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Summary. Diggle's test for complete spatial randomness of a given point pattern uses the discrepancy between the estimated and the theoretical form of a summary function as the test statistic. One commonly used discrepancy measure is the supremum of the pointwise differences over a suitably chosen range; the upper bound of the range is an arbitrary but sometimes crucial parameter. This paper shows that when we use Ripley's K-function as the summary function, it is possible to avoid using an arbitrary upper bound by using adapted distance dependent intensity estimators.

Key words: Complete spatial randomness; K-function; Spatial point pattern; Edge-correction; Intensity estimator.

1 Introduction

A spatial point pattern consists of a finite number of points in a bounded sampling window in $\mathbb{R}^2$ (Boots and Getis, 1988; Diggle, 2003; Stoyan *et al.*, 1995). An often asked question in the analysis of such patterns is whether a given point pattern is completely spatially random (CSR). A typical approach is to compare the theoretical form of a summary function under the CSR hypothesis with its estimate. The most popular summary function of a stationary point process is perhaps Ripley's K-function, which is defined as

$$K(t) = \frac{\text{Mean number of further points within distance } t \text{ of an arbitrary point}}{\lambda}, \quad \text{for } t \geq 0,$$

where λ is the intensity of the point process. However, the estimation of the K-function is intervened by the edge-effects because point patterns are observed via a bounded sampling window W, which is only a part of a larger, probably infinite, region, where the underlying point process takes place. As a result, whenever a disk of radius t centred at an observed point is not entirely contained in W, the number of further points within distance t of that observed point will be censored. Various edge-corrected estimators have been proposed in the literature.

Many authors (e.g., Diggle, 1979; Gignoux *et al.*, 1999; Haase, 1995; Ripley, 1979; Yamada and Rogerson, 2003) investigated the power of different edge-corrected or uncorrected estimators $\hat{K}$ of the K-function in testing the CSR hypothesis by using the test statistic

$$\sup_{t \leq t_0} \left| \sqrt{K(t)} - \sqrt{\hat{K}(t)} \right|, \tag{1}$$

where t_0 is a suitably chosen upper limit. The choice of t_0 sometimes is crucial for testing

CSR because typically the variance of $\sqrt{\hat{K}(t)}$ increases as t increases. For data in a unit square, Ripley (1979) suggested that t_0 should be 0.25 for 25 points and 0.125 for 100 points or in this proportion, and Diggle (2003, p. 87) suggested that t_0 should be no bigger than 0.25. However, these suggestions are somewhat arbitrary and, as we will see in our simulation, may lead to low power tests. Moreover, for a pattern with many points, as the ecology data considered in Section 4, the corresponding t_0 is quite small, meaning that we may not be able to detect medium-range interactions. This paper suggests that if we use the λ^2-estimator proposed by Stoyan and Stoyan (2000), we can avoid this arbitrary upper limit.

2 Testing procedure

Denote by Φ the random set, as well as the counting measure, of a stationary point process in $\mathbb{R}^2$ with intensity λ, and $v_2(\cdot)$ the area measure in $\mathbb{R}^2$. Estimators of $K(t)$ are usually of the form

$$\hat{K}(t) = \frac{\hat{\kappa}(t)}{\hat{\lambda^2}}, \tag{2}$$

where $\hat{\kappa}(t)$ is an estimator of $\lambda^2 K(t)$ and $\hat{\lambda^2}$ is an estimator of λ^2. For a point pattern observed in the sampling window W, two popular edge-corrected estimators of $\lambda^2 K(t)$ are

$$\begin{aligned}\hat{\kappa}_s(t) &= \sum_{\substack{x,y\in\Phi\\ x\neq y}} \mathbf{1}_W(x)\mathbf{1}_W(y)\frac{\mathbf{1}_{b(0,t)}(y-x)}{v_2(W_x\cap W_y)},\\ \hat{\kappa}_i(t) &= \sum_{\substack{x,y\in\Phi\\ x\neq y}} \mathbf{1}_W(x)\mathbf{1}_W(y)\frac{\mathbf{1}_{[0,t]}(\|y-x\|)k(x,y)}{v_2(W^{(\|y-x\|)})},\end{aligned}$$

