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L(j, k)-labeling numbers of square of paths

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Abstract

For j ≤ k, the L(j, k)-labeling arose from code assignment problem. That is, let j, k and m be positive numbers, an m-L(j, k)-labeling of a graph G is a mapping f : V(G) → [0, m] such that |f(u) − f(v)| ≥ j if d(u, v) = 1, and |f(u) − f(v)| ≥ k if d(u, v) = 2. The span of f is the difference between the maximum and the minimum numbers assigned by f. The L(j, k)-labeling number of G, denoted by λj,k(G), is the minimum span over all L(j, k)-labelings of G. The kth power Gk of an undirected graph G is the graph with the vertex set of G in which two vertices are adjacent when their distance in G is at most k. In this paper, the L(j, k)-labeling numbers of Pn2 are determined for j ≤ k.

Keywords: L(j, k)-labeling; Path; Square of path; Code assignment

1. Introduction

The rapid growth of computer wireless networks highlighted the scarcity of available codes (such as radio frequencies) for communication with minimum interference. For example, the Packet Radio Network (PRN) is a computer network that uses radio frequencies to transmit packet among computers. The two major types of interference in PRN are Direct collision (or interference), which is caused by the transmission of adjacent stations (computers), and Hidden terminal collision (or interference), which is caused by distance-two stations that transmit to the same receiving station or receive the data from the same transmitting station.

Let G be a graph and let V(G) and E(G) be its vertex set and edge set, respectively. For any two vertices u and v, let dG(u, v) (or simply d(u, v)) denote the distance (length of a shortest path) between u and v in G. Noted that all graphs considered in this article are simple connected and undirected. All notation not defined in this article can be found in the book [1].
For positive numbers \(j\) and \(k\), an \(L(j, k)\)-labeling \(f\) of \(G\) is an assignment of numbers to vertices of \(G\) such that \(|f(u) - f(v)| \geq j\) if \(uv \in E(G)\), and \(|f(u) - f(v)| \geq k\) if \(d(u, v) = 2\). The span of \(f\) is the difference between the maximum and the minimum numbers assigned by \(f\). In other words, if we list the image of \(f\) as a non-decreasing sequence \(\{f(u_1)\}_{i=1}^n\), then the span of \(f\) is \(f(u_n) - f(u_1)\) which is called the span of the sequence, where \(n\) is the order of \(G\). The \(L(j, k)\)-labeling number of \(G\), denoted by \(\lambda_{j,k}(G)\), is the minimum span over all \(L(j, k)\)-labeling of \(G\). Without loss of generality, we may assume the minimum value of each labeling \(f\) is 0.

In a computer network, suppose direct collision is so weak that it can be ignored, and two distance-two stations can generate a hidden terminal collision. In order to avoid the hidden terminal collision, Bertossi and Bonuccelli [2] introduced an optimal code assignment, that is, two distance-two stations have different codes. By corresponding codes to labels, this code assignment problem is equivalent to the \(L(0, 1)\)-labeling problem, that is, two distance-two vertices must be assigned different labels.

In general, the direct collision cannot be ignored. Based on this premise, Jin and Yeh [3] generalized the code assignment problem to \(L(j, k)\)-labeling problem with \(j \leq k\). That is, to avoid direct collision, any two adjacent stations are required to be assigned at least \(j\) apart codes. Additionally, to avoid hidden terminal collision, any two distance-two stations need to be assigned at least \(k\) apart codes. Therefore, we face on the \(L(j, k)\)-labeling problem with \(j \leq k\).

On the other hand, \(L(j, k)\)-labeling numbers of graphs for \(j \geq k\) have been studied in many articles. Interested readers were referred to the surveys [4,5].

By now, about the \(L(j, k)\)-labeling numbers of graphs for \(j \leq k\), there already exist some results. For example, Jin and Yeh determined \(L(0, 1)\), \(L(1, 1)\), \(L(1, 2)\)-labeling numbers of paths, cycles and grids in [3]. Furthermore, Niu [6] introduced \(L(j, k)\)-labeling numbers of paths and cycles, and Griggs and Jin studied \(L(j, k)\)-labeling numbers of lattices (grids) in [7]. Moreover, Jayasree and Nicholas [8] mentioned \(L(1, 2)\)-labeling numbers of certain generalizes Petersen graphs and \(n\)-star. In [9], the authors introduced the \(L(j, k)\)-labeling numbers of trees and stars with maximum degree. Lam, Lin and Wu [10] worked on \(L(j, k)\)-labeling numbers of product of completed graphs. Recently, Shiu and Wu determined \(L(j, k)\)-labeling numbers of direct and Cartesian product of path and cycle in [11] and [12], respectively. Moreover, the authors studied circular \(L(j, k)\)-labeling numbers of tree and Cartesian products of graphs, direct product of path and cycle, and square of paths in [13–15], respectively.

The \(k\)th power \(G^k\) of a graph \(G\) is the graph with the vertex set of \(G\) in which two vertices are adjacent when their distance in \(G\) is at most \(k\). \(G^2\) is called the square of \(G\).

**Lemma 1.1.** Let \(j\) and \(k\) be two positive numbers with \(j \leq k\). Suppose \(G\) is a graph and \(H\) is an induced subgraph of \(G\). Then \(\lambda_{j,k}(G) \geq \lambda_{j,k}(H)\).

Note that Lemma 1.1 is not true if \(H\) is not an induced subgraph. Throughout this paper, \(P_n = v_0v_1 \cdots v_{n-1}\) denotes the path of order \(n\).

## 2. \(L(j, k)\)-labeling numbers of \(P_4^2\) and \(P_5^2\)

**Theorem 2.1.** For \(j \leq k\), \(\lambda_{j,k}(P_4^2) = \max\{k, 3j\}\).

**Proof.** Let \(\lambda = \max\{k, 3j\}\). Let \(f\) be the labeling of \(P_4^2\) defined by \(f(v_0) = 0, f(v_1) = j, f(v_2) = 2j\) and \(f(v_3) = \lambda\). It is easy to verify that \(f\) is a \(\lambda\)-\(L(j, k)\)-labeling of \(P_4^2\). Hence \(\lambda_{j,k}(P_4^2) \leq \lambda\).

On the other hand, since any two vertices of \(P_4^2\) are adjacent or of distance two, \(\lambda_{j,k}(P_4^2) \geq 3j\). Moreover, since \(d(v_0, v_3) = 2\), \(\lambda_{j,k}(P_4^2) \geq k\). It implies that \(\lambda_{j,k}(P_4^2) \geq \max\{k, 3j\} = \lambda\). Hence, \(\lambda_{j,k}(P_4^2) = \max\{k, 3j\}\).

