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Wai Chee Shiu
Hong Kong Baptist University, wcshiu@hkbu.edu.hk

Richard M. Low
San Jose State University, richard.low@sjsu.edu

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The integer-magic spectra and null sets of the Cartesian product of trees

WAI CHEE SHIU*

Department of Mathematics
Hong Kong Baptist University
224 Waterloo Road, Hong Kong
P.R. China
wcshiu@math.hkbu.edu.hk

RICHARD M. LOW

Department of Mathematics
San José State University
San José, CA 95192
U.S.A.
richard.low@sjsu.edu

Abstract
Let $A$ be a non-trivial, finitely-generated abelian group and $A^* = A \setminus \{0\}$. A graph is $A$-magic if there exists an edge labeling $f$ using elements of $A^*$ which induces a constant vertex labeling of the graph. Such a labeling $f$ is called an $A$-magic labeling and the constant value of the induced vertex labeling is called the $A$-magic value. The integer-magic spectrum of a graph $G$ is the set

$$\text{IM}(G) = \{k \in \mathbb{N} \mid G \text{ is } \mathbb{Z}_k\text{-magic}\},$$

where $\mathbb{N}$ is the set of natural numbers. The null set of $G$ is the set of integers $k \in \mathbb{N}$ such that $G$ has a $\mathbb{Z}_k$-magic labeling with magic value 0. In this paper, we determine the integer-magic spectra and null sets of the Cartesian product of two trees.

1 Introduction

All concepts and notation not explicitly defined in this paper can be found in [2]. Let $G = (V, E)$ be a connected simple graph. For any non-trivial, finitely generated

* This work was supported by Tianjin Research Program of Application Foundation and Advanced Technology (No.14JCYBJC43100), National Natural Science Foundation of China (No.11601391).
abelian group $A$ (written additively), let $A^* = A \setminus \{0\}$. A mapping $f : E \rightarrow A^*$ is called an edge labeling of $G$. Any such edge labeling induces a vertex labeling $f^+ : V \rightarrow A$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$. If there exists an edge labeling $f$ whose induced mapping on $V$ is a constant map, we say that $f$ is an $A$-magic labeling of $G$ and that $G$ is an $A$-magic graph. The corresponding constant is called an $A$-magic value. The integer-magic spectrum of a graph $G$ is the set $\text{IM}(G) = \{k \in \mathbb{N} \mid G$ is $\mathbb{Z}_k$-magic$\}$, where $\mathbb{N}$ is the set of natural numbers. Here, $\mathbb{Z}_1$ is understood to be the set of integers. Generally speaking, it is quite difficult to determine the integer-magic spectrum of a graph. Note that the integer-magic spectrum of a graph is not to be confused with the set of achievable magic values.

Group-magic graphs were studied in [7,9,15,16,26] and $\mathbb{Z}_k$-magic graphs were investigated in [4,6,8,10–14,17–22,27,28]. $\mathbb{Z}$-magic graphs were considered by Stanley [29,30], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. They were also considered in [1,23].

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an $A$-magic graph is due to J. Sedlacek [24,25], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [5] had introduced yet another definition of a magic graph. Over the years, there has been considerable interest in graph labeling problems. The interested reader is directed to Wallis’ [31] monograph on magic graphs and to Gallian’s [3] excellent dynamic survey of graph labelings.

2 Cartesian product of a tree with a path

Some work on group-magic labelings of trees and their related graphs appear within the literature [11–14,17,21,22]. With regards to Cartesian products, Low and Lee [15] showed the following: If $G$ and $H$ are $\mathbb{Z}_k$-magic, then $G \times H$ is $\mathbb{Z}_k$-magic. In this section, we study the group-magicness of the Cartesian product of trees with paths.

With the purpose of constructing large classes of $\mathbb{Z}_k$-magic graphs, Salehi [19,20] introduced the concept of a null set of a graph. The null set of a graph $G$, denoted by $N(G)$, is the set of integers $k \in \mathbb{N}$ such that $G$ has a $\mathbb{Z}_k$-magic labeling with magic value 0. Hence, $N(G) \subseteq \text{IM}(G)$.

It is easy to see that a graph $G$ is $\mathbb{Z}_2$-magic if and only if the degrees of the vertices are of the same parity. Moreover, $2 \in N(G)$ if and only if the degree of each vertex of $G$ is even.

