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CIRCULAR $L(j,k)$-LABELING NUMBERS OF SQUARE OF PATHS*

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Abstract

Let j, k and σ be positive numbers, a circular σ-$L(j,k)$-labeling of a graph G is a function $f : V(G) \to [0,\sigma)$ such that $|f(u) - f(v)|_\sigma \geq j$ if u and v are adjacent, and $|f(u) - f(v)|_\sigma \geq k$ if u and v are at distance two, where $|a-b|_\sigma = \min\{|a-b|, \sigma - |a-b|\}$. The minimum σ such that there exist a circular σ-$L(j,k)$-labeling of G is called the circular-$L(j,k)$-labeling number of G and is denoted by $\sigma_{j,k}(G)$. The k-th power G^k of an undirected graph G is a graph with the same set of vertices and an edge between two vertices when their distance in G is at most k. In this paper, the circular $L(j,k)$-labeling numbers of P_n^2 are determined.

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1. Introduction

The rapid growth of computer wireless networks highlighted the scarcity of available codes (such as radio frequencies) for communication with minimum interference. For example, the Packet Radio Network (PRN) is a computer network that using radio frequencies to transmit packet among computers. The two major types of interference in PRN are direct collision (or interference), which caused by the transmission of adjacent stations (computers), and hidden terminal collision (or interference), which caused by distance-two stations that transmit to the same receiving station or receive the data from the same transmitting station.

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Bertossi and Bonuccelli [1] introduced an optimal code assignment to avoid hidden terminal interference and Jin and Yeh [10] further generalized it to the $L(j,k)$-labeling problem with $j \leq k$.

Let G be a graph and let $V(G)$ and $E(G)$ be its vertex set and edge set, respectively. For any two vertices u and v, let $d_G(u,v)$ (or simply $d(u,v)$) denote the distance (length of a shortest path) between u and v in G. All notation not defined in this article can be found in the book [2].

For positive numbers j and k, an *$L(j,k)$-labeling* f of G is an assignment of numbers to vertices of G such that $|f(u)-f(v)| \geq j$ if $uv \in E(G)$, and $|f(u)-f(v)| \geq k$ if $d(u,v)=2$. The *span* of f is the difference between the maximum and the minimum numbers assigned by f. The *$L(j,k)$-labeling number* of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labeling of G.

The introduction of $L(j,k)$-labeling with $j \geq k$ was motivated from the channel assignment problem [7]. Afterwards, Heuvel, Leese and Shepherd initiated the investigation of circular $L(j,k)$-labeling of a graph in [8]. They used the "circular distance" (will be defined below) to instead of the "linear distance".

For a positive real number σ, let $S(\sigma)$ denote the circle obtained from the closed interval $[0,\sigma]$ by identifying 0 and σ into a single point. $S(\sigma)$ is called a *σ-cycle*. For any $x \in \mathbb{R}$, $[x]_\sigma \in [0,\sigma)$ denotes the remainder of x upon division of σ. The *circular distance* of two points p and q on $S(\sigma)$ is defined as $|p-q|_\sigma = \min\{|p-q|, \sigma-|p-q|\}$.

A *circular σ-$L(j,k)$-labeling* of G is a function $f: V(G) \to S(\sigma)$ satisfying $|f(x)-f(y)|_\sigma \geq j$ if $d(x,y)=1$ and $|f(x)-f(y)|_\sigma \geq k$ if $d(x,y)=2$. The minimum σ such that there exists a circular σ-$L(j,k)$-labeling of G is called the *circular $L(j,k)$-labeling number* of G and is denoted by $\sigma_{j,k}(G)$. $L(j,k)$-labeling numbers or circular $L(j,k)$-labeling numbers of graphs were studied in many articles, please refer to [4–6, 11, 12, 14–17, 19, 25] for $j \geq k$ and [1, 3, 10, 13, 20–24] for $j \leq k$.

2. Main Results

The *k-th power* G^k of an undirected graph G is a graph with the same set of vertices and an edge between two vertices when their distance in G is at most k. G^2 is called the *square* of G.