where $b(x,a)$ is a ball centred at x with radius a, $\partial b(x,a)$ is the boundary of $b(x,a)$, $W^{(a)} = \{x \in W : \partial b(x,a) \cap W \neq \emptyset\}$, $W_x = W + x = \{y + x : y \in W\}$, $\mathbf{1}_W(\cdot)$ is the indicator function of the set W and $k(x, y)$ is equal to 2π divided by the sum of all angles of the arcs in $\partial b(x, \|x - y\|) \cap W$ (Stoyan *et al.*, 1995, pp. 134-135). The estimators $\hat{\kappa}_s$ and $\hat{\kappa}_i$ are unbiased if the point process is, respectively, stationary (invariant in distribution under translation) and stationary as well as isotropic (invariant in distribution under rotation). Thus, the edge-correction used in $\hat{\kappa}_s$ is known as the translational correction, whilst that used in $\hat{\kappa}_i$ the isotropic correction.

For λ^2, a natural estimator is $\hat{\lambda}_1^2 = \{\Phi(W)/v_2(W)\}^2$ (Ripley, 1976). It also has been suggested (e.g., Diggle, 2003, p. 50) to use $\hat{\lambda}_2^2 = \Phi(W)\{\Phi(W) - 1\}/v_2(W)^2$. However, estimators of $K(t)$ with either one of these two λ^2-estimator as the denominator usually have large variance for large t. Besag (1977) proposed to take the square root transform to stabilize the variance and so the statistic given in (1) is the discrepancy between the square root of the hypothesized K and that of the estimated K. However, as can be seen in Figure 1, the variance of $\sqrt{\hat{K}(t)}$ still increases as t increases and so it has been recommended to take the supremum of $\left|\sqrt{K(t)} - \sqrt{\hat{K}(t)}\right|$ over $0 \leq t \leq t_0$ for a suitably chosen t_0.

Figure 1 about here

Stoyan and Stoyan (2000) established a class of adapted distance dependent λ^2-estimator. Two particular examples are the volume weighted estimator

$$\hat{\lambda}_3^2(t) = \left\{ \sum_{x \in \Phi} \frac{\mathbf{1}_W(x) v_2(W \cap b(x, t))}{\pi t^2 - \frac{8}{3} t^3 + \frac{1}{2} t^4} \right\}^2,$$

and the surface weighted estimator

$$\hat{\lambda}_4^2(t) = \left\{ \sum_{x \in \Phi} \frac{\mathbf{1}_W(x) v_1(W \cap \partial b(x, t))}{2\pi t(1 - \frac{4}{\pi} t + \frac{t^2}{\pi})} \right\}^2,$$

where $v_1(\cdot)$ is the arc length measure. Simulation reveals that using $\hat{\lambda}_3^2(t)$ as the denominator of the estimator of $K(t)$, the variance of $\sqrt{\hat{K}(t)}$ can be stabilized further. Thus, it is reasonable to use $\hat{\kappa}_s(t)/\hat{\lambda}_3^2(t)$ as $\hat{K}(t)$ in the test statistics given in (1), hoping that the arbitrary choice of t_0 will then be less crucial and we may simply take t_0 as the largest t such that $K(t)$ can be estimated.

3 Simulation

In our simulation study, the CSR hypothesis was tested against two alternative models, namely, the conditional Poisson cluster process and the Strauss process with N points.

The Poisson cluster process generates clustered patterns in which cluster centres (parents) follow a stationary Poisson process and then an independent and identically distributed cluster of points (daughters) are assigned to each parent. To condition the cluster process to have exactly N points in a unit square, we followed Diggle (1979) but instead of the normal distribution we used the uniform distribution in a compact set (cluster), for the locations of daughter points. First, N_c independent parents are distributed uniformly in a unit square and then N daughter points are assigned randomly to these clusters and such that each daughter is located uniformly within her parent's cluster under the periodic boundary condition. We consider isotropic cluster processes in which all clusters are disks with radius R and anisotropic processes in which all clusters are square with area πR^2 and their orientations are fixed and the same. The latter is anisotropic because the squares are not invariant under rotation.

The Strauss process is a pairwise interaction point process that produces, by self-inhibiting, patterns in which points are more spread out than they would be in CSR; such patterns exhibit regularity and so are often called regular patterns. We start with a Poisson process in a bounded region and then define the Strauss process by giving a probability density with respect to the Poisson process. The probability density of the Strauss process has a single parameter C that controls the strength of inhibition and a parameter R for the range of inhibition so that $C = 0$ and $C = 1$ correspond to the hard core process with hard core distance R and the Poisson process, respectively; for $0 < C < 1$ we have a self-inhibiting point process. For details, see Kelly and Ripley (1976) and Strauss (1975).

