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$L(j, k)$-labeling numbers of square of paths☆

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Abstract

For $j \leq k$, the $L(j,k)$-labeling arose from code assignment problem. That is, let j, k and m be positive numbers, an m-$L(j,k)$-labeling of a graph G is a mapping $f : V(G) \to [0,m]$ such that $|f(u)-f(v)| \geq j$ if $d(u,v) = 1$, and $|f(u)-f(v)| \geq k$ if $d(u,v) = 2$. The span of f is the difference between the maximum and the minimum numbers assigned by f. The $L(j,k)$-labeling number of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labelings of G. The kth power G^k of an undirected graph G is the graph with the vertex set of G in which two vertices are adjacent when their distance in G is at most k. In this paper, the $L(j,k)$-labeling numbers of P_n^2 are determined for $j \leq k$.

Keywords: $L(j,k)$-labeling; Path; Square of path; Code assignment

1. Introduction

The rapid growth of computer wireless networks highlighted the scarcity of available codes (such as radio frequencies) for communication with minimum interference. For example, the *Packet Radio Network* (PRN) is a computer network that uses radio frequencies to transmit packet among computers. The two major types of interference in PRN are *Direct collision (or interference)*, which is caused by the transmission of adjacent stations (computers), and *Hidden terminal collision (or interference)*, which is caused by distance-two stations that transmit to the same receiving station or receive the data from the same transmitting station.

Let G be a graph and let $V(G)$ and $E(G)$ be its vertex set and edge set, respectively. For any two vertices u and v, let $d_G(u,v)$ (or simply $d(u,v)$) denote the distance (length of a shortest path) between u and v in G. Noted that all graphs considered in this article are simple connected and undirected. All notation not defined in this article can be found in the book [1].

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For positive numbers j and k, an $L(j,k)$-*labeling* f of G is an assignment of numbers to vertices of G such that $|f(u)-f(v)| \geq j$ if $uv \in E(G)$, and $|f(u)-f(v)| \geq k$ if $d(u,v)=2$. The *span* of f is the difference between the maximum and the minimum numbers assigned by f. In other words, if we list the image of f as a non-decreasing sequence $\{f(u_i)\}_{i=1}^{n}$, then the span of f is $f(u_n)-f(u_1)$ which is called the *span of the sequence*, where n is the order of G. The $L(j,k)$-*labeling number* of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labeling of G. Without loss of generality, we may assume the minimum value of each labeling f is 0.

In a computer network, suppose direct collision is so weak that it can be ignored, and two distance-two stations can generate a hidden terminal collision. In order to avoid the hidden terminal collision, Bertossi and Bonuccelli [2] introduced an optimal code assignment, that is, two distance-two stations have different codes. By corresponding codes to labels, this code assignment problem is equivalent to the $L(0,1)$-labeling problem, that is, two distance-two vertices must be assigned different labels.

In general, the direct collision cannot be ignored. Based on this premise, Jin and Yeh [3] generalized the code assignment problem to $L(j,k)$-labeling problem with $j \leq k$. That is, to avoid direct collision, any two adjacent stations are required to be assigned at least j apart codes. Additionally, to avoid hidden terminal collision, any two distance-two stations need to be assigned at least k apart codes. Therefore, we face on the $L(j,k)$-labeling problem with $j \leq k$.

On the other hand, $L(j,k)$-labeling numbers of graphs for $j \geq k$ have been studied in many articles. Interested readers were referred to the surveys [4,5].

By now, about the $L(j,k)$-labeling numbers of graphs for $j \leq k$, there already exist some results. For example, Jin and Yeh determined $L(0,1)$, $L(1,1)$, $L(1,2)$-labeling numbers of paths, cycles and grids in [3]. Furthermore, Niu [6] introduced $L(j,k)$-labeling numbers of paths and cycles, and Griggs and Jin studied $L(j,k)$-labeling numbers of lattices (grids) in [7]. Moreover, Jayasree and Nicholas [8] mentioned $L(1,2)$-labeling numbers of certain generalizes Petersen graphs and n-star. In [9], the authors introduced the $L(j,k)$-labeling numbers of trees and stars with maximum degree. Lam, Lin and Wu [10] worked on $L(j,k)$-labeling numbers of product of completed graphs. Recently, Shiu and Wu determined $L(j,k)$-labeling numbers of direct and Cartesian product of path and cycle in [11] and [12], respectively. Moreover, the authors studied circular $L(j,k)$-labeling numbers of tree and Cartesian products of graphs, direct product of path and cycle, and square of paths in [13–15], respectively.

The kth *power* G^k of a graph G is the graph with the vertex set of G in which two vertices are adjacent when their distance in G is at most k. G^2 is called the *square* of G.

Lemma 1.1. *Let j and k be two positive numbers with $j \leq k$. Suppose G is a graph and H is an induced subgraph of G. Then $\lambda_{j,k}(G) \geq \lambda_{j,k}(H)$.*

Note that Lemma 1.1 is not true if H is not an induced subgraph. Throughout this paper, $P_n = v_0v_1\cdots v_{n-1}$ denotes the path of order n.

2. $L(j,k)$-labeling numbers of P_4^2 and P_5^2

Theorem 2.1. *For $j \leq k$, $\lambda_{j,k}(P_4^2) = \max\{k, 3j\}$.*

Proof. Let $\lambda = \max\{k, 3j\}$. Let f be the labeling of P_4^2 defined by $f(v_0)=0$, $f(v_1)=j$, $f(v_2)=2j$ and $f(v_3)=\lambda$. It is easy to verify that f is a λ-$L(j,k)$-labeling of P_4^2. Hence $\lambda_{j,k}(P_4^2) \leq \lambda$.

On the other hand, since any two vertices of P_4^2 are adjacent or of distance two, $\lambda_{j,k}(P_4^2) \geq 3j$. Moreover, since $d(v_0,v_3)=2$, $\lambda_{j,k}(P_4^2) \geq k$. It implies that $\lambda_{j,k}(P_4^2) \geq \max\{k,3j\} = \lambda$. Hence, $\lambda_{j,k}(P_4^2) = \max\{k,3j\}$. □

Theorem 2.2. *For $j \leq k$, $\lambda_{j,k}(P_5^2) = \max\{j+k, 4j\}$.*

Proof. Let $\eta = \max\{k, 3j\}$ and $\lambda = \eta + j$. Let f be the labeling of P_5^2 defined by $f(v_0)=0$, $f(v_1)=j$, $f(v_2)=2j$, $f(v_3)=\eta$ and $f(v_3)=\eta+j$. It is easy to verify that f is a λ-$L(j,k)$-labeling of P_5^2. Hence $\lambda_{j,k}(P_5^2) \leq \lambda$.