Lemma 2.1. *Let j and k be two positive numbers with $j \leq k$. Suppose G is a graph and H is an induced subgraph of G. Then $\sigma_{j,k}(G) \geq \sigma_{j,k}(H)$.*

Note that Lemma 2.1 is not true if H is not an induced subgraph. Suppose f is a circular σ-$L(j,k)$-labeling of G. For any $a \in \mathbb{R}$, define $g(v) = [f(v)-a]_\sigma$ for $v \in V(G)$. Here g is still a circular σ-$L(j,k)$-labeling of G. In this paper, the path is denoted by $P_n = v_0v_1\cdots v_{n-1}$. Without loss of generality, we may always assume v_0 is labeled by 0 when we consider circular $L(j,k)$-labelings of P_n^2. Also, in this paper we shall use $[a,b]$ to denote the close interval from a to b. Similar definition are applied for semi-open intervals $(a,b]$ and $[a,b)$.

Lemma 2.2. *Let $a,b \in \mathbb{R}$ and let σ, h and l be positive numbers. Suppose $|a-b| \geq h$ and $|a-b|_\sigma \geq l$. Then $\sigma \geq h+l$.*

Proof. The lemma follows from $l \le |a-b|_\sigma = \min\{|a-b|, \sigma-|a-b|\} \le \sigma-|a-b| \le \sigma-h$. □

Theorem 2.1. *For $j \le k$, $\sigma_{j,k}(P_4^2) = \max\{2k,4j\}$.*

Proof. Let $\eta = \max\{2k,4j\}$. Define a circular labeling f for P_4^2 as follows:
If $k \ge 2j$, let $f(v_0)=0$, $f(v_1)=j+k$, $f(v_2)=j$ and $f(v_3)=k$.
If $j \le k < 2j$, let $f(v_0)=0$, $f(v_1)=3j$, $f(v_2)=j$ and $f(v_3)=2j$.

It is clearly that f is a circular η-$L(j,k)$-labeling of P_4^2 for $j \le k$. Then $\sigma_{j,k}(P_4^2) \le \max\{2k,4j\}$.

On the other hand, let f be a circular σ-$L(j,k)$-labeling for P_4^2. Note that by our assumption, $f(v_0)=0$. Since $d(v_0,v_3)=2$, $f(v_3)\ge k$. By Lemma 2.2, this implies that $\sigma \ge 2k$. Moreover, since any two vertices of P_4^2 are adjacent or of distance two, $\max\limits_{1\le i\le 3} f(v_i) \ge 3j$. Since $|f(v_s)-f(v_t)|_\sigma \ge j$ and $\max\{|f(v_s)-f(v_t)|\} \ge 3j$ for $0 \le s < t \le 3$, by Lemma 2.2 we have $\sigma \ge 4j$. Thus, $\sigma \ge \max\{2k,4j\}$.

Hence, $\sigma_{j,k}(P_4^2) = \max\{2k,4j\}$. □

Theorem 2.2. *For $j \le k$, $\sigma_{j,k}(P_5^2) = \max\{j+2k,5j\}$.*

Proof. Let $\eta = \max\{j+2k,5j\}$. Given a circular labeling f for P_5^2 as follows:
If $k \ge 2j$, let $f(v_0)=0$, $f(v_1)=2j+k$, $f(v_2)=j$, $f(v_3)=j+k$ and $f(v_4)=k$.
If $j \le k < 2j$, let $f(v_0)=0$, $f(v_1)=4j$, $f(v_2)=j$, $f(v_3)=3j$ and $f(v_4)=2j$.

It is easy to verify that f is a circular η-$L(j,k)$-labeling of P_5^2 for $j \le k$. Then $\sigma_{j,k}(P_5^2) \le \max\{j+2k,5j\}$.

On the other hand, let f be a circular σ-$L(j,k)$-labeling for P_5^2. Since $d(v_0,v_3) = d(v_0,v_4) = 2$ and v_3 and v_4 are adjacent, $\max\limits_{i=3,4} f(v_i) \ge j+k$. Again since $d(v_0,v_i)=2$ for $i=3,4$, by Lemma 2.2 we have $\sigma \ge j+2k$. Meantime, since any two vertices of P_5^2 are adjacent or of distance two, $\max\limits_{1\le i\le 4} f(v_i) \ge 4j$. Since $|f(v_s)-f(v_t)|_\sigma \ge j$ and $\max\{|f(v_s)-f(v_t)|\} \ge 4j$ for $0 \le s < t \le 4$, by Lemma 2.2 we have $\sigma \ge 5j$. Thus, $\sigma \ge \max\{j+2k,5j\}$.