**Theorem 2.2.** For \(j \leq k\), \(\lambda_{j,k}(P_5^2) = \max\{j + k, 4j\}\).

**Proof.** Let \(\eta = \max\{k, 3j\}\) and \(\lambda = \eta + j\). Let \(f\) be the labeling of \(P_5^2\) defined by \(f(v_0) = 0, f(v_1) = j, f(v_2) = 2j, f(v_3) = \eta\) and \(f(v_4) = \eta + j\). It is easy to verify that \(f\) is a \(\lambda\)-\(L(j, k)\)-labeling of \(P_5^2\). Hence \(\lambda_{j,k}(P_5^2) \leq \lambda\).

On the other hand, since any two vertices of \(P_5^2\) are adjacent or of distance two, \(\lambda_{j,k}(P_5^2) \geq 4j\). Moreover, since two adjacent vertices \(v_3\) and \(v_4\) are at distance two from vertex \(v_0\), \(\lambda_{j,k}(P_5^2) \geq j + k\). It implies that \(\lambda_{j,k}(P_5^2) \geq \max\{j + k, 4j\}\). Hence, \(\lambda_{j,k}(P_5^2) = \max\{j + k, 4j\}\).
3. $L(j, k)$-labeling numbers of $P_n^2$ for $n \geq 6$

In this section, we shall study the $L(j, k)$-number of $P_n^2$ by separating the condition $j \leq k$ into three cases which are $j \leq k < 2j$, $3j \leq k$ and $2j \leq k < 3j$, where $n \geq 6$.

We consider $j \leq k < 2j$ first. Define a labeling $f$ for $P_n^2$ by $f(v_i) = [i]_6 j$ for $0 \leq i \leq n$. Clearly, $f$ is a $(5j)$-$L(j, k)$-labeling for $P_n^2$ when $j \leq k < 2j$. So

$$\lambda_{j,k}(P_n^2) \leq 5j \text{ for } n \geq 6. \tag{3.1}$$

**Lemma 3.1.** Let $j$ and $k$ be two positive numbers with $j \leq k < 3j$. Then $\lambda_{j,k}(P_6^2) \geq \min\{5j, 3j + k\}$.

**Proof.** Suppose $f$ is a $\lambda$-$L(j, k)$-labeling of $P_6^2$ and $\lambda < 5j$. Let $I_0 = [0, j)$, $I_1 = [j, (\lambda + j)/3]$, $I_2 = ((\lambda + j)/3, (2\lambda - j)/3)$, $I_3 = [(2\lambda - j)/3, \lambda - j]$ and $I_4 = (\lambda - j, \lambda]$. Here, each $I_i$ is of length less than $j$, $0 \leq i \leq 4$. Since any two of vertices $v_0, v_1, v_2, v_3, v_4$ are adjacent or of distance two, each interval $I_i$ contains exactly one label of $f(v_0), f(v_1), f(v_2), f(v_3), f(v_4)$. Similarly, each interval $I_i$ contains exactly one labels of $f(v_1), f(v_2), f(v_3), f(v_4), f(v_5), f(v_6)$. By pigeonhole principle, two labels among $f(v_0), f(v_1), f(v_2), f(v_3), f(v_4), f(v_5)$ fall into the same interval. By considering the distance between those vertices, we can see that only $f(v_0)$ and $f(v_5)$ lie in the same interval. By symmetry of the graph, it suffices to consider $f(v_0), f(v_5) \in I_i$ for $0 \leq i \leq 2$. Let $A = \{v_0, v_5\}$.

**Case 1.** Suppose $f(v_0), f(v_5) \in I_0$. Here $f(v_1), f(v_2), f(v_3), f(v_4) \in [k, \lambda]$. Since $j \geq k$, by Theorem 2.1 we have $\lambda - k \geq 3j$. Hence $\lambda \geq 3j + k$.

**Case 2.** Suppose $f(v_0), f(v_5) \in I_1$. We have $f(v_1), f(v_2), f(v_3), f(v_4) \in [0, (\lambda + j)/3) \cup [j + k, \lambda]$. Since $k \geq j$ and $\lambda < 5j$, the length of $[0, (\lambda + j)/3 - k)$ is less than $j$. Thus, $[j + k, \lambda]$ contains three of $f(v_1), f(v_2), f(v_3)$ and $f(v_4)$. Now $\lambda - k - j \geq 2j$, i.e., $\lambda \geq 3j + k$.

**Case 3.** Suppose $f(v_0), f(v_5) \in I_2$. Let $f(v_i) \in I_i$ for some $w_i$, where $i = 0, 1, 3, 4$. Hence $\{w_0, w_1, w_3, w_4\} = \{v_1, v_2, v_3, v_4\}$. There exists $v \in A$ such that $d(v, w_1) = 2$. So $f(v) - f(w_1) \geq k$. Then the span of the increasing sequence $f(w_0) < f(w_1) < f(v) < f(w_3) < f(w_4)$ is at least $3j + k$. Hence $\lambda \geq 3j + k$.

Thus $\lambda_{i,j}(P_6^2) \geq \min\{5j, 3j + k\}$. $\square$

**Theorem 3.2.** Suppose $6 \leq n \leq 10$. Let $j$ and $k$ be two positive numbers. If $j \leq k < 2j$, then $\lambda_{j,k}(P_n^2) = 3j + k$.

**Proof.** Define a labeling $f$ for $P_{10}^2$ as follows:

$f(v_0) = f(v_5) = 0$, $f(v_1) = k, f(v_2) = j + k, f(v_3) = 2j + k, f(v_4) = 3j + k, f(v_6) = j$, $f(v_7) = 2j, f(v_8) = 3j, f(v_9) = 4j$. It is easy to verify that $f$ is a $(3j + k)$-$L(j, k)$-labeling of $P_{10}^2$ when $j \leq k < 2j$.

By Lemma 1.1, we have $\lambda_{j,k}(P_n^2) \leq 3j + k$ for $n \leq 10$.

Since $P_n^2$ is an induced subgraph of $P_6^2$, it suffices to show that $\lambda = \lambda_{j,k}(P_6^2) \geq 3j + k$. By Lemma 3.1 and $j \leq k < 2j$, we have $\lambda_{i,j}(P_6^2) = 3j + k$. Thus $\lambda_{i,j}(P_n^2) = 3j + k$. By Lemma 1.1 we get that $\lambda_{j,k}(P_n^2) \geq \lambda_{i,j}(P_6^2) = 3j + k$ for $n \geq 6$.

Combining the discussion above, we have $\lambda(P_n^2) = 3j + k$ for $6 \leq n \leq 10$. $\square$

For any integer $a$, $[a]_m \in \{0, 1, \ldots, m - 1\}$ denotes the residue of $a$ modulo $m$, where $m$ is a positive integer greater than 1. For convenience, we let $V_i = \{v \in V(P_n^2) \mid l \equiv i \pmod{5}\}, 0 \leq i \leq 4$, where $n \geq 11$. And also let $E(A, B)$ be the set of edges from $A$ to $B$ and $f(A) = \{f(v) \mid v \in A\}$ for a labeling $f$ of $P_n^2$, where $A$ and $B$ are subsets of $V(P_n^2)$.

