's theorem [Berstein, 1929; Widder, 1941, Chapter IV,Theorem 12b]: A function f is completely monotone on $(0,\infty)$ if and only if there exists a non-decreasing function ϕ such that the Laplace–Stieltjes integral $f(x) = \int_0^\infty \mathrm{e}^{-sx}\mathrm{d}\phi(s)$ converges for $0 < x < \infty$. Thus, any completely monotone function is both non-increasing and log-convex. Examples of completely monotone density functions include the exponential, the Weibull with shape parameter less than 1, the Pareto, and the gamma with shape parameter not greater than 1. Originally, completely monotonic functions arose as Laplace transforms of nonnegative functions, see e.g. Feller (1966, Chapter XIII). Now they appear in many different areas; see e.g. Sumita &

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Masuda (1987), Pillai & Sandhya (1990), Hara (2008) and Yu (2009). For the application of completely monotone functions in risk theory, see Loeffen (2008, 2009) and Yin & Wang (2009).

Let $\{X_k\}_{k\geq 1}$ be a sequence of nonnegative independent and identically distributed random variables with common distribution function H with $H(0) = 0$. Further, let N be a geometric random variable with $\mathbf{P}(N = n) = (1-\rho)\rho^n$, $n = 0, 1, \ldots$, for $0 < \rho < 1$, which is independent of $\{X_k\}$. The distribution of the random sum $\sum_{k=1}^{N} X_k$ (with the convention that $\sum_{k=1}^{0} X_k = 0$), known as the compound geometric distribution, is given by

$$C(x) = (1-\rho)\sum_{n=0}^{\infty} \rho^n H^{*n}(x), \quad x \geq 0, \tag{1.1}$$

where H^{*n} denotes the nth convolution of H with itself, and $H^{*0}(x) = 1$ if $x \geq 0$, and 0 otherwise. For the application of compound geometric distribution in risk theory, see e.g. Willmot & Lin (2001, Chapter 7) and Willmot (2002).

Let W be the further convolution of C with another distribution G supported on $[0, \infty)$, i.e.

$$W = C * G = (1-\rho)\sum_{n=0}^{\infty} \rho^n H^{*n} * G. \tag{1.2}$$

This compound geometric convolution is the distribution of $\sum_{k=1}^{N} X_k$ above plus another independent random variable having the distribution G. Such a random sum arises in many applied probability models in reliability, queueing and risk theory. For example, the waiting time of a specified event A in a renewal process where between any two consecutive arrivals the event A may occur with probability ρ. See Brown (1990), Gertsbakh (1984), Willmot & Lin (1996), Willmot & Cai (2004) and references therein.

A special form of (1.2) often appearing in queueing and risk theory is that H itself is a convolution of the distribution G with a distribution F supported on $(0, \infty)$. Then the compound geometric convolution given in (1.2) becomes

$$W = (1-\rho)\sum_{n=0}^{\infty} \rho^n G^{*(n+1)} * F^{*n}. \tag{1.3}$$

For example, the survival probability in the perturbed risk processes can often be deduced to the form of W; see e.g. Dufresne & Gerber (1991) and Furrer (1998).

The purpose of this paper is to consider the complete monotonicity of the distribution of the random sum $X_0 + \sum_{k=1}^{N}(X_k + Y_k)$, where $\{X_k\}_{k\geq 0}$ is a sequence of independent exponential random variables, $\{Y_k\}_{k\geq 1}$ a sequence of independent nonnegative random variables with a common completely monotone density, and N the mixed geometric distribution with $\mathbf{P}(N = n) = \int_0^1 (1-\rho)\rho^n \mathrm{d}B(\rho)$, $n = 0, 1, \ldots$, in which B is a distribution on $(0, 1)$. Moreover, $\{X_k\}$, $\{Y_k\}$ and N are independent. In particular, we prove that the complete monotonicity is preserved under mixed geometric compounding. Similar kinds of preserving have been observed in the literature. Shanthikumar (1988) proved that the decreasing failure rate property is preserved under geometric sum; Keilson (1978) (see also Szekli, 1986) proved that the complete monotonicity is preserved under geometric compounding; Cai & Kalashnikov (2000) proved that if N is discrete new worse strongly than used, then the sum $\sum_{k=1}^{N} X_k$ is new worse than used; Willmot & Cai (2004) proved that the residual lifetime of a compound geometric convolution is again a compound geometric convolution. After showing the preserving under mixed geometric compounding, we then apply our results to establish a sufficient condition for the complete monotonicity of the ruin probability, the Laplace transform of the ruin time, and the density of the tail of the joint distribution of ruin and the deficit at ruin in the Sparre Andersen model.