On the other hand, since any two vertices of P_5^2 are adjacent or of distance two, $\lambda_{j,k}(P_5^2) \geq 4j$. Moreover, since two adjacent vertices v_3 and v_4 are at distance two from vertex v_0, $\lambda_{j,k}(P_5^2) \geq j+k$. It implies that $\lambda_{j,k}(P_5^2) \geq \max\{j+k, 4j\}$. Hence, $\lambda_{j,k}(P_5^2) = \max\{j+k, 4j\}$. □

3. $L(j,k)$-labeling numbers of P_n^2 for $n \ge 6$

In this section, we shall study the $L(j,k)$-number of P_n^2 by separating the condition $j \le k$ into three cases which are $j \le k < 2j$, $3j \le k$ and $2j \le k < 3j$, where $n \ge 6$.

We consider $j \le k < 2j$ first. Define a labeling f for P_n^2 by $f(v_i) = [i]_6 j$ for $0 \le i \le n$. Clearly, f is a $(5j)$-$L(j,k)$-labeling for P_n^2 when $j \le k < 2j$. So

$$\lambda_{j,k}(P_n^2) \le 5j \text{ for } n \ge 6. \tag{3.1}$$

Lemma 3.1. *Let j and k be two positive numbers with $j \le k < 3j$. Then $\lambda_{j,k}(P_6^2) \ge \min\{5j, 3j+k\}$.*

Proof. Suppose f is a λ-$L(j,k)$-labeling of P_6^2 and $\lambda < 5j$. Let $I_0 = [0, j)$, $I_1 = [j, (\lambda+j)/3]$, $I_2 = ((\lambda+j)/3, (2\lambda-j)/3)$, $I_3 = [(2\lambda-j)/3, \lambda-j]$ and $I_4 = (\lambda-j, \lambda]$. Here, each I_i is of length less than j, $0 \le i \le 4$. Since any two of vertices v_0, v_1, v_2, v_3, v_4 are adjacent or of distance two, each interval I_i contains exactly one labels of $f(v_0), f(v_1), f(v_2), f(v_3), f(v_4)$. Similarly, each interval I_i contains exactly one labels of $f(v_1), f(v_2), f(v_3), f(v_4), f(v_5)$. By pigeonhole principle, two labels among $f(v_0), f(v_1), f(v_2), f(v_3), f(v_4), f(v_5)$ fall into the same interval. By considering the distance between those vertices, we can see that only $f(v_0)$ and $f(v_5)$ lie in the same interval. By symmetry of the graph, it suffices to consider $f(v_0), f(v_5) \in I_i$ for $0 \le i \le 2$. Let $A = \{v_0, v_5\}$.

Case 1. Suppose $f(v_0), f(v_5) \in I_0$. Here $f(v_1), f(v_2), f(v_3), f(v_4) \in [k, \lambda]$. Since $k \ge j$, by Theorem 2.1 we have $\lambda - k \ge 3j$. Hence $\lambda \ge 3j+k$.

Case 2. Suppose $f(v_0), f(v_5) \in I_1$. We have $f(v_1), f(v_2), f(v_3), f(v_4) \in [0, (\lambda+j)/3-k) \cup [j+k, \lambda]$. Since $k \ge j$ and $\lambda < 5j$, the length of $[0, (\lambda+j)/3-k)$ is less than j. Thus, $[j+k, \lambda]$ contains three of $f(v_1)$, $f(v_2)$, $f(v_3)$ and $f(v_4)$. Now $\lambda - k - j \ge 2j$, i.e., $\lambda \ge 3j+k$.

Case 3. Suppose $f(v_0), f(v_5) \in I_2$. Let $f(w_i) \in I_i$ for some w_i, where $i = 0, 1, 3, 4$. Hence $\{w_0, w_1, w_3, w_4\} = \{v_1, v_2, v_3, v_4\}$. There exists $v \in A$ such that $d(v, w_1) = 2$. So $f(v) - f(w_1) \ge k$. Then the span of the increasing sequence $f(w_0) < f(w_1) < f(v) < f(w_3) < f(w_4)$ is at least $3j+k$. Hence $\lambda \ge 3j+k$.

Thus $\lambda_{i,j}(P_6^2) \ge \min\{5j, 3j+k\}$. $\square$

Theorem 3.2. *Suppose $6 \le n \le 10$. Let j and k be two positive numbers. If $j \le k < 2j$, then $\lambda_{j,k}(P_n^2) = 3j+k$.*

Proof. Define a labeling f for P_{10}^2 as follows:
$f(v_0) = f(v_5) = 0$, $f(v_1) = k$, $f(v_2) = j+k$, $f(v_3) = 2j+k$, $f(v_4) = 3j+k$, $f(v_6) = j$, $f(v_7) = 2j$, $f(v_8) = 3j$, $f(v_9) = 4j$. It is easy to verify that f is a $(3j+k)$-$L(j,k)$-labeling of P_{10}^2 when $j \le k < 2j$.

By Lemma 1.1, we have $\lambda_{j,k}(P_n^2) \le 3j+k$ for $n \le 10$.

Since P_6^2 is an induced subgraph of P_n^2, it suffices to show that $\lambda = \lambda_{j,k}(P_6^2) \ge 3j+k$. By Lemma 3.1 and $j \le k < 2j$, we have $\lambda_{i,j}(P_6^2) \ge 3j+k$. Thus $\lambda_{i,j}(P_6^2) = 3j+k$. By Lemma 1.1 we get that $\lambda_{j,k}(P_n^2) \ge \lambda_{i,j}(P_6^2) = 3j+k$ for $n \ge 6$.

Combining the discussion above, we have $\lambda(P_n^2) = 3j+k$ for $6 \le n \le 10$. $\square$

For any integer a, $[a]_m \in \{0, 1, \dots, m-1\}$ denotes the residue of a modulo m, where m is a positive integer greater than 1. For convenience, we let $V_i = \{v_l \in V(P_n^2) \mid l \equiv i \pmod 5\}$, $0 \le i \le 4$, where $n \ge 11$. And also let $E(A, B)$ be the set of edges from A to B and $f(A) = \{f(v) \mid v \in A\}$ for a labeling f of P_n^2, where A and B are subsets of $V(P_n^2)$.

Theorem 3.3. *Suppose $11 \le n \le 15$. Let j and k be two positive numbers with $j \le k < 2j$. Then $\lambda_{j,k}(P_n^2) = \min\{2j+2k, 5j\}$.*

Proof. Let $(0, k, 2k, j+2k, 2j+2k, 0, k, j+k, 2j+k, 2j+2k, 0, j, 2j, 2j+k, 2j+2k)$ be the list of the values of $(g(v_i))_{0 \le i \le 14}$. Hence this defines a $(2j+2k)$-$L(j,k)$-labeling g for P_{15}^2. By Lemma 1.1 and (3.1), we have $\lambda_{j,k}(P_n^2) \le \min\{2j+2k, 5j\}$ for $11 \le n \le 15$.

Similar to the proof of Theorem 3.2, in order to obtain the theorem it suffices to show that $\lambda = \lambda_{j,k}(P_{11}^2) \ge \min\{2j+2k, 5j\}$. Thus we have to show that "if $\lambda < 5j$, then $\lambda \ge 2j+2k$".