**Theorem 3.3.** Suppose $11 \leq n \leq 15$. Let $j$ and $k$ be two positive numbers with $j \leq k < 2j$. Then $\lambda_{j,k}(P_n^2) = \min\{2j + 2k, 5j\}$.

**Proof.** Let $(0, k, 2k, j + 2k, 2j + 2k, 0, k, j + k, 2j + k, 2j + 2k, 0, j, 2j, 2j + k, 2j + 2k)$ be the list of the values of $(g(v_i))_{0 \leq i \leq 14}$. Hence this defines a $(2j + 2k)$-$L(j, k)$-labeling $g$ for $P_{15}^2$. By Lemma 1.1 and (3.1), we have $\lambda_{j,k}(P_n^2) \leq \min\{2j + 2k, 5j\}$ for $11 \leq n \leq 15$.

Similar to the proof of Theorem 3.2, in order to obtain the theorem it suffices to show that $\lambda = \lambda_{j,k}(P_{15}^2) \geq \min\{2j + 2k, 5j\}$. Thus we have to show that “if $\lambda < 5j$, then $\lambda \geq 2j + 2k$”.


Now suppose $\lambda < 5j$. Let $f$ be a $\lambda$-$L(j, k)$-labeling of $P_{11}^2$. Let $I_i$ be intervals defined in the proof of Theorem 3.2. Note that the length of each interval is less than $j$. By pigeonhole principle, at least one interval contains three vertex labels. Note that such labels may be the same. Let $H$ be a graph with the vertex set $V(P_{11}^2)$ in which two vertices are adjacent if they are of distance at least 3 in $P_{11}^2$. Note that, $H$ is a compatibility graph, in which two vertices are adjacent if and only if their assigned labels can lie in the same interval $I_i$ for some $i$. We can see that $H$ contains only one 3-cycle which is $v_0v_5v_1v_0$. Thus, only $f(v_0)$, $f(v_5)$ and $f(v_{10})$ lie in the same interval $I_{h_0}$ for some $h_0$. Thus, each of other interval contains exactly two labels. By symmetry of the graph, we may assume that $0 \leq h_0 \leq 2$.

By considering the subgraph induced by $\{v_i \mid 0 \leq i \leq 5\}$ and the same argument in the proof of Theorem 3.2, each of $f(v_1)$, $f(v_2)$, $f(v_3)$, $f(v_4)$ lies in exactly one different interval. Let $f(v_i) \in I_{h_i}$ for $1 \leq i \leq 4$. Now $\{h_0, h_1, h_2, h_3, h_4\} = \{0, 1, 2, 3, 4\}$. Similarly, by considering the subgraph induced by $\{v_i \mid 5 \leq i \leq 10\}$, each of $f(v_6)$, $f(v_7)$, $f(v_8)$ lies in exactly one different interval $I_{h_i}$, $1 \leq i \leq 4$.

By considering another compatibility graph $H - V_0$ (Fig. 1), we can see that $f(V_1) \subset I_{h_1}$ and $f(V_4) \subset I_{h_4}$. This forces that $f(V_2) \subset I_{h_2}$ and $f(V_3) \subset I_{h_3}$.  

**Case 1.** Suppose $h_0 = 0$. We want to determine the span of the set $S = \{f(v_i) \mid 1 \leq i \leq 9, i \neq 5\}$, i.e., the maximum difference between each pair of labels in $S$. For each $w \in V_1$, $1 \leq i \leq 4$, there is a $v \in V_0$ such that $d(w, v) = 2$. Thus, $S \subset [k, \lambda]$. We shall face on all permutations of $h_1h_2h_3h_4$. For example, suppose $h_1h_2h_3h_4 = 1234$. That means $f(V_1) \subset I_1$, $f(V_2) \subset I_2$, $f(V_3) \subset I_3$ and $f(V_4) \subset I_4$. Considering the path $v_6v_2v_3v_4$ at the graph $H_2$ shown in Fig. 2, we have an increasing subsequence $f(v_6) < f(v_2) < f(v_3) < f(v_4)$. Thus the span of $S$ is at least $k + 2j$. The reflection case of this case is $h_1h_2h_3h_4 = 4321$. By means of reflection there are $4!/2 = 12$ permutations we have to deal with. Combining all cases, we have $\lambda - k \geq 2j + k$. Hence $\lambda \geq 2j + 2k$.

**Case 2.** Suppose $h_0 = 1$. Similar to Case 2 of the proof of Theorem 3.2, $[j + k, \lambda]$ contains three of $f(V_1)$, $f(V_2)$, $f(V_3)$ and $f(V_4)$. No matter which case, the span of the union of these three subsets is at least $k + j$. So $\lambda - j - k \geq k + j$. Hence $\lambda \geq 2j + 2k$.

**Case 3.** Suppose $h_0 = 2$. Now $\{h_1, h_2, h_3, h_4\} = \{0, 1, 3, 4\}$. Consider the graph $H_3$. There is always a hard edge in $E(V_s, V_t)$, where $1 \leq s < t \leq 4$. Moreover, each vertex in $H_2$ is of distance either 1 or 2 to $v_5$ in $P_{11}^2$. Thus, for each permutation of $h_1h_2h_3h_4$, there always exists an increasing subsequence of $f(V(P_{11}^2))$ involving $f(v_5)$ with the span at least $2j + 2k$. For example, when $h_1h_2h_3h_4 = 0314$, the required subsequence is $f(v_6) < f(v_5) < f(v_3) < f(v_7) < f(v_4)$. So the span of $S$ is at least $2j + 2k$.

Combining the discussion above, we have $\lambda = 2j + 2k$ for $11 \leq n \leq 15$. □

**Theorem 3.4.** Suppose $16 \leq n \leq 20$. Let $j, k$ be two positive numbers with $j \leq k < 2j$. Then $\lambda_{j,k}(P_n^2) = \min \{j + 3k, 5j\}$.

**Proof.** Let $(0, k, 2k, 3k, j + 3k, 0, k, 2k, j + 2k, 2j + 2k, 0, k, j + k, 2j + k, 2j + 2k, 0, k, j + 2j, 2j + k, 2j + 2k)$ be the list of the values of $(g(v_i))_{0 \leq i \leq 19}$. Hence this defines a $(j + 3k)$-$L(j, k)$-labeling $g$ for $P_n^2$ if $16 \leq n \leq 20$. By Lemma 1.1 and (3.1), we have $\lambda_{j,k}(P_n^2) \leq \min \{j + 3k, 5j\}$ for $16 \leq n \leq 20$.