2 Complete monotonicity of compound geometric convolution

Throughout this section we assume that G is the exponential distribution function and F is a distribution function on $[0,\infty)$ with $F(0)=0$. We start with a rather obvious lemma.

Lemma 2.1. *(Alzer & Berg, 2006, Lemma 2.4) The sum, the product and the pointwise limit of completely monotone functions are also completely monotone.*

The following is the main result of this section.

Theorem 2.2. *Assume that F has a density f on $(0,\infty)$. Then the compound geometric convolution W given in (1.3) has a completely monotone density on $(0,\infty)$ if and only if f is a completely monotone function on $(0,\infty)$.*

Proof. Denote by ξ^{-1} the mean of G. Let $\hat{h}(\alpha)=\int_0^\infty \mathrm{e}^{-\alpha x}\mathrm{d}H(x)$ denote the Laplace–Stieltjes transform of a distribution function H supported on $(0,\infty)$. Taking the Laplace–Stieltjes transforms on both sides of (1.3) yields

$$\hat{w}(\alpha)=\frac{(1-\rho)\hat{g}(\alpha)}{1-\rho\hat{g}(\alpha)\hat{f}(\alpha)}=\frac{(1-\rho)\xi}{\alpha+\xi-\rho\xi\hat{f}(\alpha)}=\frac{\frac{1}{\alpha}(1-\rho)\xi}{1+\frac{\xi}{\alpha}-\frac{\rho\xi}{\alpha}\hat{f}(\alpha)}.$$

Notice that

$$\int_0^\infty \mathrm{e}^{-\alpha x}W(x)\mathrm{d}x=\frac{1}{\alpha}\int_0^\infty \mathrm{e}^{-\alpha x}W'(x)\mathrm{d}x=\frac{1}{\alpha}\hat{w}(\alpha)=\frac{\frac{1}{\alpha^2}(1-\rho)\xi}{1+\frac{\xi}{\alpha}-\frac{\rho\xi}{\alpha}\hat{f}(\alpha)}.$$

It follows that W satisfies the following Volterra integral equation of the second kind

$$W(x)+\int_0^x a(x-z)W(z)\mathrm{d}z=q(x),\quad x>0, \tag{2.1}$$

which can be easily verified by taking the Laplace transform of both sides, where $q(x) = x(1-\rho)\xi$, and

$$a(x) = \xi\{1 - \rho F(x)\}. \tag{2.2}$$

The solution of Eq. (2.1) has the form (see e.g. Gripenberg, 1978, Eqs. (1.1)–(1.3))

$$W(x) = q(x) - \int_0^x r(x-y)q(y)\mathrm{d}y, \tag{2.3}$$

where the resolvent kernel r is the solution of the equation

$$r(x) + \int_0^x a(x-z)r(z)\mathrm{d}z = a(x), \quad x \geq 0. \tag{2.4}$$

It follows from Miller (1968) that Eq. (2.4) has a unique solution r such that r is continuous on $(0,\infty)$ and satisfies

$$0 \leq r(x) \leq a(x), \qquad 0 < x < \infty, \qquad \int_0^\infty r(z)dz \leq 1.$$

