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# Coefficients of the characteristic polynomial of the (signless, normalized) Laplacian of a graph\*

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## Abstract

In this paper, we give a combinatorial expression for the fifth coefficient of the (signless) Laplacian characteristic polynomial of a graph. The first five normalized Laplacian coefficients are also given.

**Keywords:** (Signless, normalized) Laplacian matrix; (signless, normalized) Laplacian characteristic polynomial; coefficient; graph

**AMS classification:** 05C50; 15A18.

## 1 Introduction

Let  $G = (V, E)$  be a simple graph with vertex set  $V = \{v_1, \dots, v_n\}$  and edge set  $E$ . Let  $d_G(v_i)$  (or simply  $d(v_i)$  or  $d_i$ ) denote the degree of the vertex  $v_i \in V$  ( $i = 1, 2, \dots, n$ ), and  $D = D(G) = \text{diag}(d(v_1), d(v_2), \dots, d(v_n))$  be the diagonal matrix of vertex degrees. The *Laplacian matrix*, the *signless Laplacian matrix* and the *normalized Laplacian matrix* of  $G$  are defined by  $L(G) = D(G) - A(G)$ ,  $Q(G) = D(G) + A(G)$  and  $\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$  (with the convention that if the degree of  $v$  is 0 then  $d(v)^{-1/2} = 0$ ), respectively, where  $A(G)$  denotes the adjacency matrix of  $G$ .

Throughout this paper, Let the *Laplacian characteristic polynomial*, the *signless Laplacian characteristic polynomial* and the *normalized Laplacian characteristic polynomial* of  $G$  be

$$\Phi(L(G)) = \det(xI - L(G)) = \sum_{i=0}^n q_i(G)x^{n-i},$$

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$$\Phi(Q(G)) = \det(xI - Q(G)) = \sum_{i=0}^n p_i(G)x^{n-i},$$

and

$$\Phi(\mathcal{L}(G)) = \det(xI - \mathcal{L}(G)) = \sum_{i=0}^n r_i(G)x^{n-i},$$

respectively.

The search for isomorphism invariants has led to consideration of various algebraic properties of the (adjacency, signless Laplacian, normalized Laplacian) matrix of a graph. In particular, interest has focused on the coefficients of the characteristic polynomial of the (adjacency, signless Laplacian, normalized Laplacian) matrix.

Collatz and Sinogowitz [1] investigated the relationship between the coefficients of the characteristic polynomial of the adjacency matrix of a graph and certain subgraphs. In [5], a formula for the coefficients of the characteristic polynomial of the adjacency matrix of an arbitrary digraph was derived and it was shown that the coefficients of the polynomial of a tree count matchings.

In [6], Oliveira et al. gave the formulas for the first four coefficients of the Laplacian characteristic polynomial of a graph. In [2], Cvetković et al. gave the first three coefficients of the signless Laplacian characteristic polynomial of a graph. In [7], Wang et al. gave the fourth coefficient of the signless Laplacian characteristic polynomial of a graph. The coefficients of the (adjacency, signless Laplacian, normalized Laplacian) matrix can be used to distinguish non-isomorphic graphs in some class of graphs. By using the fifth Laplacian coefficient of trees, Lepović and Gutman [4] proved that no starlike trees are cospectral. The (signless) Laplacian coefficients were used to discuss the (signless) Laplacian spectral characterizations of two kind of graphs:  $\infty$ -graphs and 3-rose graphs [8,9]. In this paper, we give the fifth coefficient of the Laplacian characteristic polynomial and the signless Laplacian characteristic polynomial and the first five coefficients of the normalized Laplacian characteristic polynomial of a graph, respectively.

## 2 The Laplacian coefficient

Laplacian coefficients can be expressed in terms of subtree structures of  $G$  by the following result of Kelmans [3].