Let $G$ be a graph of order $s$ and $P_t$ be the path of order $t$. Let $V(G) = \{g_1, \ldots, g_s\}$ and $V(P_t) = \{p_1, \ldots, p_t\}$. Consider the Cartesian product graph $G \times P_t$. For a fixed $i$, the subgraph induced by $\{(g_i, p_j) \mid 1 \leq j \leq t\}$ is called a vertical path (or more precisely, the $g_i$-path). For a fixed $j$, the subgraph induced by $\{(g_i, p_j) \mid 1 \leq i \leq s\}$ is called a horizontal graph (or more precisely, the $j$-th graph).
Remark 2.1. For \(P_2 \times P_2 \cong C_4\), we label the edges (clockwise) 1, \(-1\), 1 and \(-1\). Thus, \(N(P_2 \times P_2) = \mathbb{N} = \text{IM}(P_2 \times P_2)\).

Lemma 2.1. Let \(s \geq 2\) and \(t \geq 3\). Then, \(N(P_s \times P_t) = \mathbb{N} \setminus \{2\} = \text{IM}(P_s \times P_t)\).

Proof: Since \(P_s \times P_t\) contains vertices of even and odd degrees, it is not \(\mathbb{Z}_2\)-magic. Let \(P_s = g_1 \cdots g_s\). Label the vertical \(g_1\)-path and \(g_s\)-path by 1 and the other vertical \(g_j\)-paths (if any) by 2, where \(2 \leq j \leq s - 1\); label the horizontal 1-st and \(t\)-th paths by \(-1\) and the other horizontal paths by \(-2\). This yields a \(\mathbb{Z}_k\)-magic labeling with magic value 0, for \(k \in \mathbb{N} \setminus \{2\}\). □

For \(s \geq 3\), \(t \geq 2\) and \(1 \leq r \leq s\), let \(B(r; s, t)\) be the graph obtained from \(P_s \times P_t\) by deleting all edges of the \(r\)-th vertical path. Note that \(B(r; s, t) \cong B(s - r + 1; s, t)\).

Remark 2.2. Observe that \(B(2; 3, 2) \cong C_6\). In this case, we label the edges (clockwise) 1, \(-1\), 1, \(-1\), 1 and \(-1\). Thus, \(N(B(2; 3, 2)) = \mathbb{N} = \text{IM}(B(2; 3, 2))\).

Lemma 2.2. Let \(s \geq 3\), \(t \geq 2\) and \(2 \leq r \leq s - 1\). If \((s, t) \neq (3, 2)\), then \(N(B(r; s, t)) = \mathbb{N} \setminus \{2\} = \text{IM}(B(r; s, t))\).

Proof: Clearly, \(B(r; s, t)\) is not \(\mathbb{Z}_2\)-magic. To obtain a \(\mathbb{Z}_k\)-magic labeling for \(B(r; s, t)\) with magic value 0 (for \(k \neq 2\)), we perform the following steps:

1. Label \(P_s \times P_t\) using the labeling found in the proof of Lemma 2.1.
2. Delete the edges of the \(r\)-th vertical path.
3. Multiply all edge labels that are to the left (or right) of the (former) \(r\)-th vertical path by \(-1\). □

Example 2.1. Here are some labelings (see Figure 1) which illustrate the proofs of Lemmas 2.1 and 2.2 for \(P_3 \times P_3\), \(B(2; 5, 3)\) and \(B(3; 5, 3)\), respectively:

![Figure 1](image)

Definition 2.1. Let \(T\) be a tree, \(u \in V(T)\) and \(\deg(u) \geq 3\). We say that \(u\) has the 2-\(\text{pendant paths property}\) to mean the following:
• There exists two paths $uv_1v_2\cdots v_a$ and $uw_1w_2\cdots w_b$.

• $T$ is the edge-disjoint union of $[T - \{v_i \mid 1 \leq i \leq a\} \cup \{w_j \mid 1 \leq j \leq b\}]$ and path $w_b\cdots w_1uv_1\cdots v_a$, through identification of vertex $u$.

**Lemma 2.3.** Let $T$ be a tree which is not a path. Then, there exists a vertex $u \in V(T)$ which has the 2-pendant paths property.

**Proof:** View $T$ as a rooted tree. Since $T$ is not a path, there is a vertex $u$ furthest away from the root, where $\deg(u) \geq 3$. Then, there are at least two subtrees of $u$ which are paths. Hence, $u$ has the 2-pendant paths property. $\square$

**Lemma 2.4.** Let $s \geq 2$. If $T_s$ is a tree of order $s$, then $\mathbb{N} \setminus \{2\} \subseteq N(T_s \times P_2) \subseteq \text{IM}(T_s \times P_2)$.