For each alternative we performed 100 times the Monte Carlo test suggested by Diggle (1979) to estimate the power: the CSR hypothesis would be rejected whenever the test statistic calculated from a pattern generated according to the alternative model, when pooled together with the values of the same statistic calculated from 99 simulated patterns of a binomial process, has been ranked in the top 5%. A binomial process is a conditional Poisson process with a fixed number N of points in the sampling window W, i.e. a point process in which N independent points are uniformly distributed in W.

We have eight different $\hat{K}(t)$'s to insert in the test statistic given in (1) and all of them are of the form given in (2). We have either the translational correction $\hat{\kappa}_s(t)$ or the isotropic correction $\hat{\kappa}_i(t)$ as the numerator and one of the four λ^2-estimators as the denominator. For the upper limit t_0, we first used the value recommended by Ripley (1979), i.e. $t_0 = 0.25$ for $N = 25$ and $t_0 = 0.125$ for $N = 100$, which proportionally imply $t_0 = 0.177$ for $N = 50$, and then we considered, somewhat arbitrarily, $t_0 = 0.5$ and $t_0 = 0.6$. Finally we considered the half of the diagonal of the unit square $t_0 = \sqrt{2}/2$. Estimated powers for isotropic cluster processes with circular clusters, anisotropic cluster processes with square clusters and Strauss processes are given in Tables 1, 2 and 3, respectively, and from which we have the following conclusions.

Table 1 about here

Table 2 about here

Table 3 about here

If λ^2 is estimated by $\hat{\lambda}_1^2$ and $\hat{\lambda}_2^2$, then no matter the cluster process is isotropic or anisotropic, the isotropic correction works as good as or, when $t_0 \geq 0.5$, even better than the translational correction. The same is true for the Strauss process. An explanation is that for the binomial process the variance of the square root of the estimated $K(t)$ with the isotropic correction as the numerator is smaller than that with the translational correction as the numerator (see Figure 1). Consider the choice of the upper limit t_0. For each particular Strauss process simulated, using the values recommended by Ripley (1979) yields a much higher power that using 0.5, 0.6 or $\sqrt{2}/2$. This is, however, not always true for the cluster processes, especially when R is larger than the recommended t_0.

Suppose that λ^2 is estimated by $\hat{\lambda}_3^2$ and $\hat{\lambda}_4^2$, which are then used as the denominator in the test statistic. We can observe that, unlike using $\hat{\lambda}_1^2$ or $\hat{\lambda}_2^2$ as the denominator, the translational correction usually works better than the isotropic correction. Furthermore, using $\hat{\lambda}_3^2$ as the denominator typically gives a more powerful test statistic than using $\hat{\lambda}_4^2$.

These two observations can be explained by Figure 1, which shows that $\sqrt{\hat{\kappa}_s(t)/\hat{\lambda}_3^2(t)}$ has the smallest variance. More importantly, a major advantage of using $\hat{\kappa}_s(t)/\hat{\lambda}_3^2(t)$ as $\hat{K}(t)$ in (1) is that since the variance of $\sqrt{\hat{K}(t)}$ is stabilized and is the smallest, the choice of t_0 is much less crucial than other estimators, while the powers, at any t_0, are often as high as, if not higher than, the powers at t_0 equal to the Ripley's recommended values.

4 Real data

We consider the data studied in Grabarnik and Chiu (2002), which are the locations of trees in a 75m × 75m region of broad-leaved multispecies old-growth forest in the south-east of Central Russia (Smirnova, 1994), excluding the trees which are fully overshadowed by neighbouring trees. When we rescale the study region to a unit square, the recommended value of t_0 for 270 points in a unit square should be 0.0761, which is rather small.

Grabarnik and Chiu (2002) showed that their Q^2-statistic provides strong evidence against the CSR hypothesis (p-value = 0.0041) for these data. We obtained the Monte Carlo p-value for each of the eight statistics at $t_0 = 0.0761$, 0.5, 0.6 and $\sqrt{2}/2$ on the basis of 99 independent realizations of the binomial process with the same number of points. The p-values are given in Table 4, from which we can see that only the test statistic that uses the translational correction and the volume weighted λ^2-estimator suggests rejection of the CSR hypothesis at the 0.05 significance level.