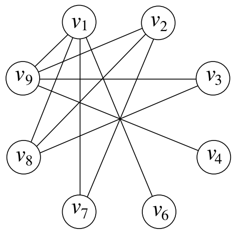

Fig. 1. The graph $H - V_0$.

Now suppose $\lambda < 5j$. Let f be a λ-$L(j,k)$-labeling of P_{11}^2. Let I_i be intervals defined in the proof of Theorem 3.2. Note that the length of each interval is less than j. By pigeonhole principle, at least one interval contains three vertex labels. Note that such labels may be the same. Let H be a graph with the vertex set $V(P_{11}^2)$ in which two vertices are adjacent if they are of distance at least 3 in P_{11}^2. Note that, H is a compatibility graph, in which two vertices are adjacent if and only if their assigned labels can lie in the same interval I_i for some i. We can see that H contains only one 3-cycle which is $v_0v_5v_{10}v_0$. Thus, only $f(v_0)$, $f(v_5)$ and $f(v_{10})$ lie in the same interval I_{h_0} for some h_0. Thus, each of other interval contains exactly two labels. By symmetry of the graph, we may assume that $0 \le h_0 \le 2$.

By considering the subgraph induced by $\{v_l \mid 0 \le l \le 5\}$ and the same argument in the proof of Theorem 3.2, each of $f(v_1)$, $f(v_2)$, $f(v_3)$, $f(v_4)$ lies in exactly one different interval. Let $f(v_i) \in I_{h_i}$ for $1 \le i \le 4$. Now $\{h_0, h_1, h_2, h_3, h_4\} = \{0, 1, 2, 3, 4\}$. Similarly, by considering the subgraph induced by $\{v_l \mid 5 \le l \le 10\}$, each of $f(v_6)$, $f(v_7)$, $f(v_8)$, $f(v_9)$ lies in exactly one different interval I_{h_i}, $1 \le i \le 4$.

By considering another compatibility graph $H - V_0$ (Fig. 1), we can see that $f(V_1) \subset I_{h_1}$ and $f(V_4) \subset I_{h_4}$. This forces that $f(V_2) \subset I_{h_2}$ and $f(V_3) \subset I_{h_3}$.

Case 1. Suppose $h_0 = 0$. We want to determine the span of the set $S = \{f(v_i) \mid 1 \le i \le 9, i \ne 5\}$, i,e., the maximum difference between each pair of labels in S. For each $w \in V_i$, $1 \le i \le 4$, there is a $v \in V_0$ such that $d(w, v) = 2$. Thus, $S \subset [k, \lambda]$. We shall face on all permutations of $h_1h_2h_3h_4$. For example, suppose $h_1h_2h_3h_4 = 1234$. That means $f(V_1) \subset I_1$, $f(V_2) \subset I_2$, $f(V_3) \subset I_3$ and $f(V_4) \subset I_4$. Considering the path $v_6v_2v_3v_4$ at the graph H_2 shown in Fig. 2, we have an increasing subsequence $f(v_6) < f(v_2) < f(v_3) < f(v_4)$. Thus the span of S is at least $k + 2j$. The reflection case of this case is $h_1h_2h_3h_4 = 4321$. By means of reflection there are $4!/2 = 12$ permutations we have to deal with.

Combining all cases, we have $\lambda - k \ge 2j + k$. Hence $\lambda \ge 2j + 2k$.

Case 2. Suppose $h_0 = 1$. Similar to Case 2 of the proof of Theorem 3.2, $[j + k, \lambda]$ contains three of $f(V_1)$, $f(V_2)$, $f(V_3)$ and $f(V_4)$. No matter which case, the span of the union of these three subsets is at least $k + j$. So $\lambda - j - k \ge k + j$. Hence $\lambda \ge 2j + 2k$.

Case 3. Suppose $h_0 = 2$. Now $\{h_1, h_2, h_3, h_4\} = \{0, 1, 3, 4\}$. Consider the graph H_2. There is always a hard edge in $E(V_s, V_t)$, where $1 \le s < t \le 4$. Moreover, each vertex in H_2 is of distance either 1 or 2 to v_5 in P_{11}^2. Thus, for each permutation of $h_1h_2h_3h_4$, there always exists an increasing subsequence of $f(V(P_{11}^2))$ involving $f(v_5)$ with the span at least $2j + 2k$. For example, when $h_1h_2h_3h_4 = 0314$, the required subsequence is $f(v_6) < f(v_3) < f(v_5) < f(v_7) < f(v_4)$. So the span of f is at least $2j + 2k$.

Combining the discussion above, we have $\lambda = 2j + 2k$ for $11 \le n \le 15$. □

Theorem 3.4. *Suppose* $16 \le n \le 20$. *Let* j, k *be two positive numbers with* $j \le k < 2j$. *Then* $\lambda_{j,k}(P_n^2) = \min\{j + 3k, 5j\}$.

Proof. Let $(0, k, 2k, 3k, j + 3k, 0, k, 2k, j + 2k, 2j + 2k, 0, k, j + k, 2j + k, 2j + 2k, 0, j, 2j, 2j + k, 2j + 2k)$ be the list of the values of $(g(v_i))_{0 \le i \le 19}$. Hence this defines a $(j + 3k)$-$L(j,k)$-labeling g for P_n^2 if $16 \le n \le 20$. By Lemma 1.1 and (3.1), we have $\lambda_{j,k}(P_n^2) \le \min\{j + 3k, 5j\}$ for $16 \le n \le 20$.

Conversely, we consider $\lambda = \lambda_{j,k}(P_{16}^2)$. Similar to the proof of Theorem 3.3 we assume $\lambda < 5j$ and show that $\lambda \ge j + 3k$ in the following.

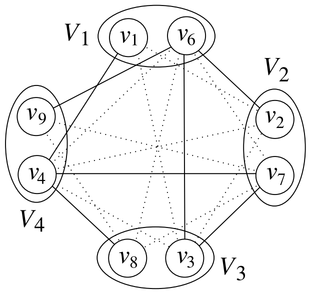

Fig. 2. Graph H_2: Dot and hard edges indicate distance 1 and 2 between the involved vertices in P_{11}^2, respectively.

Table 1
Lower bound of the span S.