Conversely, we consider $\lambda = \lambda_{j,k}(P_n^2)$. Similar to the proof of Theorem 3.3 we assume $\lambda < 5j$ and show that $\lambda \geq j + 3k$ in the following.
Let \( I_i \) be defined in Theorem 3.2. By considering the subgraphs induced by \( \{v_i \mid 0 \leq i \leq 10 \} \) and \( \{v_i \mid 5 \leq i \leq 15 \} \), we obtain that \( f(V_0) \subset I_{h_0} \) for some \( h_0 \in \{0, 1, 2\} \) (without loss of generality), \( f(V_i) \subset I_{h_i}, 1 \leq i \leq 4 \), where \( \{h_0, h_1, h_2, h_3, h_4\} = \{0, 1, 2, 3, 4\} \).

**Case 1.** Suppose \( h_0 = 0 \). By Table 1, we only need to consider the case when \( h_1h_2h_3h_4 = 1234 \). In this case, there is a subsequence \( f(v_{11}) < f(v_7) < f(v_3) < f(v_4) \). Now the span of \( S \) is at least \( j + k \).

**Case 2.** Suppose \( h_0 = 1 \). Similar to Case 2 of the proof of Theorem 3.2, \([j + k, \lambda] \) contains three of \( f(V_1), f(V_2), f(V_3) \) and \( f(V_4) \). No matter which case (see Fig. 3), the span of the union of these three subsets is at least \( 2k \). So \( \lambda - j - k \geq 2k \). Hence \( \lambda \geq j + 3k \).

**Case 3.** Suppose \( h_0 = 2 \). Now \( \{h_1, h_2, h_3, h_4\} = \{0, 1, 3, 4\} \). We have to deal with the 12 permutations of \( h_1h_2h_3h_4 \). We only provide the discussion of the case when \( h_1h_2h_3h_4 = 0314 \) here. Other cases are similarly to show. For this case, we have an increasing subsequence \( f(v_{11}) < f(v_8) < f(v_5) < f(v_7) < f(v_4) \). So the span of \( f \) is at least \( j + 3k \).

Combining the discussion above, we have \( \lambda = j + 3k \) for \( 16 \leq n \leq 20 \). \( \square \)

**Lemma 3.5.** Let \( W_i \) be a set consisting of 4 vertices of a graph \( G \), \( 1 \leq i \leq 4 \). Assume that \( E(W_i, W_{i+1}) \) contains at least 3 disjoint edges, for \( 1 \leq i \leq 3 \). Then there is a path \( w_1w_2w_3w_4 \) with \( w_i \in W_i \).

**Proof.** Let \( w_{1j}w_{2j} \in E(W_1, W_2) \) for \( 1 \leq j \leq 3 \), \( w_{ij} \in W_i \). Since at most one vertex in \( W_2 \) is not incident with edge of \( E(W_1, W_2) \), at least two edges of \( E(W_2, W_3) \) are adjacent with \( w_{1j}w_{2j} \), \( 1 \leq j \leq 3 \). After renaming if necessary, we may assume that such two edges of \( E(W_2, W_3) \) are \( w_{21}w_{31} \) and \( w_{22}w_{32} \). Since only one vertex in \( W_4 \) is not incident with edge in \( E(W_3, W_4) \), there is an edge, say \( w_{31}w_{41} \in E(W_3, W_4) \). Now we have a path \( w_{11}w_{21}w_{31}w_{41} \). \( \square \)

**Corollary 3.6.** Let \( W_i \) be a set consisting of 4 vertices of a graph \( G \), \( 1 \leq i \leq 3 \). Assume that \( E(W_i, W_{i+1}) \) contains at least 3 disjoint edges, for \( 1 \leq i \leq 2 \). Then there are two disjoint paths \( w_1w_2w_3 \) with \( w_i \in W_i \).
Graph $H_3$: Two vertices are adjacent if they are of distance 2 in $P_{16}$.

**Theorem 3.7.** Suppose $n \geq 21$. Let $j, k$ be two positive numbers with $j \leq k < 2j$. Then $\lambda_{j,k}(P_n^2) = \min\{4k, 5j\}$.

**Proof.** Define a labeling $g$ for $P_n^2$ by $g(v_i) = [i]_5 k$. Clearly $g$ is a $(4k)$-$L(j, k)$-labeling of $P_n^2$. By (3.1), we have $\lambda_{j,k}(P_n^2) \leq \min\{4k, 5j\}$.

Consider the graph $P_{21}^2$ and assume $\lambda < 5j$. Let $I_i$ be defined in Theorem 3.2. By a similar argument of the proof of Theorem 3.4, we have $f(V_0) \subset I_{h_0}$ for some $h_0 \in \{0, 1, 2\}$, $f(V_i) = \{f(v_1), f(v_6), f(v_{11}), f(v_{16})\} \subset I_{h_i}$, $1 \leq i \leq 4$, where $\{h_0, h_1, h_2, h_3, h_4\} = \{0, 1, 2, 3, 4\}$. Let $H_4$ be the graph in Fig. 4. Two vertices are adjacent if they are of distance 2 in $P_{21}^2$.

**Remark 1.** For $1 \leq s < t \leq 4$, there are at least 3 disjoint edges in $E(V_s, V_t) \subset E(H_4)$. For each $w \in V_i$ with $1 \leq i \leq 4$, there is a unique $v \in V_0$ such that $wv \in E(V_0, V_i)$.

**Case 1.** Suppose $h_0 = 0$. Let $\sigma = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ h_1 & h_2 & h_3 & h_4 \end{array}\right)$ be a permutation of $\{1, 2, 3, 4\}$. Let $k_i = \sigma^{-1}(i)$. By Remark 1 and Lemma 3.5, there is a path $w_1w_2w_3w_4$ in which $w_i \in V_{k_i}$. By Remark 1 again, there is $w_0 \in V_0$ such that $d(w_0, w_1) = 2$. Hence the path $w_0 \cdots w_5$ induces a subsequence $f(w_0) < f(w_1) < f(w_2) < f(w_3) < f(w_4)$ and the span of this subsequence is $4k$. Hence we have $\lambda \geq 4k$.

**Case 2.** Suppose $h_0 = 1$. Let $\sigma = \left(\begin{array}{cccc} 0 & 2 & 3 & 4 \\ h_1 & h_2 & h_3 & h_4 \end{array}\right)$ be a permutation of $\{0, 2, 3, 4\}$. Let $k_i = \sigma^{-1}(i)$, $2 \leq i \leq 4$ and $\sigma^{-1}(0) = \ell$. By Corollary 3.6, there are two disjoint paths $w_2w_3w_4$ and $u_2u_3u_4$ such that $w_i, u_i \in V_{k_i}$,
2 ≤ i ≤ 4. By Remark 1 there are u₀, w₀ ∈ V₀ such that w₀u₂ and u₀u₂ are edges of E(V₀, V₂). Since only one vertex in V₀ is not adjacent to vertex V₁, there is a w₁ ∈ V₁ such that either w₁w₀ ∈ E(V₁, V₀) or w₁u₀ ∈ E(V₁, V₀). Let us say w₁u₀ ∈ E(V₁, V₀). There is a path w₁u₁w₂u₃w₄ that induces a subsequence f(w₁) < f(w₀) < f(w₂) < f(w₃) < f(w₄). Hence we have λ ≥ 4k.