Table 4 about here

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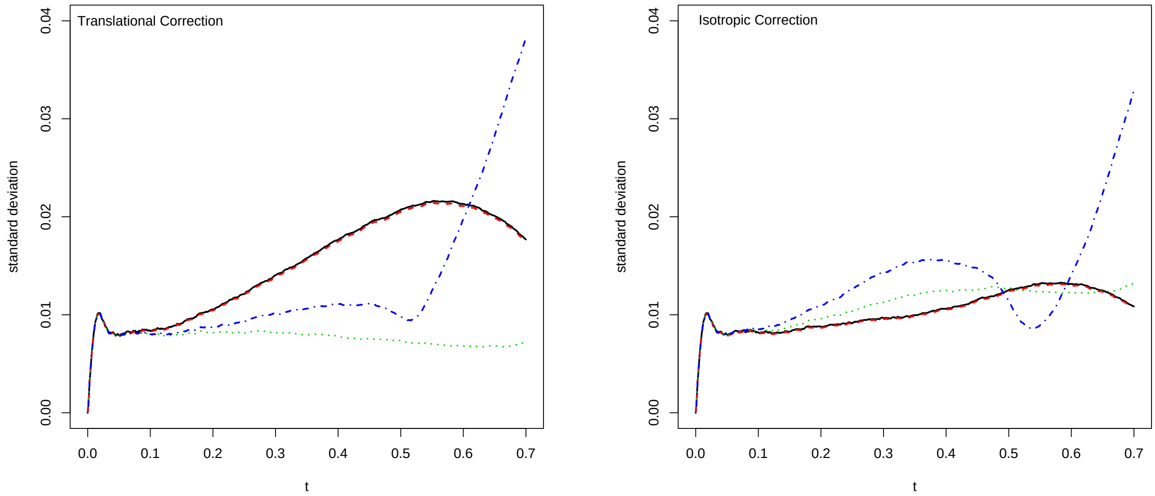

Figure 1: Estimated standard deviations of $\sqrt{\hat{K}(t)}$ for the binomial process with 100 points in a unit square by the translational edge-correction $\hat{\kappa}_s$ and the isotropic edge-correction $\hat{\kappa}_i$ with four different λ^2-estimators (—, $\hat{\lambda}_1^2$; - - -, $\hat{\lambda}_2^2$; $\cdots$, $\hat{\lambda}_3^2$; -·-, $\hat{\lambda}_4^2$).

Table 1: Estimated power (in percent) for testing CSR against cluster processes that generate N points uniformly in N_c circular clusters with radius R in a unit square under the periodic boundary condition.

				Translational				Isotropic			
N	N_c	R	t_0	$\hat{\lambda}_1^2$	$\hat{\lambda}_2^2$	$\hat{\lambda}_3^2$	$\hat{\lambda}_4^2$	$\hat{\lambda}_1^2$	$\hat{\lambda}_2^2$	$\hat{\lambda}_3^2$	$\hat{\lambda}_4^2$
25	10	0.1	0.25	77	82	77	75	77	81	73	71
			0.5	59	71	79	74	80	84	73	63
			0.6	59	71	79	74	80	84	73	63
			$\sqrt{2}/2$	56	66	78	43	70	81	73	44
50	10	0.2	0.177	64	70	60	58	61	65	56	50
			0.5	54	56	70	68	64	73	61	55
			0.6	50	53	70	50	64	70	64	55
			$\sqrt{2}/2$	52	56	74	33	66	73	69	33
	5	0.2	0.177	92	92	92	91	93	96	91	89
			0.5	80	82	92	92	94	94	86	82
			0.6	78	80	98	81	91	96	89	93
			$\sqrt{2}/2$	77	77	93	57	89	76	92	60
100	10	0.2	0.125	87	89	91	92	91	92	91	93
			0.5	73	75	97	90	90	97	93	95
			0.6	72	73	97	76	86	91	93	81
			$\sqrt{2}/2$	72	74	97	51	86	90	93	57
	10	0.3	0.125	32	36	35	38	33	40	38	38
			0.5	45	44	59	49	56	62	48	44
			0.6	43	44	59	37	51	56	56	45
			$\sqrt{2}/2$	42	43	67	37	50	54	63	36

Table 2: Estimated power (in percent) for testing CSR against cluster processes that generate N points uniformly in N_c square clusters with area πR^2 each in a unit square under the periodic boundary condition.