$h_1h_2h_3h_4$	Subsequence of labels	Lower bound of the span
1234	$f(v_6) < f(v_2) < f(v_3) < f(v_4)$	$2j + k$
1243	$f(v_6) < f(v_7) < f(v_4) < f(v_8)$	$j + 2k$
1324	$f(v_6) < f(v_3) < f(v_7) < f(v_4)$	$3k$
1342	$f(v_1) < f(v_4) < f(v_7) < f(v_3)$	$3k$
1423	$f(v_6) < f(v_3) < f(v_4) < f(v_7)$	$3k$
1432	$f(v_1) < f(v_4) < f(v_3) < f(v_7)$	$j + 2k$
2134	$f(v_2) < f(v_6) < f(v_3) < f(v_4)$	$j + 2k$
2143	$f(v_2) < f(v_6) < f(v_9) < f(v_8)$	$j + 2k$
2314	$f(v_3) < f(v_6) < f(v_2) < f(v_4)$	$j + 2k$
2413	$f(v_3) < f(v_6) < f(v_9) < f(v_7)$	$j + 2k$
3124	$f(v_7) < f(v_3) < f(v_6) < f(v_9)$	$3k$
3214	$f(v_3) < f(v_7) < f(v_6) < f(v_9)$	$j + 2k$

Let I_i be defined in Theorem 3.2. By considering the subgraphs induced by $\{v_i \mid 0 \le i \le 10\}$ and $\{v_i \mid 5 \le i \le 15\}$, we obtain that $f(V_0) \subset I_{h_0}$ for some $h_0 \in \{0, 1, 2\}$ (without loss of generality), $f(V_i) \subset I_{h_i}$, $1 \le i \le 4$, where $\{h_0, h_1, h_2, h_3, h_4\} = \{0, 1, 2, 3, 4\}$.

Case 1. Suppose $h_0 = 0$. By Table 1, we only need to consider the case when $h_1h_2h_3h_4 = 1234$. In this case, there is a subsequence $f(v_{11}) < f(v_7) < f(v_3) < f(v_4)$. Now the span of S is at least $j + 2k$.

Case 2. Suppose $h_0 = 1$. Similar to Case 2 of the proof of Theorem 3.2, $[j + k, \lambda]$ contains three of $f(V_1)$, $f(V_2)$, $f(V_3)$ and $f(V_4)$. No matter which case (see Fig. 3), the span of the union of these three subsets is at least $2k$. So $\lambda - j - k \ge 2k$. Hence $\lambda \ge j + 3k$.

Case 3. Suppose $h_0 = 2$. Now $\{h_1, h_2, h_3, h_4\} = \{0, 1, 3, 4\}$. We have to deal with the 12 permutations of $h_1h_2h_3h_4$. We only provide the discussion of the case when $h_1h_2h_3h_4 = 0314$ here. Other cases are similarly to show. For this case, we have an increasing subsequence $f(v_{11}) < f(v_8) < f(v_5) < f(v_7) < f(v_4)$. So the span of f is at least $j + 3k$.

Combining the discussion above, we have $\lambda = j + 3k$ for $16 \le n \le 20$. □

Lemma 3.5. *Let W_i be a set consisting of 4 vertices of a graph G, $1 \le i \le 4$. Assume that $E(W_i, W_{i+1})$ contains at least 3 disjoint edges, for $1 \le i \le 3$. Then there is a path $w_1w_2w_3w_4$ with $w_i \in W_i$.*

Proof. Let $w_{1j}w_{2j} \in E(W_1, W_2)$ for $1 \le j \le 3$, $w_{ij} \in W_i$. Since at most one vertex in W_2 is not incident with edge of $E(W_1, W_2)$, at least two edges of $E(W_2, W_3)$ are adjacent with $w_{1j}w_{2j}$, $1 \le j \le 3$. After renaming if necessary, we may assume that such two edges of $E(W_2, W_3)$ are $w_{21}w_{31}$ and $w_{22}w_{32}$. Since only one vertex in W_4 is not incident edges in $E(W_3, W_4)$, there is an edge, say $w_{31}w_{41} \in E(W_3, W_4)$. Now we have a path $w_{11}w_{21}w_{31}w_{41}$. □

Corollary 3.6. *Let W_i be a set consisting of 4 vertices of a graph G, $1 \le i \le 3$. Assume that $E(W_i, W_{i+1})$ contains at least 3 disjoint edges, for $1 \le i \le 2$. Then there are two disjoint paths $w_1w_2w_3$ with $w_i \in W_i$.*

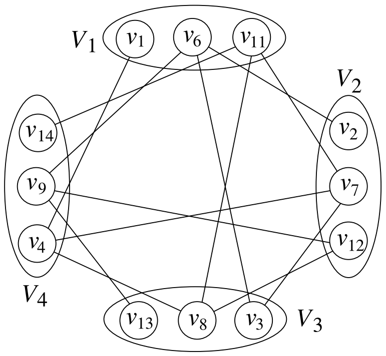

Fig. 3. Graph H_3: Two vertices are adjacent if they are of distance 2 in P_{16}^2.

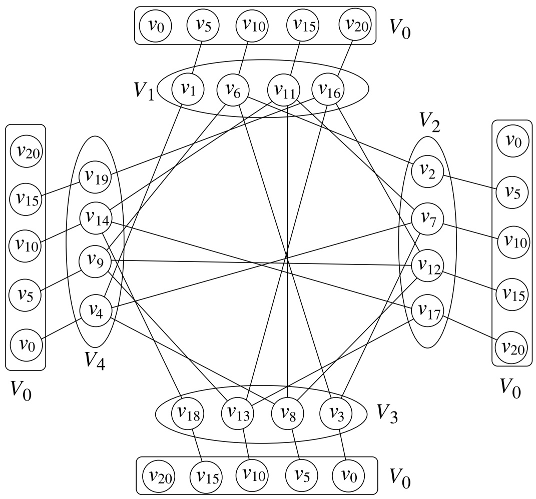

Fig. 4. Graph H_4.

Theorem 3.7. *Suppose $n \geq 21$. Let j, k be two positive numbers with $j \leq k < 2j$. Then $\lambda_{j,k}(P_n^2) = \min\{4k, 5j\}$.*

Proof. Define a labeling g for P_n^2 by $g(v_i) = [i]_5 k$. Clearly g is a $(4k)$-$L(j,k)$-labeling of P_n^2. By (3.1), we have $\lambda_{j,k}(P_n^2) \leq \min\{4k, 5j\}$.

Consider the graph P_{21}^2 and assume $\lambda < 5j$. Let I_i be defined in Theorem 3.2. By a similar argument of the proof of Theorem 3.4, we have $f(V_0) \subset I_{h_0}$ for some $h_0 \in \{0, 1, 2\}$, $f(V_i) = \{f(v_1), f(v_6), f(v_{11}), f(v_{16})\} \subset I_{h_i}$, $1 \leq i \leq 4$, where $\{h_0, h_1, h_2, h_3, h_4\} = \{0, 1, 2, 3, 4\}$. Let H_4 be the graph in Fig. 4. Two vertices are adjacent if they are of distance 2 in P_{21}^2.

Remark 1. For $1 \leq s < t \leq 4$, there are at least 3 disjoint edges in $E(V_s, V_t) \subset E(H_4)$. For each $w \in V_i$ with $1 \leq i \leq 4$, there is a unique $v \in V_0$ such that $wv \in E(V_0, V_i)$.