**Theorem 2.1** ([3]) *Let  $F$  be a spanning forest of  $G$  with components  $T_i, i = 1, 2, \dots, k$ , having  $n_i$  vertices each, and let  $\gamma(F) = \prod_{i=1}^k n_i$ . The Laplacian coefficient  $q_{n-k}$  of a graph  $G$  is given by*

$$(-1)^{n-k} q_{n-k} = \sum_{F \in \mathcal{F}_k} \gamma(F),$$

where  $\mathcal{F}_k$  is the set of all spanning forests of  $G$  with exactly  $k$  components.

Let  $A^k = (a_{ij}^{(k)})$ , where  $A = (a_{ij})$  is the adjacency matrix of a connected graph  $G$ . It is a well-known fact in graph theory that  $a_{ij}^{(k)}$  is the number of walks from  $v_i$  to  $v_j$  with length  $k$  of  $G$ . By using Theorem 2.1, it is easy to see that  $q_0(G) = 1$ ,  $q_1(G) = -2m$ , where  $m$  is the number of edges of  $G$ . Furthermore, Oliveira et al. [6] gave the third and the fourth Laplacian coefficients, as follows:

**Theorem 2.2** [6] Let  $G$  be a simple graph with  $n$  vertices and  $m$  edges and let  $d = (d_1, d_2, \dots, d_n)$  be its degree sequence. Then

$$q_2(G) = 2m^2 - m - \frac{1}{2} \sum_{i=1}^n d_i^2;$$

$$q_3(G) = \frac{1}{3}[-4m^3 + 6m^2 + 3(m-1) \sum_{i=1}^n d_i^2 - \sum_{i=1}^n d_i^3 + \text{tr}(A^3)],$$

where  $A$  is the adjacency matrix of  $G$ .

Note that  $\text{tr}(A^3) = 6n_3(G)$ , where  $n_3(G)$  is the number of  $C_3$ , a cycle with length 3, in  $G$ . Thus from Theorem 2.2, we also have

$$q_3(G) = \frac{1}{3}[-4m^3 + 6m^2 + (3m-1) \sum_{i=1}^n d_i^2 - \sum_{i=1}^n d_i^3 + 6n_3(G)]. \quad (2.1)$$

We define an  $H_2$ -spanning forest in  $G$  as a spanning graph with only one component isomorphic to  $P_3$  and  $(n-3)$  components isomorphic to  $K_1$ . Just as Oliveira et al. pointed that in order to find the third Laplacian coefficient, all one needs to do is to count the number of  $H_2$ -spanning forests in  $G$ . On the other hand, determining  $q_3(G)$  is not that easy [6]. In order to obtain the fifth Laplacian coefficient, we need the following:

**Lemma 2.3** [10] Let  $B = (b_{ij})$  be a matrix with characteristic polynomial

$$\Phi(B) = \det(xI - B) = x^n + \sum_{i=1}^n a_i x^{n-i}.$$

Let  $s_k = \text{tr}(B^k)$ . Then the coefficients of  $\Phi(B)$  satisfy the following:

$$a_1 = -s_1 \quad \text{and} \quad ka_k = -s_k - a_1 s_{k-1} - a_2 s_{k-2} - \dots - a_{k-1} s_1, \quad (k = 2, 3, \dots, n).$$

**Lemma 2.4** Let  $A = (a_{ij})$  and  $B = (b_{ij})$  be two  $n \times n$  symmetric real matrices and  $C = \text{diag}(c_1, c_2, \dots, c_n)$  be the real diagonal matrix. Then we have

$$\text{tr}(ACB) = \text{tr}(CAB).$$

**Proof.** By direct computation, it is easy to see that

$$\text{tr}(ACB) = \sum_{i=1}^n \sum_{k=1}^n a_{ik} c_k b_{ki} = \sum_{i=1}^n c_i \sum_{k=1}^n a_{ki} b_{ik}$$

and

$$\text{tr}(CAB) = \sum_{i=1}^n c_i \sum_{k=1}^n a_{ik} b_{ki}.$$

Note that  $A = A^T$  and  $B = B^T$ . The result follows.  $\square$

In the following, we will give a simpler proof of Theorem 2.2. Furthermore, we will give the fifth Laplacian coefficient.