Case 3. Suppose h₀ = 2. Let σ = \((0, 1, 3, 4, 2)\) be a permutation of \([0, 1, 3, 4]\). Let kᵢ = σ⁻¹(i), i ∈ \([0, 1, 3, 4]\). Note that E(Vₖ₁₁, Vₖ₁) and E(Vₖ₃ᵣ, Vₖ₃) contain 3 disjoint edges, respectively; E(Vₖ₁, V₀) and E(V₀, Vₖ₃) contain 4 disjoint edges, respectively. By a similar argument as above, there are three paths u₁u₂u₃u₄ with u₁ ∈ V₁, u₂ ∈ V₃ and u₀ ∈ V₀ and three paths u₀u₃u₄ with u₃ ∈ V₃, u₄ ∈ V₅ and u₀ ∈ V₀. By pigeonhole principle, there are two paths u₁u₂u₀ and u₀u₃u₄ such that u₀ = w₀. Here we have a path u₁u₂u₃u₄. Hence we have λ ≥ 4k. □

Now, we consider the case when k ≥ 3j.

**Theorem 3.8.** Let j and k be two positive numbers. If k ≥ 3j, then \(λ_j,k(P₆^n) = 2j + k\).

**Proof.** Let \(λ = λ_j,k(P₆^n)\). Define \(g(v₀) = 0, g(v₁) = j, g(v₂) = 2j, g(v₃) = k, g(v₄) = j + k, g(v₅) = 2j + k\). It is easy to verify that g is a \((2j + k)\)-L(j, k)-labeling for \(P₆^n\). Hence \(λ ≤ 2j + k\).

On the other hand, let f be a \(λ\)-L(j, k)-labeling of \(P₆^n\). Since v₃, v₄ are distance two apart from v₀, \(f(v₃), f(v₄) \in [0, f(v₀) - k] \cup [f(v₀) + k, λ]\). If \(f(v₃) < f(v₀) < f(v₄)\) or \(f(v₄) < f(v₀) < f(v₃)\), then \(λ ≥ 2k > 2j + k\).

Since we have just known that \(λ ≤ 2j + k\), this is not a case. So both \(f(v₃)\) and \(f(v₄)\) are either greater than or less than \(f(v₀)\). Without loss of generality, we may assume \(f(v₃)\) and \(f(v₄)\) are greater than \(f(v₀)\), otherwise consider the labeling \(f' = λ - f\). That is, \(f(v₃), f(v₄) \in [f(v₀) + k, λ]\). Hence \(λ - k ≥ f(v₀)\). Moreover, since \(d(v₁, v₄) = 2, f(v₁) \in [0, λ - k]\). Similarly, \(d(v₁, v₃) = 2, f(v₃) \in [k, λ]\). Since \(d(v₂, v₅) = 2, f(v₂) \in [0, λ - k]\). Now, we can conclude that \(f(v₀), f(v₁), f(v₂) \in [0, λ - k]\). Since \(v₀, v₁, v₂\) are adjacent to each other, \(λ - k ≥ 2j\). Hence \(λ ≥ 2j + k\). Hence \(λ_j,k(P₆^n) = 2j + k\). □

**Theorem 3.9.** Suppose \(n ≥ 7\). Let j, k be two positive numbers. If \(k ≥ 3j\), then \(λ_j,k(P₆^n) = 2k\).

**Proof.** Let \(λ = λ_j,k(P₆^n)\). Define \(g(v₀) = 0, g(v₁) = j, g(v₂) = 2j, g(v₃) = k, g(v₄) = j + k, g(v₅) = 2j + k, g(v₆) = 2k\) and \(g(v₇) = g(vᵢ_{₇})\) for \(i ≥ 7\). It is easy to verify that g is a \(2k\)-L(j, k)-labeling of \(P₆^n\). Hence \(λ ≤ 2k\).

Since \(P₆^n\) is an induced subgraph of \(P₆^n\), it suffices to show that \(λ ≥ λ_j,k(P₆^n) ≥ 2k\). Let f be a \(λ\)-L(j, k)-labeling of \(P₆^n\). Consider the labels of \(v₀, v₃\) and \(v₄, v₅\). By the same argument of the proof of **Theorem 3.8**, we only need to consider the case when \(f(v₃), f(v₄) \in [f(v₀) + k, λ]\). By the proof of **Theorem 3.8** again, we have \(f(v₀), f(v₁), f(v₂) \in [0, λ - k]\).

Now, if \(f(v₃) < f(v₀)\), then \(f(v₀) < f(v₃) < f(v₆)\) implies \(λ ≥ 2k\). The remaining case is \(f(v₆) < f(v₃)\). But this implies that \(f(v₀) \in [0, λ - k]\). Now, \(f(v₂), f(v₆) \in [0, λ - k]\). Hence \(λ ≥ 2k\).

This completes the proof.

Finally, we consider the case when \(2j ≤ k < 3j\).

**Theorem 3.10.** Let j and k be two positive numbers. If \(2j ≤ k < 3j\), then \(λ_j,k(P₆^n) = 5j\).

**Proof.** Define a labeling g for \(P₆^n\) by \(g(vᵢ) = [i]_{j} j\). Clearly g is a \((5j)\)-L(j, k)-labeling of \(P₆^n\). Thus, we have \(λ_j,k(P₆^n) ≤ 5j\).

Moreover, by **Lemma 3.1**, we have \(λ_j,k(P₆^n) ≥ |5j, 3j + k| = 5j\). Hence \(λ_j,k(P₆^n) = 5j\). □

**Theorem 3.11.** Let j and k be two positive numbers. If \(2j ≤ k < 3j\) and \(7 ≤ n ≤ 12\), then \(λ_j,k(P₆^n) = 3j + k\).

**Proof.** Let \(g(v₀) = 0, g(v₁) = j, g(v₂) = k, g(v₃) = j + k, g(v₄) = 2j + k, g(v₅) = 3j + k, g(v₆) = 0, g(v₇) = j, g(v₈) = 2j, g(v₉) = 3j, g(v₁₀) = 4j, g(v₁₁) = 5j\). It is easy to check that g is a \((3j + k)\)-L(j, k)-labeling of \(P₆^n\).

Thus, By **Lemma 1.1**, \(λ_j,k(P₆^n) ≤ 3j + k\) for \(7 ≤ n ≤ 12\).