Case 1. Suppose $h_0 = 0$. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ h_1 & h_2 & h_3 & h_4 \end{pmatrix}$ be a permutation of $\{1, 2, 3, 4\}$. Let $k_i = \sigma^{-1}(i)$. By Remark 1 and Lemma 3.5, there is a path $w_1 w_2 w_3 w_4$ in which $w_i \in V_{k_i}$. By Remark 1 again, there is $w_0 \in V_0$ such that $d(w_0, w_1) = 2$. Hence the path $w_0 \cdots w_5$ induces a subsequence $f(w_0) < f(w_1) < f(w_2) < f(w_3) < f(w_4)$ and the span of this subsequence is $4k$. Hence we have $\lambda \geq 4k$.

Case 2. Suppose $h_0 = 1$. Let $\sigma = \begin{pmatrix} 0 & 2 & 3 & 4 \\ h_1 & h_2 & h_3 & h_4 \end{pmatrix}$ be a permutation of $\{0, 2, 3, 4\}$. Let $k_i = \sigma^{-1}(i)$, $2 \leq i \leq 4$ and $\sigma^{-1}(0) = \ell$. By Corollary 3.6, there are two disjoint paths $w_2 w_3 w_4$ and $u_2 u_3 u_4$ such that $w_i, u_i \in V_{k_i}$,

$2 \le i \le 4$. By Remark 1 there are $u_0, w_0 \in V_0$ such that w_0w_2 and u_0u_2 are edges of $E(V_0, V_{k_2})$. Since only one vertex in V_0 is not adjacent to vertex of V_ℓ, there is a $w_1 \in V_\ell$ such that either $w_1w_0 \in E(V_\ell, V_0)$ or $w_1u_0 \in E(V_\ell, V_0)$. Let us say $w_1w_0 \in E(V_\ell, V_0)$. There is a path $w_1w_0w_2w_3w_4$ that induces a subsequence $f(w_1) < f(w_0) < f(w_2) < f(w_3) < f(w_4)$. Hence we have $\lambda \ge 4k$.

Case 3. Suppose $h_0 = 2$. Let $\sigma = \begin{pmatrix} 0 & 1 & 3 & 4 \\ h_1 & h_2 & h_3 & h_4 \end{pmatrix}$ be a permutation of $\{0, 1, 3, 4\}$. Let $k_i = \sigma^{-1}(i)$, $i \in \{0, 1, 3, 4\}$. Note that $E(V_{k_0}, V_{k_1})$ and $E(V_{k_3}, V_{k_4})$ contain 3 disjoint edges, respectively; $E(V_{k_1}, V_0)$ and $E(V_0, V_{k_3})$ contain 4 disjoint edges, respectively. By a similar argument as above, there are three paths $u_1u_2u_0$ with $u_1 \in V_{k_1}$, $u_2 \in V_{k_1}$ and $u_0 \in V_0$ and three paths $w_0u_3u_4$ with $u_3 \in V_{k_3}$, $u_4 \in V_{k_4}$ and $w_0 \in V_0$. By pigeonhole principle, there are two paths $u_1u_2u_0$ and $w_0u_3u_4$ such that $u_0 = w_0$. Here we have a path $u_1u_2u_0u_3u_4$. Hence we have $\lambda \ge 4k$. □

Now, we consider the case when $k \ge 3j$.

Theorem 3.8. *Let j and k be two positive numbers. If $k \ge 3j$, then $\lambda_{j,k}(P_6^2) = 2j + k$.*

Proof. Let $\lambda = \lambda_{j,k}(P_6^2)$. Define $g(v_0) = 0$, $g(v_1) = j$, $g(v_2) = 2j$, $g(v_3) = k$, $g(v_4) = j + k$, $g(v_5) = 2j + k$. It is easy to verify that g is a $(2j + k)$-$L(j, k)$-labeling for P_6^2. Hence $\lambda \le 2j + k$.

On the other hand, let f be a λ-$L(j, k)$-labeling of P_6^2. Since v_3, v_4 are distance two apart from v_0, $f(v_3), f(v_4) \in [0, f(v_0) - k] \cup [f(v_0) + k, \lambda]$. If $f(v_3) < f(v_0) < f(v_4)$ or $f(v_4) < f(v_0) < f(v_3)$, then $\lambda \ge 2k > 2j + k$. Since we have just known that $\lambda \le 2j + k$, this is not a case. So both $f(v_3)$ and $f(v_4)$ are either greater than or less than $f(v_0)$. Without loss of generality, we may assume $f(v_3)$ and $f(v_4)$ are greater than $f(v_0)$, otherwise consider the labeling $\overline{f} = \lambda - f$. That is, $f(v_3), f(v_4) \in [f(v_0) + k, \lambda]$. Hence $\lambda - k \ge f(v_0)$. Moreover, since $d(v_1, v_4) = 2$, $f(v_1) \in [0, \lambda - k]$. Similarly, since $d(v_1, v_5) = 2$, $f(v_5) \in [k, \lambda]$. Since $d(v_2, v_5) = 2$, $f(v_2) \in [0, \lambda - k]$. Now, we can conclude that $f(v_0), f(v_1), f(v_2) \in [0, \lambda - k]$. Since v_0, v_1, v_2 are adjacent to each other, $\lambda - k \ge 2j$. Hence $\lambda \ge 2j + k$. Hence $\lambda_{j,k}(P_6^2) = 2j + k$. □

Theorem 3.9. *Suppose $n \ge 7$. Let j, k be two positive numbers. If $k \ge 3j$, then $\lambda_{j,k}(P_n^2) = 2k$.*

Proof. Let $\lambda = \lambda_{j,k}(P_n^2)$. Define $g(v_0) = 0$, $g(v_1) = j$, $g(v_2) = 2j$, $g(v_3) = k$, $g(v_4) = j + k$, $g(v_5) = 2j + k$, $g(v_6) = 2k$ and $g(v_i) = g(v_{[i]_7})$ for $i \ge 7$. It is easy to verify that g is a $2k$-$L(j, k)$-labeling of P_n^2. Hence $\lambda \le 2k$.

Since P_7^2 is an induced subgraph of P_n^2, it suffices to show that $\lambda = \lambda_{j,k}(P_7^2) \ge 2k$. Let f be a λ-$L(j, k)$-labeling of P_7^2. Consider the labels of v_0, v_3 and v_4. By the same argument of the proof of Theorem 3.8, we only need to consider the case when $f(v_3), f(v_4) \in [f(v_0) + k, \lambda]$. By the proof of Theorem 3.8 again, we have $f(v_0), f(v_1), f(v_2) \in [0, \lambda - k]$.

Now, if $f(v_3) < f(v_6)$, then $f(v_0) < f(v_3) < f(v_6)$ implies $\lambda \ge 2k$. The remaining case is $f(v_6) < f(v_3)$. But this implies that $f(v_6) \in [0, \lambda - k]$. Now, $f(v_2), f(v_6) \in [0, \lambda - k]$. Hence $\lambda \ge 2k$.

This completes the proof. □

Finally, we consider the case when $2j \le k < 3j$.