Substituting $y(x) = 1 - \int_0^x r(s)\mathrm{d}s$ in Eq. (2.4), we have

$$y(x) = 1 - \int_0^x a(x-s)y(s)\mathrm{d}s, \quad x \geq 0. \tag{2.5}$$

If f is completely monotone on $(0,\infty)$, then a in Eq. (2.2) is completely monotone on $(0,\infty)$ and hence, from Friedman (1963, Theorem 8), y satisfying Eq. (2.5) is also completely monotone on $(0,\infty)$, and so is r. It is now easy to see from (2.3) that

$$W'(x) = (1-\rho)\xi\left\{1 - \int_0^x r(z)dz\right\} \geq 0,$$
$$W^{(n)}(x) = -(1-\rho)\xi r^{(n-2)}(x), \quad n \geq 2,$$

which implies that for $n \geq 1$,

$$(-1)^n W^{(n+1)}(x) = (1-\rho)\xi(-1)^{n-1}r^{(n-1)}(x) \geq 0.$$

Thus, W has a completely monotone density on $(0,\infty)$.

Now we show the second statement is also a necessary condition. It follows from (2.3) that

$$W'(x) = W'(0)\left\{1 - \int_0^x r(y)\mathrm{d}y\right\} > 0,$$
$$W''(x) = -W'(0)r(x),$$

from which we see that if W has a completely monotone density, then r is completely monotone. Furthermore, $\int_0^\infty r(y)\mathrm{d}y = 1 - W'(\infty)/W'(0) \leq 1$. It follows from Gripenberg (1978, Theorem 3) that a is also completely monotone, where a is defined by (2.2). The result follows since a is completely monotone if and only if f is completely monotone. □

The "if" part of the following result was first proved by Keilson (1978, Theorem 3.1).

Corollary 2.3. *The compound geometric distribution C defined in (1.1) has a completely monotone density on $(0, \infty)$ if and only if F has a completely monotone density on $(0, \infty)$.*

Proof. By letting the mean of G in (1.3) go to zero, the result follows from Theorem 2.2 and the dominated convergence theorem. □

From Lemma 2.1 and Theorem 2.2 we immediately have the following two corollaries.

Corollary 2.4. *Suppose B is a probability distribution on $(0, 1)$. If F has a completely monotone density on $(0, \infty)$, then*

$$\sum_{n=0}^{\infty} \int_0^1 (1-\rho)\rho^n \mathrm{d}B(\rho) \cdot G^{*(n+1)} * F^{*n}(x), \quad x \geq 0, \tag{2.6}$$

has a completely monotone density on $(0, \infty)$.

Corollary 2.5. *For any positive integer k, if F has a completely monotone density on $(0, \infty)$, then*

$$\sum_{n=0}^{\infty} \sum_{i=1}^{k} A_i(1-\rho_i)\rho_i^n \cdot G^{*(n+1)} * F^{*n}(x), \quad x \geq 0, \tag{2.7}$$

and

$$\sum_{n=0}^{\infty} \int_0^1 (1-\rho)\rho^n \mathrm{d}B(\rho) \cdot F^{*n}(x), \quad x \geq 0, \tag{2.8}$$

have completely monotone densities on $(0, \infty)$, where $A_i \geq 0$, $0 < \rho_i < 1$ for $i = 1, 2, \ldots, k$, $\sum_{i=1}^{k} A_i = 1$, and B is a probability distribution on $(0, 1)$.

Proof. If B in (2.6) is a discrete distribution, we get the expression in (2.7). If G in (2.6) is the degenerate distribution (by letting its mean go to zero), then we get the expression (2.8) from the dominated convergence theorem. The result then follows from Corollary 2.4. □

Remark 2.6. *From Theorem 2.2 and Corollary 2.3 we can see that C defined in (1.1) has a completely monotone density on $(0, \infty)$ if and only if W has a completely monotone density on $(0, \infty)$.*

The random sum $\sum_{k=1}^{N}(X_k + Y_k)$ has the distribution that is also in the form of compound geometric convolution:

$$W_0(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n G^{*n} * F^{*n}(x), \quad x \geq 0. \tag{2.9}$$

However, it does not necessarily have a completely monotone density, even when G is an exponential distribution and F has a completely monotone density on $(0,\infty)$. The following is a counterexample.