Let \(λ = λ_j,k(P₆^n)\) and let f be a \(λ\)-L(j, k)-labeling of \(P₆^n\). We have \(λ ≤ 3j + k\). As the proofs of those previous theorems, we only need to show \(λ ≥ 3j + k\).
Let \( J_0 = [0, j] \), \( J_1 = [j, 2j] \), \( J_2 = [2j, \lambda/2] \), \( J_3 = (\lambda/2, \lambda - 2j] \), \( J_4 = (\lambda - 2j, \lambda - j] \) and \( J_5 = (\lambda - j, \lambda] \). Since \( \lambda \leq 3j + k \) and \( k < 3j \), the length of each interval is less than \( j \). Thus, if \( f(u), f(w) \in J_i \) for some \( i \), then \( d(u, w) > 2 \). Hence \([u, w] \) is either \( A_0 = [v_0, v_3] \), \( A_1 = [v_1, v_3] \) or \( A_2 = [v_0, v_6] \). Also, each \( J_i \) cannot contain more than 2 vertices. Since the length of each \( J_i \cup J_{i+1} \) is less than \( 2j < k \), for \( 0 \leq i \leq 4 \), if \([u, w] \) is none of \( A_0 \), \( A_1 \) and \( A_2 \) but \( f([u, w]) \subset J_i \cup J_{i+1} \) for some \( i \), then \( d(u, w) = 1 \). For this case, \( f([u, w]) \subset J_i \cup J_{i+1} \) associates a path \( uv \) of length 1. Furthermore, \( J_i \cup J_{i+1} \) cannot contain three labels of \( \{ f(v_0), f(v_1), f(v_3), f(v_6) \} \).

Following we want to find an increasing sequence of labels with span at least \( 3j + k \). It is easy to get the following claim.

**Claim 1.** Suppose \( v \in \{v_2, v_3, v_4\} \). For each \( i, i = 0, 1, 2 \), there exists \( w_i \in A_i \) such that \( d(v, w_i) = 2 \).

Now, by pigeonhole principle, there is at least one \( J_q \) containing two labels. By symmetry we may assume that \( q = 0, 1, 2 \), otherwise consider the labeling \( \bar{f} = f - \). Thus, \( J_q \) contains either \( f(A_0) \), \( f(A_1) \) or \( f(A_2) \).

**Case A.** Suppose there are two intervals, say \( J_q \) and \( J_r \), containing 2 labels, where \( 0 \leq q < 2 \) and \( q < r \). In this case, \( f(A_2) \) does not contain in \( J_q \cup J_r \). By renumbering the vertex if necessary, we may assume that \( f(A_1) \subset J_q \) and \( f(A_0) \subset J_r \). Let \( \{ f(u_1), f(u_2), f(u_3) \} = \{ f(v_2), f(v_3), f(v_4) \} \), where \( f(u_1) < f(u_2) < f(u_3) \).

**A-1.** Suppose \( f(v_0) < f(u_1) \). By Claim 1 there is \( w_0 \in A_0 \) such that \( d(w_0, u_1) = 2 \). Now we have \( f(v_1) < f(w_0) < f(u_3) \). By \( f(v_0) < f(u_1) \) and \( f(u_1) < f(u_3) \), \( f(u_2) < f(u_3) \) with span at least \( 3j + k \).

**A-2.** Suppose \( f(v_1) < f(u_1) < f(v_0) \) or \( f(u_1) < f(v_1) < f(u_2) \). By Claim 1 there is \( w_1 \in A_1 \) such that \( d(w_1, u_1) = 2 \). Thus, \( f(w_0), f(u_2), f(u_3) \) lie in \( [k + j, \lambda] \). So \( \lambda - k \geq 3j + k \). Hence \( \lambda \geq 3j + k \).

**A-3.** Suppose \( f(u_2) < f(v_1) \) and \( f(u_3) > f(v_1) \). In this case, we have \( f(v_1) < f(u_3) \). This the reflexive case of Case A-2.

**A-4.** Suppose \( f(u_3) < f(v_1) \). This is the reflexive case of Case A-1.

**Case B.** Suppose there is only one interval \( J_q \) containing 2 labels, where \( 0 \leq q \leq 2 \).

**B-1.** Suppose \( J_0 \) contains two labels.

a. \( f(A_0) \subset J_0 \). The span of the set \( \{ f(v_1), f(v_2), f(v_3), f(v_4) \} \) is at least \( 3j \). No matter which label is the minimum, there always exists a vertex \( w \in A_0 \) such that \( f(w) \) is less than this minimum by at least \( k \). Hence, the span of the set \( \{ f(w), f(v_1), f(v_2), f(v_3), f(v_4) \} \) is at least \( 3j + k \).

b. \( f(A_1) \subset J_0 \). Consider the set \( \{ f(v_2), f(v_3), f(v_4) \} \). Similar to Case a, we will get the same result.

c. \( f(A_2) \subset J_0 \). Since the length of \( J_0 \cup J_1 \) is less than \( k \), only \( f(v_1) \) or \( f(v_3) \) lies in \( J_1 \). Renaming the vertex if necessary, we may assume \( f(v_1) \in J_1 \). By the same reason, only \( f(v_2) \) or \( f(v_4) \) lies in \( J_2 \). Suppose \( f(v_3) \in J_3 \). Let \( f(w_4) \in J_4 \) and \( f(v_5) \in J_5 \). We have \( f(w_0) < f(v_1) < f(v_2) < f(v_4) < f(w_3) \) with span at least \( 3j + k \). We will get the same result, if we replace \( v_4 \) by \( v_5 \).

Now the remaining cases are \( f(v_2) < f(v_3) < f(v_4) < f(v_5) < f(v_2) \). The span of the last three sequences are at least \( j + k \). So combining with the sequence \( f(v_0) < f(v_1) \) we have a sequence with span \( 3j + k \). Finally, we consider the case \( f(v_0) < f(v_1) < f(v_2) < f(v_3) < f(v_4) < f(v_5) \). Since \( f(v_0) \) also in \( J_0 \), we have the sequence \( f(v_0) < f(v_2) < f(v_3) < f(v_4) < f(v_5) \) with span \( 3j + k \).

**B-2.** Suppose \( J_1 \) contains two labels. Let \( f(A_1) \subset J_1 \). Then \( f(A_1) \cap (J_0 \cup J_2) = \emptyset \) for all \( l \). Let \( f(w_r) \in J_r \), for \( 0 \leq r \leq 5 \) and \( r \neq 1 \). There is a \( v \in A_1 \) such that \( d(v, w_2) = 2 \). Hence the sequence \( f(w_0) < f(v) < f(w_2) < f(w_3) < f(w_4) < f(w_5) \) is of span at least \( 3j + k \) (as the span of \( \{ f(w_2), f(w_3), f(w_4), f(w_5) \} \) is at least \( 2j \), \( f(v) - f(w_0) \geq j \) and \( f(w_2) - f(v) \geq k \).