**Proof:** For $s = 2$, the claim holds by Remark 2.1. Now, let $s \geq 3$. Using mathematical induction, we assume that the claim holds for any tree of order less than $s$, where $s \geq 3$. Now consider $T_s$, a tree of order $s$. If $T_s = P_s$, then we are done by Lemma 2.1. Suppose that $T_s$ is not a path. Then by Lemma 2.3, there exists a vertex $u$ of $T_s$ which has the 2-pendant paths property. Let $uv_1\cdots v_a$ and $uw_1\cdots w_b$ be two such pendant paths. Let $T = T_s - \{v_i \mid 1 \leq i \leq a\} \cup \{w_j \mid 1 \leq j \leq b\}$ and $G = T \times P_2$. Let $P$ be the path $w_b\cdots w_1uv_1\cdots v_a$, which is isomorphic to $P_{a+b+1}$. Let $B$ be the graph obtained from $P \times P_2$ by deleting the edges of the $(b+1)$-st vertical path. Here, $B$ is isomorphic to $B(b+1; a+b+1, 2)$. Now, $G$ and $B$ are edge-disjoint and $T_s \times P_2 = G \cup B$, (via identification of the copies of $u$ in $G$ with the vertices of the edge-deleted $(b+1)$-st vertical path in $B$). By the inductive hypothesis and Lemma 2.2 (or Remark 2.2, if $B \cong B(2; 3, 2) \cong C_6$), we know that $G$ and $B$ have $\mathbb{Z}_k$-magic labelings with magic value 0, for $k \neq 2$. Combining these two $\mathbb{Z}_k$-magic labelings, we get the required $\mathbb{Z}_k$-magic labeling of $T_s \times P_2$, for $k \neq 2$. Hence by mathematical induction, the claim is established. $\square$

**Theorem 2.5.** Let $s \geq 2$ and $t \geq 3$. If $T_s$ is a tree of order $s$, then $N(T_s \times P_t) = \mathbb{N} \setminus \{2\} = \text{IM}(T_s \times P_t)$.

**Proof:** Since $T_s \times P_t$ contains vertices of even and odd degrees, it is not $\mathbb{Z}_2$-magic. From Lemma 2.1, the claim holds when $s = 2$ or $s = 3$. Using mathematical induction, we assume that the claim holds for any tree of order less than $s$, where $s \geq 4$. Now consider $T_s$, a tree of order $s$. If $T_s = P_s$, then we are done by Lemma 2.1. Suppose that $T_s$ is not a path. Then by Lemma 2.3, there exists a vertex $u$ of $T_s$ which has the 2-pendant paths property. Let $uv_1\cdots v_a$ and $uw_1\cdots w_b$ be two such pendant paths. Let $T = T_s - \{v_i \mid 1 \leq i \leq a\} \cup \{w_j \mid 1 \leq j \leq b\}$ and $G = T \times P_t$. Let $P$ be the path $w_b\cdots w_1uv_1\cdots v_a$, which is isomorphic to $P_{a+b+1}$. Let $B$ be the graph obtained from $P \times P_t$ by deleting the edges of the $(b+1)$-st vertical path. Here, $B$ is isomorphic to $B(b+1; a+b+1, t)$. Now, $G$ and $B$ are edge-disjoint and $T_s \times P_t = G \cup B$, (via identification of the copies of $u$ in $G$ with the vertices of the edge-deleted $(b+1)$-st vertical path in $B$). By the inductive hypothesis and Lemma 2.2,
we know that \( G \) and \( B \) have \( \mathbb{Z}_k \)-magic labelings with magic value 0, for \( k \neq 2 \). Combining these two \( \mathbb{Z}_k \)-magic labelings, we get the required \( \mathbb{Z}_k \)-magic labeling of \( T_s \times P_t \), for \( k \neq 2 \). Hence by mathematical induction, the claim is established.

Example 2.2. Here are \( \mathbb{Z}_k \)-magic labelings (see Figure 2), where \( k \neq 2 \) for \( T_5 \times P_3 \) and \( T_7 \times P_3 \), respectively:

![Figure 2](image)

Example 2.3. Note that \( K_{1,3} \times P_2 \) is an Eulerian graph with an even number of edges. Traveling along an Eulerian circuit of \( K_{1,3} \times P_2 \), we can label the edges 1, -1, 1, -1, ..., 1, -1. This is \( \mathbb{Z}_k \)-magic labeling with magic value 0, for \( k \in \mathbb{N} \). See Figure 3.

![Figure 3](image)

3 Cartesian product of two trees

Suppose \( T \) is a tree and \( t \geq 3 \). Let \( B_T(r; t) \) be the graph obtained from \( T \times P_t \) by deleting all the edges of the \( r \)-th horizontal tree, where \( 2 \leq r \leq t - 1 \).