**Theorem 2.5** Let  $G$  be a simple graph with  $n$  vertices and  $m$  edges and let  $d = (d_1, d_2, \dots, d_n)$  be its degree sequence. Then the fifth Laplacian coefficient

$$q_4(G) = -\frac{1}{4} \sum_{i=1}^n d_i^4 + \left(\frac{2m}{3} - 1\right) \sum_{i=1}^n d_i^3 + \frac{1}{8} \left(\sum_{i=1}^n d_i^2\right)^2 - \frac{1}{2} (2m^2 - 5m + 1) \sum_{i=1}^n d_i^2 - \sum_{v_i, v_j \in E(G)} d_i d_j \\ + 2 \sum_{i=1}^n d_i t_3(v_i) - 2n_4(G) - 4mn_3(G) + \frac{2}{3}m^4 - 2m^3 + \frac{1}{2}m^2 + \frac{1}{2}m, \quad (2.2)$$

where  $t_3(v_i)$  denotes the number of  $C_3$  of  $G$  through the vertex  $v_i$  and  $n_4(G)$  is the number of  $C_4$ , a cycle with length 4, in  $G$ .

**Proof.** Let  $s_k = \text{tr}(L^k(G))$  in Lemma 2.3. It is easy to see that  $s_1 = 2m$ . From Lemma 2.3, we have  $q_1(G) = -2m$ . By direct calculation, we have

$$(D - A)^2 = D^2 - DA - AD + A^2; \quad (2.3)$$

$$(D - A)^3 = D^3 - D^2A - DAD - AD^2 + DA^2 + ADA + A^2D - A^3; \quad (2.4)$$

$$(D - A)^4 = D^4 - D^3A - D^2AD - DAD^2 + D^2A^2 + DA^2D + DADA - DA^3 \\ - AD^3 + AD^2A + ADAD - ADA^2 + A^2D^2 - A^2DA - A^3D + A^4. \quad (2.5)$$

Since

$$\text{tr}(D^2) = \sum_{i=1}^n d_i^2, \text{tr}(DA) = \text{tr}(AD) = 0, \text{tr}(A^2) = \sum_{i=1}^n d_i = 2m,$$

from Eq. (2.3), we have  $s_2 = \sum_{i=1}^n d_i^2 + 2m$ . From Lemma 2.3, we have

$$q_2(G) = 2m^2 - m - \frac{1}{2} \sum_{i=1}^n d_i^2.$$

From Lemma 2.4 and by simple computation, we have

$$\text{tr}(D^3) = \sum_{i=1}^n d_i^3,$$

$$\text{tr}(D^2A) = \text{tr}(DAD) = \text{tr}(AD^2) = 0,$$

$$\text{tr}(DA^2) = \text{tr}(A^2D) = \text{tr}(ADA) = \sum_{i=1}^n d_i^2,$$

$$\text{tr}(A^3) = 6n_3(G).$$

From Eq. (2.4), we have  $s_3 = \sum_{i=1}^n d_i^3 + 3 \sum_{i=1}^n d_i^2 - 6n_3(G)$ . From Lemma 2.3, Eq. (2.1) holds.

Similarly, we have

$$\begin{aligned}
tr(D^4) &= \sum_{i=1}^n d_i^4, \\
tr(D^3A) &= tr(D^2AD) = tr(DAD^2) = tr(AD^3) = 0, \\
tr(D^2A^2) &= tr(DA^2D) = tr(AD^2A) = tr(A^2D^2) = \sum_{i=1}^n d_i^3, \\
tr(DADA) &= tr(ADAD) = 2 \sum_{v_i v_j \in E(G)} d_i d_j, \\
tr(DA^3) &= tr(ADA^2) = tr(A^2DA) = tr(A^3D) = 2 \sum_{v_i \in V(G)} d_i t_3(v_i), \\
tr(A^4) &= 8n_4(G) + 2 \sum_{i=1}^n d_i^2 - 2m.
\end{aligned}$$