**B-3.** Suppose \( J_2 \) contains two labels. Let \( f(A_1) \subset J_2 \). Then \( f(A_1) \cap (J_1 \cup J_5) = \emptyset \) for all \( l \). Let \( f(w_r) \in J_r \), for \( 0 \leq r \leq 5 \) and \( r \neq 2 \). There is a \( v \in A_1 \) such that \( d(v, w_1) = 2 \). Since \( w_1 \in \{v_2, v_3, v_4\} \), \( 1 \leq d(w_1, w_2) \leq 2 \). This implies that \( f(w_1) - f(w_0) \geq j \). Since \( w_1 \in \{v_2, v_3, v_4\} \), \( f(w_3) - f(v) \geq j \). Note that the span of \( \{ f(w_3), f(w_4), f(w_5) \} \) is at least \( j \). Hence the sequence \( f(w_0) < f(w_1) < f(v) < f(w_3) < f(w_4) < f(w_5) \) is of span at least \( 3j + k \).

Combining the above cases, we have \( \lambda \geq 3j + k \). Hence the proof is completed. \( \square \)
Theorem 3.12. Let $j$ and $k$ be two positive numbers with $2j \leq k < 3j$. If $n \geq 13$, then $\lambda_{j,k}(P_n^2) = \min\{j + 2k, 6j\}$.

Proof. Suppose $6j \leq j + 2k$. Define $g(v_i) = [i]_{7j}$ for all $i$. It is easy to verify that $g$ is a $(6j)$-$L(j,k)$-labeling for $P_n^2$.

Suppose $6j > j + 2k$. Define $g(v_0) = 0$, $g(v_1) = j$, $g(v_2) = k$, $g(v_3) = j + k$, $g(v_4) = 2k$, $g(v_5) = j + 2k$ and $g(v_i) = g(v_{i(\text{mod} n)})$ for $7 \leq i \leq n$. It is easy to verify that $g$ is a $(j+2k)$-$L(j,k)$-labeling for $P_n^2$.

Let $f$ be a $\lambda$-$L(j,k)$-labeling for $P_n^2$. It suffices to show that $\lambda \geq \min\{j + 2k, 6j\}$. Now, we assume $\lambda < j + 2k$.

We want to show that $\lambda \geq 6j$. We may assume $f(v_0) < f(v_3)$, otherwise consider the labeling $f = \lambda - f$.

Case A. Suppose $f(v_3) < f(v_6)$. Since $f(v_0) < f(v_3) < f(v_6)$, $f(v_6) \in [0, \lambda - 2k]$, $f(v_3) \in [k, \lambda - k]$ and $f(v_0) \in [2k, \lambda]$. Since $d(v_3, v_7) = 2$, $f(v_7) \in [0, \lambda - 2k] \cup [2k, \lambda]$. Since the length $[2k, \lambda]$ is less than $j$, $f(v_7) \in [0, \lambda - 2k]$. This implies that $f(v_4) > f(v_3)$. As the length of $[2k, \lambda]$ is less than $j$, we have $f(v_2), f(v_4)$ and $f(v_5)$ are not in $[2k, \lambda]$. By considering the distance apart from $v_6$, we have $f(v_4), f(v_5) \in [0, \lambda - j]$ and $f(v_2) \in [0, \lambda - k]$. By $f(v_0), f(v_7) \in [0, \lambda - 2k]$, we have $f(v_1), f(v_2), f(v_5) \in [j, \lambda]$ and $f(v_4) \in [k, \lambda]$. As the length of $[k, \lambda - k]$ is less than $j$, we have $f(v_1), f(v_2), f(v_4)$ and $f(v_5)$ are not in this interval. Now we summarize the range of some labels: $f(v_1) \in [j, k) \cup (\lambda - k, \lambda]$; $f(v_2) \in [j, k)$; $f(v_3) \in [k, \lambda - k]$; $f(v_4) \in (\lambda - k, \lambda - j]$ and $f(v_5) \in [j, k) \cup (\lambda - k, \lambda - j]$. Since the length of $[j, k]$ is less than $k$, $f(v_6) \notin [j, k]$ and hence $f(v_6) \in (\lambda - k, \lambda - j) \subset (\lambda - k, \lambda)$. Also since the length of $(\lambda - k, \lambda]$ is less than $k$, $f(v_6) \notin (\lambda - k, \lambda]$. Hence $f(v_1) \in [j, k]$. Now we have $f(v_0) < f(v_1), f(v_2), f(v_3) < f(v_5)$. So $f(v_3) \geq 2j + k$. That is, $f(v_3) \in [2j + k, \lambda)$. Up to now we have $f(v_0), f(v_1), f(v_2), f(v_3), f(v_4)$ and $f(v_5)$ are at most $\lambda - j$. By Theorem 3.10, $\lambda - j \geq 5j$. Hence $\lambda \geq 6j$.

Case B. Suppose $f(v_3) > f(v_6)$. We have $f(v_6) \in [0, \lambda - k]$. Suppose $f(v_7) > f(v_3)$. Since $f(v_0) < f(v_3) < f(v_7)$, $f(v_6) \in [0, \lambda - 2k]$, $f(v_3) \in [k, \lambda - k]$ and $f(v_7) \in [2k, \lambda]$. Since the lengths of $[0, \lambda - 2k]$ and $[2k, \lambda]$ are less than $j$, $f(v_7)$ does not lie in these two intervals. This implies that $f(v_0) < f(v_3) < f(v_7)$. Now, $f(v_3), f(v_4) \in [k, \lambda - k]$ which is impossible. That means $f(v_7) < f(v_5)$ and $f(v_7) \in [0, \lambda - k]$. Since $v_3, v_10$ are of distance two from $v_6$ and $v_7$ and the length of $[0, \lambda - k]$ is less than $j + k$, $f(v_3), f(v_10) \geq \max\{f(v_6), f(v_7)\} + k \geq j + k$ or $f(v_3), f(v_{10}) \leq \min\{f(v_6), f(v_7)\} - k \leq (\lambda - k - j) - k < 0$. So the last case is impossible. Hence $f(v_3), f(v_10) < f(v_3), f(v_{10})$. Similarly, we have $f(v_{11}) \in [0, \lambda - 2k] \cup [k, \lambda]$.