Theorem 3.10. *Let j and k be two positive numbers. If $2j \le k < 3j$, then $\lambda_{j,k}(P_6^2) = 5j$.*

Proof. Define a labeling g for P_6^2 by $g(v_i) = [i]_6 j$. Clearly g is a $(5j)$-$L(j, k)$-labeling of P_6^2. Thus, we have $\lambda_{j,k}(P_6^2) \le 5j$.

Moreover, by Lemma 3.1, we have $\lambda_{j,k}(P_6^2) \ge \min\{5j, 3j + k\} = 5j$. Hence $\lambda_{j,k}(P_6^2) = 5j$. □

Theorem 3.11. *Let j and k be two positive numbers. If $2j \le k < 3j$ and $7 \le n \le 12$, then $\lambda_{j,k}(P_n^2) = 3j + k$.*

Proof. Let $g(v_0) = 0$, $g(v_1) = j$, $g(v_2) = k$, $g(v_3) = j + k$, $g(v_4) = 2j + k$, $g(v_5) = 3j + k$, $g(v_6) = 0$, $g(v_7) = j$, $g(v_8) = 2j$, $g(v_9) = 3j$, $g(v_{10}) = 4j$, $g(v_{11}) = 5j$. It is easy to check that g is a $(3j + k)$-$L(j, k)$-labeling of P_{12}^2. Thus, By Lemma 1.1, $\lambda_{j,k}(P_n^2) \le 3j + k$ for $7 \le n \le 12$.

Let $\lambda = \lambda_{j,k}(P_7^2)$ and let f be a λ-$L(j, k)$-labeling of P_7^2. We have $\lambda \le 3j + k$. As the proofs of those previous theorems, we only need to show $\lambda \ge 3j + k$.

Let $J_0 = [0, j)$, $J_1 = [j, 2j)$, $J_2 = [2j, \lambda/2]$, $J_3 = (\lambda/2, \lambda - 2j]$, $J_4 = (\lambda - 2j, \lambda - j]$ and $J_5 = (\lambda - j, \lambda]$. Since $\lambda \leq 3j + k$ and $k < 3j$, the length of each interval is less than j. Thus, if $f(u), f(w) \in J_i$ for some i, then $d(u, w) > 2$. Hence $\{u, w\}$ is either $A_0 = \{v_0, v_5\}$, $A_1 = \{v_1, v_6\}$ or $A_2 = \{v_0, v_6\}$. Also, each J_i cannot contain more than 2 vertices. Since the length of each $J_i \cup J_{i+1}$ is less then $2j < k$, for $0 \leq i \leq 4$, if $\{u, w\}$ is none of A_0, A_1 and A_2 but $f(\{u, w\}) \subset J_i \cup J_{i+1}$ for some i, then $d(u, w) = 1$. For this case, $f(\{u, w\}) \subset J_i \cup J_{i+1}$ associates a path uw of length 1. Furthermore, $J_i \cup J_{i+1}$ cannot contain three labels of $\{f(v_0), f(v_1), f(v_5), f(v_6)\}$.

Following we want to find an increasing sequence of labels with span at least $3j + k$. It is easy to get the following claim.

Claim 1. *Suppose $v \in \{v_2, v_3, v_4\}$. For each i, $i = 0, 1, 2$, there exists $w_i \in A_i$ such that $d(v, w_i) = 2$.*

Now, by pigeonhole principle, there is at least one J_q containing two labels. By symmetry we may assume that $q = 0, 1, 2$, otherwise consider the labeling $\overline{f} = \lambda - f$. Thus, J_q contains either $f(A_0)$, $f(A_1)$ or $f(A_2)$.

Case A. Suppose there are two intervals, say J_q and J_r, containing 2 labels, where $0 \leq q \leq 2$ and $q < r$. In this case, $f(A_2)$ does not contain in $J_q \cup J_r$. By renumbering the vertex if necessary, we may assume that $f(A_1) \subset J_q$ and $f(A_0) \subset J_r$. Let $\{f(u_1), f(u_2), f(u_3)\} = \{f(v_2), f(v_3), f(v_4)\}$, where $f(u_1) < f(u_2) < f(u_3)$.

- **A-1.** Suppose $f(v_0) < f(u_1)$. By Claim 1 there is $w_0 \in A_0$ such that $d(w_0, u_1) = 2$. Now we have $f(v_1) < f(w_0) < f(u_1) < f(u_2) < f(u_3)$ with span at least $3j + k$.
- **A-2.** Suppose $f(v_1) < f(u_1) < f(v_0)$ or $f(u_1) < f(v_1) < f(u_2)$. By Claim 1 there is $w_1 \in A_1$ such that $d(w_1, u_1) = 2$. Thus, $f(v_0), f(u_2), f(u_3)$ lie in $[k + j, \lambda]$. So $\lambda - (k + j) \geq 2j$. Hence $\lambda \geq 3j + k$.
- **A-3.** Suppose $f(u_2) < f(v_1)$ and $f(u_3) > f(v_1)$. In this case, we have $f(v_1) < f(u_3) < f(v_0)$ or $f(u_2) < f(v_0) < f(u_3)$. This the reflexive case of Case A-2.
- **A-4.** Suppose $f(u_3) < f(v_1)$. This is the reflexive case of Case A-1.

Case B. Suppose there is only one interval J_q containing 2 labels, where $0 \leq q \leq 2$.