Lemma 3.1. Let \( T \) be a tree of order at least 3, \( t \geq 4 \) and \( 2 \leq r \leq t - 1 \). Then, \( N(B_T(r; t)) = \mathbb{N} \setminus \{2\} = \text{IM}(B_T(r; t)) \).

Proof: Since \( B_T(r; t) \) contains vertices of even and odd degrees, it is not \( \mathbb{Z}_2 \)-magic. To obtain a \( \mathbb{Z}_k \)-magic labeling for \( B_T(r; t) \) with magic value 0 (for \( k \neq 2 \)), we perform the following steps:

1. Label \( T \times P_t \), as described in the proof of Theorem 2.5.
2. Delete the edges of the \( r \)-th horizontal tree.

3. Multiply all edge labels that are above (or below) the (former) \( r \)-th horizontal tree by \(-1\).

This gives us a \( \mathbb{Z}_k \)-magic labeling of \( B_T(r; t) \) with magic value 0, for \( k \neq 2 \). \( \Box \)

**Remark 3.1.** Suppose that \( T \) is a tree of order at least 3 and \( t = 3 \). Then the procedure described in the proof of Lemma 3.1 yields \( N \setminus \{2\} \subseteq N(B_T(2; 3)) \subseteq IM(B_T(2; 3)) \). If \( T \) has no vertex of even degree, \( B_T(2; 3) \) has no vertices of odd degree. In this case, labeling all of the edges of \( B_T(2; 3) \) with 1 gives a \( \mathbb{Z}_2 \)-magic labeling with magic value 0. Thus, \( N(B_T(2; 3)) = N = IM(B_T(2; 3)) \). On the other hand, if \( T \) has a vertex of even degree, then \( B_T(2; 3) \) has vertices of even and odd degrees and hence, is not \( \mathbb{Z}_2 \)-magic. In this case, \( N(B_T(2; 3)) = N \setminus \{2\} = IM(B_T(2; 3)) \).

**Example 3.1.** Here are some labelings which illustrate Remark 3.1. The integer-magic spectrum of \( B_{K_5}(2; 3) \) is \( N \setminus \{2\} \). See Figure 4. Now, let \( T = K_{1, 3} \). Then, the integer-magic spectrum of \( B_T(2; 3) \) is \( \mathbb{N} \).

![Figure 4](image-url)

**Theorem 3.2.** Let \( s, t \geq 2 \). If \( T_s \) and \( T_t \) are trees of order \( s \) and \( t \), respectively, then \( N \setminus \{2\} \subseteq N(T_s \times T_t) \subseteq IM(T_s \times T_t) \).

**Proof:** Let \( s \geq 2 \). When \( t = 2 \), the claim holds by Lemma 2.4. When \( t = 3 \), the claim holds by Theorem 2.5. Using mathematical induction, we assume the claim holds for any tree of order less than \( t \), where \( t \geq 4 \). Now consider \( T_t \), a tree of order \( t \). If \( T_t = P_t \), then we are done by Theorem 2.5. Suppose that \( T_t \) is not a path. Then by Lemma 2.3, there exists a vertex \( u \) of \( T_t \) which has the 2-pendant paths property. Let \( uv_1 \cdots v_a \) and \( uw_1 \cdots w_b \) be two such pendant paths. Let \( T = T_t - \{v_i \mid 1 \leq i \leq a\} \cup \{w_j \mid 1 \leq j \leq b\} \) and \( G = T_s \times T_t \). Let \( P \) be the path \( w_b \cdots w_1 uv_1 \cdots v_a \) which is isomorphic to \( P_{a + b + 1} \). Let \( B \) be the graph obtained from \( T_s \times P \) by deleting the edges of the \((b + 1)\)-st horizontal tree. Here, \( B \) is isomorphic to \( B_{T_t}(b + 1; t) \). Now, \( G \) and \( B \) are edge-disjoint and \( T_s \times T_t = G \cup B \). By the inductive hypothesis and Lemma 3.1, we know that \( G \) and \( B \) have \( \mathbb{Z}_k \)-magic...
labelings with magic-value 0, for $k \neq 2$. Combining these two $Z_k$-magic labelings, we get the required $Z_k$-magic of $T_s \times T_t$, for $k \neq 2$. Hence by mathematical induction, the claim is established.

**Remark 3.2.** Theorem 3.2 establishes the entire integer-magic spectra and null sets of the Cartesian product of two trees, for all $k \neq 2$. To determine if 2 is contained in the integer-magic spectrum or null set of $T_s \times T_t$, one merely examines the parities of the degrees of the vertices in $T_s \times T_t$.