From Eq. (2.5), we have

$$s_4 = \sum_{i=1}^n d_i^4 + 4 \sum_{i=1}^n d_i^3 + 4 \sum_{v_i v_j \in E(G)} d_i d_j - 8 \sum_{v_i \in V(G)} d_i t_3(v_i) + 8n_4(G) + 2 \sum_{i=1}^n d_i^2 - 2m.$$

From Lemma 2.3, we find that Eq. (2.2) holds.  $\square$

### 3 The signless Laplacian coefficient

A  $TU$ -subgraph of a graph  $G$  is a spanning subgraph whose components are trees or odd-unicyclic graphs. If  $H$  is a  $TU$ -subgraph of  $G$  consisting of  $c$  unicyclic graphs and trees  $T_1, T_2, \dots, T_s$ , then we define the quantity  $W(H) = 4^c \prod_{i=1}^s (1 + |E(T_i)|)$  as the weight of  $H$ . In [2], Cvetkovic et al. expressed the signless Laplacian coefficients by the weights of  $TU$ -graphs of  $G$ , and they proved the following result.

**Theorem 3.1** ([2]) *Let  $\aleph_i$  be the set of all  $TU$ -subgraph of  $G$  with  $i$  edges. Then, for each  $1 \leq i \leq n$ ,*

$$p_i(G) = \sum_{H \in \aleph_i} (-1)^i W(H).$$

By using the above result, Cvetković et al. and Wang et al. gave the first three and the fourth signless Laplacian coefficients of a graph, respectively. Their results are as follows.

**Theorem 3.2** [2] *Let  $G$  be a simple graph with  $n$  vertices and  $m$  edges and let  $d = (d_1, d_2, \dots, d_n)$  be its degree sequence. Then*

$$p_0(G) = 1, \quad p_1(G) = -2m, \quad p_2(G) = 2m^2 - m - \frac{1}{2} \sum_{i=1}^n d_i^2;$$

**Theorem 3.3** [7] *Let  $G$  be a simple graph with  $n$  vertices and  $m$  edges and let  $d = (d_1, d_2, \dots, d_n)$  be its degree sequence. Then*

$$p_3(G) = \frac{1}{3}[-4m^3 + 6m^2 + 3(m-1) \sum_{i=1}^n d_i^2 - \sum_{i=1}^n d_i^3 - 6n_3(G)].$$

In the following, we will give a simpler proof of Theorems 3.2 and 3.3. Furthermore, we will give the fifth signless Laplacian coefficient.

**Theorem 3.4** *Let  $G$  be a simple graph with  $n$  vertices and  $m$  edges and let  $d = (d_1, d_2, \dots, d_n)$  be its degree sequence. Then the fifth signless Laplacian coefficient*

$$\begin{aligned} p_4(G) = & -\frac{1}{4} \sum_{i=1}^n d_i^4 + \left(\frac{2m}{3} - 1\right) \sum_{i=1}^n d_i^3 + \frac{1}{8} \left(\sum_{i=1}^n d_i^2\right)^2 - \frac{1}{2} (2m^2 - 5m + 1) \sum_{i=1}^n d_i^2 - \sum_{v_i v_j \in E(G)} d_i d_j \\ & - 2 \sum_{i=1}^n d_i t_3(v_i) - 2n_4(G) + 4mn_3(G) + \frac{2}{3}m^4 - 2m^3 + \frac{1}{2}m^2 + \frac{1}{2}m. \end{aligned} \quad (3.6)$$

**Proof.** Let  $s_k = \text{tr}(Q^k(G))$  in Lemma 2.3. By direct calculation, we have

$$\begin{aligned} (D + A)^2 &= D^2 + DA + AD + A^2; \\ (D + A)^3 &= D^3 + D^2A + DAD + DA^2 + AD^2 + ADA + A^2D + A^3; \\ (D + A)^4 &= D^4 + D^3A + D^2AD + DAD^2 + D^2A^2 + DADA + DA^2D + DA^3 \\ &\quad + AD^3 + AD^2A + ADAD + ADA^2 + A^2D^2 + A^2DA + A^3D + A^4. \end{aligned}$$

From the proof of Theorem 2.5, we have  $s_1 = 2m$ ,  $s_2 = \sum_{i=1}^n d_i^2 + 2m$ ,  $s_3 = \sum_{i=1}^n d_i^3 + 3 \sum_{i=1}^n d_i^2 + 6n_3(G)$ ,  $s_4 = \sum_{i=1}^n d_i^4 + 4 \sum_{i=1}^n d_i^3 + 4 \sum_{v_i v_j \in E(G)} d_i d_j + 8 \sum_{v_i \in V(G)} d_i t_3(v_i) + 8n_4(G) + 2 \sum_{i=1}^n d_i^2 - 2m$ .