Example. Let $G(x) = F(x) = 1 - \mathrm{e}^{-x}$ for $x \geq 0$. Thus, F has a completely monotone density. Taking the Laplace transform on both sides of (2.9) gives

$$\int_0^\infty \mathrm{e}^{-\beta x}\mathrm{d}W_0(x) = (1-\rho)\sum_{n=0}^\infty \frac{\rho^n}{(1+\beta)^{2n}} = 1-\rho+\frac{(1-\rho)\rho}{(1+\beta)^2-\rho}.$$

By inverting the Laplace transform we get

$$W_0(x) = (1-\rho)+(1-\rho)\frac{\sqrt{\rho}}{2}\left[\frac{1}{\sqrt{\rho}-1}\left\{\mathrm{e}^{(\sqrt{\rho}-1)x}-1\right\}+\frac{1}{\sqrt{\rho}+1}\left\{\mathrm{e}^{-(\sqrt{\rho}+1)x}-1\right\}\right].$$

When $x > 0$,

$$W_0'(x) = (1-\rho)\frac{\sqrt{\rho}}{2}\left\{\mathrm{e}^{(\sqrt{\rho}-1)x}-\mathrm{e}^{-(\sqrt{\rho}+1)x}\right\},$$

which is not completely monotone.

Nevertheless, if we consider

$$W(x) = W_0 * G(x) = \frac{1-\rho}{2}\left[\frac{1}{1-\sqrt{\rho}}\left\{1-\mathrm{e}^{-(1-\sqrt{\rho})x}\right\}+\frac{1}{1+\sqrt{\rho}}\left\{1-\mathrm{e}^{-(1+\sqrt{\rho})x}\right\}\right],$$

we can see that it has a completely monotone density on $(0,\infty)$:

$$W'(x) = \frac{1-\rho}{2}\left\{\mathrm{e}^{-(1-\sqrt{\rho})x}+\mathrm{e}^{-(1+\sqrt{\rho})x}\right\}.$$

3 Complete monotonicity of ruin probabilities

The Sparre Andersen model generalizes the classical risk model by replacing the Poisson arrivals of claims by a renewal process, so that the surplus at time t is given by

$$U(t) := x + ct - \sum_{k=1}^{N(t)} Z_k, \quad t \geq 0, \tag{3.1}$$

where $x \geq 0$ is the initial surplus, c the positive constant premium income rate, $\{N(t), t \geq 0\}$ a renewal process with intensity $\lambda > 0$, and $\{Z_k\}_{k\geq 1}$ a sequence of positive i.i.d. random variables. We assume that $\{N(t)\}$ and $\{Z_k\}$ are independent. Denote by H and μ the distribution function and the mean, respectively, of the claim sizes Z_k, by T_k the time between the $(k-1)$th and the kth claim, for $k \geq 2$ and by T_1 the time from zero to the first claim. Assume the net profit condition holds, which means that $\lim_{t\to\infty} U(t) = \infty$ almost surely. Let F^+ be the (defective) distribution of the ascending ladder height (see e.g. Asmussen & Albrecher, 2010, p. 522, or Rolski *et al.*, 1999, p. 235) of the random walk $\{S_n\}_{n\geq 1}$, where $S_n = \sum_{k=1}^n (Z_k - cT_k)$.

Lemma 3.1. *If the claim size distribution H has a completely monotone density on $(0,\infty)$, then F^+ has a completely monotone density on $(0,\infty)$.*

Proof. Let F^- denote by the weak descending ladder height distribution function of the random walk $\{S_n\}$. It follows from Rolski *et al.* (1999, Corollary 6.4.1) or Asmussen & Albrecher (2010, Theorem A2.1) that for $x > 0$ we have

$$F^+(x) = \int_{-\infty}^{0} K(x-y) \sum_{n=0}^{\infty} (F^-)^{*n}(\mathrm{d}y),$$

where K is the distribution function of the increment $Z_k - cT_k$, i.e. $K(x) = \int_0^\infty H(x+cy)\mathrm{d}L(y)$, in which L is the distribution of T_k. The statement follows from direct verification. □

The ruin probability, defined as

$$\psi(x) = \mathbf{P}(\inf_{t>0} U(t) < 0 | U(0) = x), \quad x \geq 0.$$

satisfies the Pollaczeck–Khinchine type formula (Rolski *et al.*, 1999, Theorem 6.5.1)

$$1 - \psi(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n F^{*n}(x),$$

where $F(x) = \rho^{-1}F^+(x)$ and $\rho = F^+(\infty)$. Thus, from Lemma 3.1 and Corollary 2.3, we have the following result.