- **B-1.** Suppose J_0 contains two labels.
  - **a.** $f(A_0) \subset J_0$. The span of the set $\{f(v_1), f(v_2), f(v_3), f(v_4)\}$ is at least $3j$. No matter which label is the minimum, there always exists a vertex $w \in A_0$ such that $f(w)$ is less than this minimum by at least k. Hence, the span of the set $\{f(w), f(v_1), f(v_2), f(v_3), f(v_4)\}$ is at least $3j + k$.
  - **b.** $f(A_1) \subset J_0$. Consider the set $\{f(v_2), f(v_3), f(v_4), f(v_5)\}$. Similar to Case a, we will get the same result.
  - **c.** $f(A_2) \subset J_0$. Since the length of $J_0 \cup J_1$ is less than k, only $f(v_1)$ or $f(v_5)$ lies in J_1. Renaming the vertex if necessary, we may assume $f(v_1) \in J_1$. By the same reason, only $f(v_2)$ or $f(v_3)$ lies in J_2. Suppose $f(v_4) \in J_3$. Let $f(w_4) \in J_4$ and $f(w_5) \in J_5$. We have $f(v_0) < f(v_1) < f(v_4) < f(w_4) < f(w_5)$ with span at least $3j + k$. We will get the same result, if we replace v_4 by v_5. Now the remaining cases are $f(v_2) < f(v_3) < f(v_4) < f(v_5)$, $f(v_2) < f(v_3) < f(v_5) < f(v_4)$, $f(v_3) < f(v_2) < f(v_4) < f(v_5)$ and $f(v_3) < f(v_2) < f(v_5) < f(v_4)$. The span of the last three sequences are at least $j + k$. So combining with the sequence $f(v_0) < f(v_1)$ we have a sequence with span $3j + k$. Finally, we consider the case $f(v_0) < f(v_1) < f(v_2) < f(v_3) < f(v_4) < f(v_5)$. Since $f(v_6)$ also in J_0, we have the sequence $f(v_6) < f(v_2) < f(v_3) < f(v_4) < f(v_5)$ with span $3j + k$.
- **B-2.** Suppose J_1 contains two labels. Let $f(A_i) \subset J_1$. Then $f(A_l) \cap (J_0 \cup J_2) = \varnothing$ for all l. Let $f(w_r) \in J_r$, for $0 \leq r \leq 5$ and $r \neq 1$. There is a $v \in A_i$ such that $d(v, w_2) = 2$. Hence the sequence $f(w_0) < f(v) < f(w_2) < f(w_3) < f(w_4) < f(w_5)$ is of span at least $3j + k$ (as the span of $\{f(w_2), f(w_3), f(w_4), f(w_5)\}$ is at least $2j$, $f(v) - f(w_0) \geq j$ and $f(w_2) - f(v) \geq k$).
- **B-3.** Suppose J_2 contains two labels. Let $f(A_i) \subset J_2$. Then $f(A_l) \cap (J_1 \cup J_3) = \varnothing$ for all l. Let $f(w_r) \in J_r$, for $0 \leq r \leq 5$ and $r \neq 2$. There is a $v \in A_i$ such that $d(v, w_1) = 2$. Since $w_1 \in \{v_2, v_3, v_4\}$, $1 \leq d(w_0, w_1) \leq 2$. This implies that $f(w_1) - f(w_0) \geq j$. Since $w_1 \in \{v_2, v_3, v_4\}$, $f(w_3) - f(v) \geq j$. Note that the span of $\{f(w_3), f(w_4), f(w_5)\}$ is at least j. Hence the sequence $f(w_0) < f(w_1) < f(v) < f(w_3) < f(w_4) < f(w_5)$ is of span at least $3j + k$.

Combining the above cases, we have $\lambda \geq 3j + k$. Hence the proof is completed. □

Theorem 3.12. *Let j and k be two positive numbers with $2j \le k < 3j$. If $n \ge 13$, then $\lambda_{j,k}(P_n^2) = \min\{j + 2k, 6j\}$.*

Proof. Suppose $6j \le j + 2k$. Define $g(v_i) = [i]_7 j$ for all i. It is easy to verify that g is a $(6j)$-$L(j,k)$-labeling for P_n^2.

Suppose $6j > j + 2k$. Define $g(v_0) = 0$, $g(v_1) = j$, $g(v_2) = k$, $g(v_3) = j + k$, $g(v_4) = 2k$, $g(v_5) = j + 2k$ and $g(v_i) = g(v_{[i]_6})$ for $7 \le i \le n$. It is easy to verify that g is a $(j + 2k)$-$L(j,k)$-labeling for P_n^2.

Let f be a λ-$L(j,k)$-labeling for P_{13}^2. It suffices to show that $\lambda \ge \min\{j + 2k, 6j\}$. Now, we assume $\lambda < j + 2k$. We want to show that $\lambda \ge 6j$. We may assume $f(v_0) < f(v_3)$, otherwise consider the labeling $\overline{f} = \lambda - f$.

Case A. Suppose $f(v_3) < f(v_6)$. Since $f(v_0) < f(v_3) < f(v_6)$, $f(v_0) \in [0, \lambda - 2k]$, $f(v_3) \in [k, \lambda - k]$ and $f(v_6) \in [2k, \lambda]$. Since $d(v_3, v_7) = 2$, $f(v_7) \in [0, \lambda - 2k] \cup [2k, \lambda]$. Since the length $[2k, \lambda]$ is less than j, $f(v_7) \in [0, \lambda - 2k]$. This implies that $f(v_7) < f(v_3)$. As the length of $[2k, \lambda]$ is less than j, we have $f(v_2)$, $f(v_4)$ and $f(v_5)$ are not in $[2k, \lambda]$. By considering the distance apart from v_6, we have $f(v_4), f(v_5) \in [0, \lambda - j]$ and $f(v_2) \in [0, \lambda - k]$. By $f(v_0), f(v_7) \in [0, \lambda - 2k]$, we have $f(v_1), f(v_2), f(v_5) \in [j, \lambda]$ and $f(v_4) \in [k, \lambda]$. As the length of $[k, \lambda - k]$ is less than j, we have $f(v_1)$, $f(v_2)$, $f(v_4)$ and $f(v_5)$ are not in this interval. Now we summarize the range of some labels: $f(v_1) \in [j, k) \cup (\lambda - k, \lambda]$; $f(v_2) \in [j, k)$; $f(v_3) \in [k, \lambda - k]$; $f(v_4) \in (\lambda - k, \lambda - j]$ and $f(v_5) \in [j, k) \cup (\lambda - k, \lambda - j]$. Since the length of $[j, k)$ is less than k, $f(v_5) \notin [j, k)$ and hence $f(v_5) \in (\lambda - k, \lambda - j] \subset (\lambda - k, \lambda]$. Also since the length of $(\lambda - k, \lambda]$ is less than k, $f(v_1) \notin (\lambda - k, \lambda]$. Hence $f(v_1) \in [j, k)$. Now we have $f(v_0) < f(v_1), f(v_2), f(v_3) < f(v_5)$. So $f(v_5) \ge 2j + k$. That is, $f(v_5) \in [2j + k, \lambda - j]$. Up to now we have $f(v_0), f(v_1), f(v_2), f(v_3), f(v_4)$ and $f(v_5)$ are at most $\lambda - j$. By Theorem 3.10, $\lambda - j \ge 5j$. Hence $\lambda \ge 6j$.

Case B. Suppose $f(v_3) > f(v_6)$. We have $f(v_6) \in [0, \lambda - k]$. Suppose $f(v_7) > f(v_3)$. Since $f(v_0) < f(v_3) < f(v_7)$, $f(v_0) \in [0, \lambda - 2k]$, $f(v_3) \in [k, \lambda - k]$ and $f(v_7) \in [2k, \lambda]$. Since the lengths of $[0, \lambda - 2k]$ and $[2k, \lambda]$ are less than j, $f(v_4)$ does not lie in these two intervals. This implies that $f(v_0) < f(v_4) < f(v_7)$. Now, $f(v_3), f(v_4) \in [k, \lambda - k]$ which is impossible. That means $f(v_7) < f(v_3)$ and $f(v_7) \in [0, \lambda - k]$. Since v_3, v_{10} are of distance two from v_6 and v_7 and the length of $[0, \lambda - k]$ is less than $j + k$, $f(v_3), f(v_{10}) \ge \max\{f(v_6), f(v_7)\} + k \ge j + k$ or $f(v_3), f(v_{10}) \le \min\{f(v_6), f(v_7)\} - k \le (\lambda - k - j) - k < 0$. So the last case is impossible. Hence $f(v_3), f(v_{10}) \in [j + k, \lambda]$. Hence $f(v_6), f(v_7) < f(v_3), f(v_{10})$. Similarly, we have $f(v_{11}) \in [0, \lambda - 2k] \cup [k, \lambda]$.