				Translational				Isotropic			
N	N_c	R	t_0	$\hat{\lambda}_1^2$	$\hat{\lambda}_2^2$	$\hat{\lambda}_3^2$	$\hat{\lambda}_4^2$	$\hat{\lambda}_1^2$	$\hat{\lambda}_2^2$	$\hat{\lambda}_3^2$	$\hat{\lambda}_4^2$
25	10	0.1	0.25	77	82	76	76	75	82	74	73
			0.5	60	67	78	72	75	82	73	66
			0.6	52	63	79	63	73	79	72	67
			$\sqrt{2}/2$	50	63	77	30	69	78	71	36
50	10	0.2	0.177	62	65	67	59	63	69	62	56
			0.5	57	57	70	66	66	69	64	60
			0.6	57	57	71	59	65	70	64	60
			$\sqrt{2}/2$	58	58	74	37	68	71	67	40
	5	0.2	0.177	94	94	98	95	98	98	95	95
			0.5	88	90	98	95	98	96	97	90
			0.6	89	90	98	91	95	98	96	89
			$\sqrt{2}/2$	90	89	97	70	93	98	96	77
100	10	0.2	0.125	92	94	91	90	92	93	89	86
			0.5	71	71	96	91	95	97	91	76
			0.6	71	72	95	73	91	95	92	79
			$\sqrt{2}/2$	73	73	96	53	92	95	91	58
	10	0.3	0.125	33	37	32	32	30	38	35	33
			0.5	43	43	57	54	54	61	52	46
			0.6	45	43	56	40	57	58	55	47
			$\sqrt{2}/2$	46	45	62	33	55	59	60	34

Table 3: Estimated power (in percent) for testing CSR against Strauss processes with N points, where C is the parameter that controls the strength of interaction and R is the range of interaction.

				Translational				Isotropic			
N	C	R	t_0	$\hat{\lambda}_1^2$	$\hat{\lambda}_2^2$	$\hat{\lambda}_3^2$	$\hat{\lambda}_4^2$	$\hat{\lambda}_1^2$	$\hat{\lambda}_2^2$	$\hat{\lambda}_3^2$	$\hat{\lambda}_4^2$
25	0	0.05	0.25	51	48	53	53	50	50	51	50
			0.5	4	7	42	30	30	32	25	14
			0.6	2	1	40	8	22	18	23	10
			$\sqrt{2}/2$	1	1	35	5	20	12	16	4
50	0.1	0.05	0.177	91	89	90	89	91	90	88	86
			0.5	35	35	89	77	75	76	75	48
			0.6	21	24	86	38	60	64	66	43
			$\sqrt{2}/2$	18	23	86	2	56	61	55	2
	0.2	0.05	0.177	71	66	74	71	71	71	71	66
			0.5	17	20	69	52	52	51	49	31
			0.6	10	11	65	22	39	42	42	29
			$\sqrt{2}/2$	10	10	62	3	37	38	32	3
100	0.1	0.03	0.125	99	99	98	98	98	98	98	98
			0.5	17	17	97	83	81	82	83	41
			0.6	9	9	96	21	63	67	70	35
			$\sqrt{2}/2$	9	9	96	4	57	66	55	3
	0.2	0.03	0.125	89	88	88	88	86	88	87	88
			0.5	7	7	85	55	51	58	58	24
			0.6	5	5	83	12	34	39	49	19
			$\sqrt{2}/2$	5	4	79	5	37	31	34	6

Table 4: Estimated p-value for testing CSR of the locations of 270 trees in the south-east of Central Russia (Smirnova, 1994).

	Translational				Isotropic			
t_0	$\hat{\lambda}_1^2$	$\hat{\lambda}_2^2$	$\hat{\lambda}_3^2$	$\hat{\lambda}_4^2$	$\hat{\lambda}_1^2$	$\hat{\lambda}_2^2$	$\hat{\lambda}_3^2$	$\hat{\lambda}_4^2$
0.0761	0.17	0.16	0.15	0.16	0.11	0.08	0.09	0.10
0.5	0.51	0.49	0.01	0.26	0.12	0.11	0.22	0.47
0.6	0.60	0.57	0.01	0.74	0.22	0.15	0.25	0.57
$\sqrt{2}/2$	0.61	0.58	0.05	0.86	0.25	0.16	0.32	0.77