Theorem 3.2. *For the Sparre Andersen model, if the claim size distribution H has a completely monotone density on $(0,\infty)$, then $\psi(x)$ is completely monotone on $(0,\infty)$.*

A stronger version of Theorem 3.2 can be obtained by considering the Laplace transform of the ruin time T:

$$\Psi_\beta(x) = \mathbf{E}(\mathrm{e}^{-\beta T}\mathbf{1}_{\{T<\infty\}} | U(0) = x).$$

Note that $\psi(x) = \Psi_0(x)$. It can be shown (Willmot, 2007, Eq. (2.14)) that if H has a density h, then $\Psi_\beta(x)$ has the representation

$$\Psi_\beta(x) = (1-\theta_\beta) \sum_{n=1}^{\infty} {\theta_\beta}^n \overline{F_\beta^{*n}}(x),$$

where $\overline{F_\beta^{*n}}(x) = 1 - F_\beta^{*n}(x)$, $F_\beta(x) = \int_0^x f_\beta(y)\mathrm{d}y$, $\theta_\beta = \int_0^\infty g_\beta(x|0)\mathrm{d}x$ and

$$f_\beta(y) = {\theta_\beta}^{-1} \int_0^\infty \frac{h(x+y)}{1-H(x)} g_\beta(x|0)\mathrm{d}x, \tag{3.2}$$

in which $g_\beta(\cdot|0)$ is the discounted defective marginal density of the surplus prior to ruin, given $U(0) = 0$ (see e.g. Asmussen & Albrecher, 2010, p. 358). Hence, the next theorem follows from Corollary 2.3.

Theorem 3.3. *For the Sparre Andersen model, if the claim size distribution H has a completely monotone density on $(0, \infty)$, then $\Psi_\beta(x)$ is completely monotone on $(0, \infty)$.*

Another extension of $\psi(x)$ is the joint distribution of ruin and the deficit at ruin:

$$G(x, y) = \mathbf{P}(|U(T)| < y, T < \infty | U(0) = x),$$

so that $\psi(x) = G(x, \infty)$. It can be shown (Rolski *et al.*, 1999, Corollary 6.5.4) that the tail $\overline{G}(x, y) = \psi(x) - G(x, y)$ satisfies

$$\overline{G}(x, y) = \rho \int_0^x \overline{F}(x + y - z) \sum_{n=0}^{\infty} (F^+)^{*n}(\mathrm{d}z),$$

where $\overline{F}(x) = 1 - F(x)$. For fixed $x \geq 0$, define

$$\overline{G}_x(y) = 1 - G_x(y) = \frac{\overline{G}(x, y)}{\psi(x)}, \quad y \geq 0.$$

Clearly, G_x is a proper distribution function. Willmot (2002, Theorem 3.1) has shown that G_x is a decreasing failure rate distribution if the claim size distribution H is a decreasing failure rate distribution. By Lemma 3.1, we have the following analogous result for the complete monotonicity.

Theorem 3.4. *For the Sparre Andersen model, if the claim size distribution H has a completely monotone density on $(0, \infty)$, then G_x has a completely monotone density on $(0, \infty)$.*

Remark 3.5. *We conjecture that the converse of Lemma 3.1, and consequently the converses of Theorems 3.2–3.4, are also true.*

Corresponding results for the classic risk model perturbed by a Wiener process and for spectrally negative Lévy processes can be established by our results, as well as by properties of the Wiener–Hopf factors (Rogers, 1983, Theorem 2) or the potential theory (Loeffen, 2008, p. 1675, and Song & Vondraček, 2006) of Lévy processes.

Acknowledgments

SNC was supported by the Research Grants Council of the Hong Kong Special Administrative Region, China (Project Nos. HKBU200807 and HKBU200710) and an FRG grant of the Hong Kong Baptist University; CY was supported by the National Natural Science Foundation of China (No. 10771119) and the Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20093705110002).

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