B-1. Suppose $f(v_{11}) \in [0, \lambda - 2k]$. Since the length of $[0, \lambda - 2k]$ is less than $j$, $f(v_7), f(v_8), f(v_9), f(v_{10})$ and $f(v_{12})$ are greater than $f(v_{11})$. Combining with $f(v_7) < f(v_5), f(v_{10})$, we have $f(v_7) \in [\lambda - k, \lambda]$; $f(v_3), f(v_{10}) \in [2k, \lambda]$. Since $f(v_8)$ cannot lie in $[0, \lambda - 2k] \cup [2k, \lambda]$; $f(v_{11}) < f(v_8) < f(v_{10})$. Now we have $f(v_8) \in [k, \lambda - j]$. Since $f(v_8)$ cannot lie in $[k, \lambda - k]$; $f(v_8) > f(v_1)$ and hence $f(v_8) \in [j + k, \lambda - j]$. Comparing $f(v_4)$ with $f(v_7)$, we have $f(v_4) \in [0, \lambda - 2k] \cup [2k, \lambda]$. From the range of $f(v_3)$, we have $f(v_4) \in [0, \lambda - 2k]$. From the range of $f(v_4)$, we have $f(v_5), f(v_6) \geq j$. Hence $f(v_i) \geq j$ for $5 \leq i \leq 10$. By Theorem 3.10, $\lambda - j \geq 5j$. Hence $\lambda \geq 6j$.

B-2. Suppose $f(v_{11}) \in [k, \lambda]$. Then $f(v_8) \in [0, \lambda - k] \cup [2k, \lambda]$. Suppose $f(v_8) \in [2k, \lambda]$. Since the length of $[2k, \lambda]$ is less than $j$, $f(v_7) \in [0, \lambda - j]$, where $4 \leq i \leq 12$ and $i \neq 8$. Moreover, since $d(v_8, v_{11}) = 2$, $f(v_{11}) \in [k, \lambda - k]$. Since the length of $[k, \lambda - k]$ is less than $j$, $f(v_6), f(v_7), f(v_9)$ and $f(v_{12})$ are less than $f(v_{11})$. $f(v_6), f(v_7) \in [0, \lambda - 2k]$ and $f(v_9), f(v_{12}) \in [0, \lambda - k - j]$. But it is impossible, since the length of $[0, \lambda - k - j]$ is less than $k$. Thus, $f(v_8) \in [0, \lambda - k]$.

a. When $f(v_5) < f(v_8)$. This implies that $f(v_4) \in [0, \lambda - 2k]$ and hence $f(v_8) \in [k, \lambda - k]$ and $f(v_6), f(v_7) \in [j, \lambda - k]$. Since the length of $[k, \lambda - k]$ is less than $j$, $f(v_8)$ must be less than $f(v_{11})$ and $f(v_8)$ must be greater than $f(v_5)$ and $f(v_7)$. Thus, $f(v_{11}) \in [2k, \lambda]$. Now the length of $[2k, \lambda]$ is less than $j$, $f(v_9)$ and $f(v_{10})$ are less than $f(v_{11})$. That means, $f(v_5), f(v_6), f(v_7), f(v_9), f(v_{10})$ lie in $[0, \lambda - j]$. By Theorem 3.10, we have $\lambda \geq 6j$.

b. When $f(v_5) > f(v_8)$. Hence $f(v_5) \in [k, \lambda]$.

B-1. Suppose $f(v_2) > f(v_5)$. Since $f(v_2) > f(v_5) > f(v_8), f(v_8) \in [0, \lambda - 2k]$. Since the length of $[0, \lambda - 2k]$ is less than $j$, $f(v_4), f(v_5), f(v_6)$, $f(v_7)$ are greater than $j$. Combining with the ranges of $f(v_2)$ and $f(v_3)$ we have $f(v_i) \in [j, \lambda]$ for $2 \leq i \leq 7$. By Theorem 3.10, we have $\lambda \geq 6j$. 
b-2. Suppose \( f(v_2) < f(v_5) \). Then \( f(v_2) \in [0, \lambda - k] \). Since \( f(v_6) \in [0, \lambda - k] \) and \( d(v_2, v_6) = 2 \), \( f(v_2) \in [0, \lambda - 2k] \) or \( f(v_2) \in [k, \lambda - k] \). When \( f(v_2) \in [0, \lambda - 2k] \), we have \( f(v_6) \in [k, \lambda - k] \).

Since \( f(v_3) \) and \( f(v_{10}) \) are greater than \( f(v_6) \), \( f(v_3), f(v_{10}) \in [2k, \lambda] \). Since the length of \([2k, \lambda] \) is less than \( j \), \( f(v_i) \in [0, \lambda - j] \) for \( 4 \leq i \leq 9 \). By Theorem 3.10, we have \( \lambda \geq 6j \).

When \( f(v_2) \in [k, \lambda - k] \), now \( f(v_6) \in [0, \lambda - 2k] \) and \( f(v_5) \in [2k, \lambda] \). By considering the distances from \( v_9 \) to \( v_5 \) and \( v_6 \), we have \( f(v_6) \in [k, \lambda - k] \). Since the lengths of \([0, \lambda - 2k] \) and \([k, \lambda - k] \) are less than \( j \), then \( f(v_7), f(v_8) \in [j, \lambda - j - k] \). It implies that \( f(v_{12}) \in [2k, \lambda] \) or \( f(v_{12}) \in [0, \lambda - j - 2k] \) by considering the distances from \( v_8 \) and \( v_9 \). But the last case is impossible as \( \lambda < j + 2k \). Now \( j \leq f(v_i) \) for \( 7 \leq i \leq 12 \). By Theorem 3.10, we have \( \lambda \geq 6j \).

According to Theorems 3.2–3.12, we can obtain following conclusion.

**Corollary 3.13.** Let \( n \geq 6 \) and \( j, k \) be two positive numbers.

1. For \( j \leq k < 2j \), \( \lambda_{j,k}(P_n^2) = \left\{ \begin{align*} &3j + k, &\quad &\text{if } 6 \leq n \leq 10, \\
&\min(2j + 2k, 5j), &\quad &\text{if } 11 \leq n \leq 15, \\
&\min(j + 3k, 5j), &\quad &\text{if } 16 \leq n \leq 20, \\
&\min(4k, 5j), &\quad &\text{if } n \geq 21. \end{align*} \right. \)

2. For \( 2j \leq k < 3j \), \( \lambda_{j,k}(P_n^2) = \left\{ \begin{align*} &5j, &\quad &\text{if } n = 6, \\
&5j + k, &\quad &\text{if } 7 \leq n \leq 12, \\
&\min(j + 2k, 6j), &\quad &\text{if } n \geq 13. \end{align*} \right. \)

3. For \( k \geq 3j \), \( \lambda_{j,k}(P_n^2) = \left\{ \begin{align*} &2j + k, &\quad &\text{if } n = 6, \\
&2k, &\quad &\text{if } n \geq 7. \end{align*} \right. \)

**References**

[6] Q. Niu, \(L(j, k)\)-Labeling of Graph and Edge Span (M.Phil. thesis), Southeast University, Nanjing, China, 2007.