**Example 3.2.** Here is a construction of a $Z_k$-magic labeling with magic value 0 of $K_{1,3} \times K_{1,3}$, using the ideas in the proofs of the above results.

1. From the proof of Lemma 2.1, we obtain labelings of $P_2 \times P_3$ and $P_3 \times P_3$.
2. Perform the steps described in the proof of Lemma 2.2 on $P_3 \times P_3$ to get a labeling of $B(3; 2, 3)$.
3. From the proof of Theorem 2.5, we obtain a labeling of $K_{1,3} \times P_3$.
4. From the proof of Lemma 3.1, we get a labeling of $B_{K_{1,3}}(2; 3)$.
5. Combining the labeling of $K_{1,3} \times P_2$ obtained in Example 2.3, we get a labeling of $K_{1,3} \times K_{1,3}$.

All labelings obtained above are magic with magic value 0. Here are the resulting labelings (see Figure 5). Clearly, this is a $Z_k$-magic labeling of $K_{1,3} \times K_{1,3}$ with magic value 0, for all $k \in \mathbb{N}$.

**Theorem 3.3.** Let $s_i \geq 2$, for $1 \leq i \leq 2r$ and $T_{s_i}$ be a tree of order $s_i$. Then, $\mathbb{N} \setminus \{2\} \subseteq \text{IM}(T_{s_1} \times T_{s_2} \times T_{s_3} \times T_{s_4} \cdots \times T_{s_{2r-1}} \times T_{s_{2r}})$.

**Proof:** In [15], it was shown that the Cartesian product of two $Z_k$-magic graphs is $Z_k$-magic. This, along with Theorem 3.2, establishes our claim.

4 Miscellany

The main focus of this paper has been to determine the entire integer-magic spectra and null sets of $T_s \times T_t$. This section contains various miscellaneous results which the authors encountered along the way.

We first note that $Z_k$-magic labelings can be obtained for $P_s \times P_t$ with any number of deleted vertical paths, excluding the 1-st and $s$-th vertical paths. This is accomplished by repeatedly using the procedure described in the proof of Lemma 2.2. Thus, we have the following theorem:

**Theorem 4.1.** Let $s \geq 3$, $t \geq 2$ and $G = P_s \times P_t$ with some deleted vertical paths (excluding the 1-st and $s$-th vertical paths). Then, $\mathbb{N} \setminus \{2\} \subseteq N(G) \subseteq \text{IM}(G)$. 

Example 4.1. Here is a $\mathbb{Z}_k$-magic labeling (see Figure 6) with magic value 0 ($k \neq 2$) of $P_5 \times P_3$ with its 2-nd and the 4-th vertical paths deleted. This was obtained by using the procedure described in the proof of Lemma 2.2 twice.

One can also obtain $\mathbb{Z}_k$-magic labelings for $T_s \times P_t$ (where $T_s$ is a tree of order $s$) with any number of deleted horizontal trees, excluding the 1-st and $t$-th horizontal
trees. This is accomplished by repeatedly using the procedure described in the proof of Lemma 3.1. Thus, we have the following theorem:

**Theorem 4.2.** Let \( s \geq 3 \), \( t \geq 4 \) and \( G = T_s \times P_t \) with some deleted horizontal trees (excluding the 1-st and \( t \)-th horizontal trees). Then, \( \mathbb{N} \setminus \{2\} \subseteq N(G) \subseteq \text{IM}(G) \).

**Theorem 4.3.** Suppose that \( 2 \leq r \leq s - 1 \) and \( t \geq 2 \). Let path \( P_s = u_1 \cdots u_s \) and \( B(r; s, t) \) be the graph obtained from \( P_s \times P_t \) by deleting all edges of the \( r \)-th vertical path. Furthermore, suppose that \( G \times P_t \) has a \( \mathbb{Z}_k \)-magic labeling with magic value 0, for \( k \neq 2 \). Let \( H \) be the one point union of \( G \) and \( P_s \) by identifying a vertex of \( G \) with the vertex \( u_r \in V(P_s) \). Then, \( H \times P_t \) has a \( \mathbb{Z}_k \)-magic labeling with magic value 0.

**Proof:** Note that \( H \times P_t \cong (G \times P_t) \cup B(r; s, t) \). The claim follows immediately from this. \( \square \)

To determine if \( H \times P_t \) (in Theorem 4.3) has a \( \mathbb{Z}_2 \)-magic labeling, one merely examines the parities of the degrees of the vertices.

**Acknowledgements**

The authors wish to thank the referees for their valuable comments and suggestions.

**References**


(Received 22 May 2017; revised 3 Oct 2017)