Hence, $\sigma_{j,k}(P_5^2) = \max\{j+2k,5j\}$ for $j \le k$. □

Theorem 2.3. *Let j and k be two positive numbers with $j \le k < 3j$. For $n \ge 6$, $\sigma_{j,k}(P_n^2) = 4j+k$.*

Proof. Let $\sigma = 4j+k$. Define a circular labeling f of P_n^2 as follows.
$f(v_i) = [ij]_\sigma$, for $0 \le i \le n-1$. It is easy to verify that f is a circular σ-$L(j,k)$-labeling of P_n^2 for $j \le k < 3j$ and $n \ge 6$. Then $\sigma_{j,k}(P_n^2) \le 4j+k$ for $j \le k < 3j$ and $n \ge 6$.

On the other hand, since P_6^2 is an induced subgraph of P_n^2 for $n \ge 6$, by Lemma 2.1, we have $\sigma_{j,k}(P_n^2) \ge \sigma_{j,k}(P_6^2)$. It suffices to find a lower bound of $\sigma = \sigma_{j,k}(P_6^2)$.

Let f be a circular σ-$L(j,k)$-labeling for P_6^2. Note that $\sigma \le 4j+k$. As mentioned before $f(v_0)=0$. By considering the distances of v_i from v_0, we have $f(v_1), f(v_2) \in [j,\sigma-j]$ and $f(v_3), f(v_4) \in [k,\sigma-k]$. Without loss of generality, we may assume that $f(v_1) < f(v_4)$, otherwise consider the new labeling $[\sigma-f]_\sigma$. Here we have $f(v_1) \in [j,\sigma-2k]$ and $f(v_4) \in [2j,\sigma-k]$. Since $d(v_1,v_5)=2$, we have $|f(v_1)-f(v_5)|_\sigma \ge k$, that is, $k \le |f(v_1)-f(v_5)| \le \sigma-k$. So we have $f(v_5) \in [j+k,\sigma)$ when $f(v_1) < f(v_5)$ or $f(v_5) \in [0,\sigma-3k]$ when $f(v_1) > f(v_5)$.

Case 1. Suppose $2j < k < 3j$. Here $\sigma - 3k < 0$. Under this condition, by the same argument above, we also have $f(v_2) < f(v_5)$. Thus $f(v_5) \in [j+k, \sigma)$. This implies that $f(v_2) \in [j, \sigma - k)$. Since the length of $[j, \sigma - 2k]$ is $\sigma - j - 2k \leq 3j - k < j$, $f(v_2), f(v_3) \in [2j, \sigma - k]$. Now $f(v_2), f(v_3), f(v_4) \in [2j, \sigma - k]$. Since v_2, v_3, v_4 induce a K_3, $\sigma - k - 2j \geq 2j$. Hence $\sigma \geq 4j + k$.

Case 2. Suppose $j \leq k \leq 2j$. We have two cases:

a. Suppose $f(v_5) \in [0, \sigma - 3k]$. Similarly, we obtain that $f(v_1), f(v_2) \in [k, \sigma - j]$, and $f(v_3), f(v_4) \in [k, \sigma - k]$. We can conclude that $f(v_i) \in [k, \sigma - j]$ for $1 \leq i \leq 4$. Since the distances between any of two of v_1, v_2, v_3, v_4 are at most 2, it means that $\sigma - j - k \geq 3j$. That is, $\sigma \geq 4j + k$.

b. Suppose $f(v_5) \in [j+k, \sigma)$.
If $f(v_2) < f(v_5)$, then $f(v_2) \in [j, \sigma - k)$. We can conclude that $f(v_i) \in [j, \sigma - k]$ for $1 \leq i \leq 4$. Similar to the above case, we have $\sigma \geq 4j + k$.
If $f(v_2) > f(v_5)$, then $f(v_2) \in [j+2k, \sigma - j]$ and $f(v_5) \in [j+k, \sigma - j - k]$. Combining with $f(v_3), f(v_4) \in [k, \sigma - k]$, we can conclude that $f(v_i) \in [k, \sigma - j]$ for $2 \leq i \leq 5$. Similar to the above case, we have $\sigma \geq 4j + k$.