B-1. Suppose $f(v_{11}) \in [0, \lambda - 2k]$. Since the length of $[0, \lambda - 2k]$ is less than j, $f(v_7), f(v_8), f(v_9), f(v_{10})$ and $f(v_{12})$ are greater than $f(v_{11})$. Combining with $f(v_7) < f(v_3), f(v_{10})$, we have $f(v_7) \in [k, \lambda - k]$, $f(v_3), f(v_{10}) \in [2k, \lambda]$. Since $f(v_8)$ cannot lie in $[0, \lambda - 2k] \cup [2k, \lambda]$, $f(v_{11}) < f(v_8) < f(v_{10})$. Now we have $f(v_8) \in [k, \lambda - j]$. Since $f(v_8)$ cannot lie in $[k, \lambda - k]$, $f(v_8) > f(v_7)$ and hence $f(v_8) \in [j + k, \lambda - j]$. Comparing $f(v_4)$ with $f(v_7)$, we have $f(v_4) \in [0, \lambda - 2k] \cup [2k, \lambda]$. From the range of $f(v_3)$, we have $f(v_4) \in [0, \lambda - 2k]$. From the range of $f(v_4)$, we have $f(v_5), f(v_6) \ge j$. Hence $f(v_i) \ge j$ for $5 \le i \le 10$. By Theorem 3.10, $\lambda - j \ge 5j$. Hence $\lambda \ge 6j$.

B-2. Suppose $f(v_{11}) \in [k, \lambda]$. Then $f(v_8) \in [0, \lambda - k] \cup [2k, \lambda]$. Suppose $f(v_8) \in [2k, \lambda]$. Since the length of $[2k, \lambda]$ is less than j, $f(v_i) \in [0, \lambda - j]$, where $4 \le i \le 12$ and $i \ne 8$. Moreover, since $d(v_8, v_{11}) = 2$, $f(v_{11}) \in [k, \lambda - k]$. Since the length of $[k, \lambda - k]$ is less than j, $f(v_6), f(v_7), f(v_9)$ and $f(v_{12})$ are less than $f(v_{11})$. Moreover, $f(v_6), f(v_7) \in [0, \lambda - 2k]$ and $f(v_9), f(v_{12}) \in [0, \lambda - k - j]$. But it is impossible, since the length of $[0, \lambda - k - j]$ is less than k. Thus, $f(v_8) \in [0, \lambda - k]$.

a. When $f(v_5) < f(v_8)$. This implies that $f(v_5) \in [0, \lambda - 2k]$ and hence $f(v_8) \in [k, \lambda - k]$ and $f(v_6), f(v_7) \in [j, \lambda - k]$. Since the length of $[k, \lambda - k]$ is less than j, $f(v_8)$ must be less than $f(v_{11})$ and $f(v_8)$ must be greater than $f(v_6)$ and $f(v_7)$. Thus, $f(v_{11}) \in [2k, \lambda]$. Now the length of $[2k, \lambda]$ is less than j, $f(v_9)$ and $f(v_{10})$ are less than $f(v_{11})$. That means, $f(v_5), f(v_6), f(v_7), f(v_8), f(v_9), f(v_{10})$ lie in $[0, \lambda - j]$. By Theorem 3.10, we have $\lambda \ge 6j$.

b. When $f(v_5) > f(v_8)$. Hence $f(v_5) \in [k, \lambda]$.

b-1. Suppose $f(v_2) > f(v_5)$. Since $f(v_2) > f(v_5) > f(v_8)$, $f(v_8) \in [0, \lambda - 2k]$. Since the length of $[0, \lambda - 2k]$ is less than j, $f(v_4), f(v_5), f(v_6), f(v_7)$ are greater than j. Combining with the ranges of $f(v_2)$ and $f(v_3)$ we have $f(v_i) \in [j, \lambda]$ for $2 \le i \le 7$. By Theorem 3.10, we have $\lambda \ge 6j$.

b-2. Suppose $f(v_2) < f(v_5)$. Then $f(v_2) \in [0, \lambda - k]$. Since $f(v_6) \in [0, \lambda - k]$ and $d(v_2, v_6) = 2$, $f(v_2) \in [0, \lambda - 2k]$ or $f(v_2) \in [k, \lambda - k]$. When $f(v_2) \in [0, \lambda - 2k]$. We have $f(v_6) \in [k, \lambda - k]$. Since $f(v_3)$ and $f(v_{10})$ are greater than $f(v_6)$, $f(v_3), f(v_{10}) \in [2k, \lambda]$. Since the length of $[2k, \lambda]$ is less than j, $f(v_i) \in [0, \lambda - j]$ for $4 \le i \le 9$. By Theorem 3.10, we have $\lambda \ge 6j$. When $f(v_2) \in [k, \lambda - k]$. Now $f(v_6) \in [0, \lambda - 2k]$ and $f(v_5) \in [2k, \lambda]$. By considering the distances from v_9 to v_5 and v_6, we have $f(v_9) \in [k, \lambda - k]$. Since the lengths of $[0, \lambda - 2k]$ and $[k, \lambda - k]$ are less than j, then $f(v_7), f(v_8) \in [j, \lambda - j - k]$. It implies that $f(v_{12}) \in [2k, \lambda]$ or $f(v_{12}) \in [0, \lambda - j - 2k]$ by considering the distances from v_8 and v_9. But the last case is impossible as $\lambda < j + 2k$. Now $j \le f(v_i)$ for $7 \le i \le 12$. By Theorem 3.10, we have $\lambda \ge 6j$. □

According to Theorems 3.2–3.12, we can obtain following conclusion.

Corollary 3.13. *Let $n \ge 6$ and j, k be two positive numbers.*

1. *For $j \le k < 2j$,* $\lambda_{j,k}(P_n^2) = \begin{cases} 3j + k, & \text{if } 6 \le n \le 10, \\ \min\{2j + 2k, 5j\}, & \text{if } 11 \le n \le 15, \\ \min\{j + 3k, 5j\}, & \text{if } 16 \le n \le 20, \\ \min\{4k, 5j\}, & \text{if } n \ge 21. \end{cases}$
2. *For $2j \le k < 3j$,* $\lambda_{j,k}(P_n^2) = \begin{cases} 5j, & \text{if } n = 6, \\ 3j + k, & \text{if } 7 \le n \le 12, \\ \min\{j + 2k, 6j\}, & \text{if } n \ge 13. \end{cases}$
3. *For $k \ge 3j$,* $\lambda_{j,k}(P_n^2) = \begin{cases} 2j + k, & \text{if } n = 6, \\ 2k, & \text{if } n \ge 7. \end{cases}$

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