This completes the proof. □

Theorem 2.4. *Let j and k be two positive numbers with $k \geq 3j$. We have $\sigma_{j,k}(P_6^2) = j + 2k$.*

Proof. Define a circular labeling f for P_n^2 as follows:
$f(v_0) = 0$, $f(v_1) = j$, $f(v_2) = 2j$, $f(v_3) = k$, $f(v_4) = j + k$ and $f(v_5) = 2j + k$.

It is clearly that f is a circular $(j+2k)$-$L(j,k)$-labeling for P_6^2. Then $\sigma_{j,k}(P_6^2) \leq j + 2k$.

On the other hand, since P_5^2 is an induced subgraph of P_6^2, by Lemma 2.1 and Theorem 2.2, we have $\sigma_{j,k}(P_6^2) \geq \sigma_{j,k}(P_5^2) \geq j + 2k$. Hence, the proof completes. □

Theorem 2.5. *Let j and k be two positive numbers with $k \geq 3j$. We have $\sigma_{j,k}(P_n^2) = \frac{7}{3}k$ for $n \geq 7$.*

Proof. Let $\sigma = \frac{7}{3}k$. For $n \geq 7$, define a circular labeling f of P_n^2 by $f(v_i) = [ik/3]_\sigma$ for $0 \leq i \leq n-1$. It is easy to verify that f is a circular $\frac{7}{3}k$-$L(j,k)$-labeling of P_n^2. Thus, $\sigma_{j,k}(P_n^2) \leq \frac{7}{3}k$.

Since P_7^2 is an induced subgraph of P_n^2 for $n \geq 7$, by Lemma 2.1, we have $\sigma_{j,k}(P_n^2) \geq \sigma_{j,k}(P_7^2)$. It suffices to find a lower bound of $\sigma_{j,k}(P_7^2)$.

Let $\sigma = \sigma_{j,k}(P_7^2)$. Note that $\sigma \leq \frac{7}{3}k$. Let f be a circular σ-$L(j,k)$-labeling of P_7^2. We define *the traveling distance* from v_i to v_j, denoted by $\mathcal{L}(i,j)$, to be the length from $f(v_i)$ to $f(v_j)$ along the clockwise direction in the σ-cycle. In P_7^2, if we travel the vertices from v_0, v_4, v_1, v_5, v_2, v_6, v_3 one by one, then the total traveling distance from v_0 to v_3 is $T = \mathcal{L}(0,4) + \mathcal{L}(4,1) + \mathcal{L}(1,5) + \mathcal{L}(5,2) + \mathcal{L}(2,6) + \mathcal{L}(6,3)$. As $k \leq \mathcal{L}(u,v) \leq \sigma - k$ for $d(u,v) = 2$, we have

$$6k \leq T \leq 6(\sigma - k). \tag{2.1}$$

Since $\sigma \leq \frac{7}{3}k$, we have $2 < \frac{6k}{\sigma} \leq \frac{T}{\sigma} \leq 6 - \frac{6k}{\sigma} < 4$. It implies that we arrive $f(v_3)$ from $f(v_0)$ after rotating the σ-cycle 2 or 3 times.

Suppose we arrive $f(v_3)$ from $f(v_0)$ after rotating the σ-cycle 2 times. Since v_3 and v_0 are at distance 2 and $f(v_0) = 0$, it follows that $|T - 2\sigma|_\sigma \geq k$, that is $2\sigma + k \leq T \leq 3\sigma - k$. From (2.1), we have $6k \leq 3\sigma - k$, and thus, $\sigma \geq \frac{7}{3}k$.

Suppose we arrive $f(v_3)$ from $f(v_0)$ after rotating the σ-cycle 3 times. Since v_3 and v_0 are at distance 2 and $f(v_0) = 0$, it follows that $|T - 3\sigma|_\sigma \geq k$, that is $3\sigma + k \leq T \leq 4\sigma - k$. From (2.1), we have $3\sigma + k \leq 6(\sigma - k)$, and hence $\sigma \geq \frac{7}{3}k$.

Then, we have $\sigma_{j,k}(P_n^2) \geq \sigma_{j,k}(P_7^2) = \sigma \geq \frac{7}{3}k$. The proof completes. $\square$

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