From Lemma 2.3, we have  $p_1 = -2m$ ,  $p_2(G) = 2m^2 - m - \frac{1}{2} \sum_{i=1}^n d_i^2$ ,  $p_3(G) = \frac{1}{3}[-4m^3 + 6m^2 + 3(m-1) \sum_{i=1}^n d_i^2 - \sum_{i=1}^n d_i^3 - 6n_3(G)]$  and Eq. (3.6).

## 4 The normalized Laplacian coefficient

Suppose that  $Z = v_1 v_2 \dots v_l v_1$  is a cycle of  $G$ . Let  $d_G(Z) = \prod_{i=1}^l d(v_i)$  or simply  $d_Z = \prod_{i=1}^l d(v_i)$  if  $G$  is clear from the context. In the following, we will give the first five normalized Laplacian coefficients.

**Theorem 4.1** *Let  $G$  be a simple graph with  $n$  vertices and  $m$  edges and let  $d = (d_1, d_2, \dots, d_n)$  be its degree sequence. Then the first five normalized Laplacian coefficients are as follows*

$$\begin{aligned} r_0(G) &= 1, \\ r_1(G) &= -n, \\ r_2(G) &= \frac{n(n-1)}{2} - \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j}, \\ r_3(G) &= \frac{n(n-1)(2-n)}{6} + (n-2) \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j} + 2 \sum_{Z \in \Theta_3} \frac{1}{d_Z}, \\ r_4(G) &= \frac{n(n-1)(n-2)(n-3)}{24} - \frac{(n-2)(n-3)}{2} \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j} - (2n-6) \sum_{Z \in \Theta_3} \frac{1}{d_Z} \\ &\quad - \frac{1}{2} \sum_{i=1}^n \frac{1}{d_i^2} \left( \sum_{v_i v_j \in E(G)} \frac{1}{d_j} \right)^2 - 2 \sum_{Z \in \Theta_4} \frac{1}{d_Z} + \frac{1}{2} \sum_{v_i v_j \in E(G)} \frac{1}{d_i^2 d_j^2} + \frac{1}{2} \left( \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j} \right)^2, \end{aligned} \quad (4.7)$$

where  $\Theta_3$  and  $\Theta_4$  are the sets of 3-cycles and 4-cycles in  $G$ , respectively.

**Proof.** It is obvious that  $r_0 = 1$ . Let  $s_k = \text{tr}(\mathcal{L}^k(G))$  in Lemma 2.3. By direct calculation, we have

$$(D^{-1/2}LD^{-1/2})^2 = D^{-1/2}(D - 2A + AD^{-1}A)D^{-1/2}; \quad (4.8)$$

$$(D^{-1/2}LD^{-1/2})^3 = D^{-1/2}(D - 3A + 3AD^{-1}A - AD^{-1}AD^{-1}A)D^{-1/2}; \quad (4.9)$$

$$(D^{-1/2}L(G)D^{-1/2})^4 = D^{-1/2}(D - 4A + 6AD^{-1}A - 4AD^{-1}AD^{-1}A + AD^{-1}AD^{-1}AD^{-1}A)D^{-1/2}. \quad (4.10)$$

Since

$$\begin{aligned} \text{tr}(D^{-1/2}DD^{-1/2}) &= n, \\ \text{tr}(D^{-1/2}AD^{-1/2}) &= 0, \\ \text{tr}(D^{-1/2}AD^{-1}AD^{-1/2}) &= 2 \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j}, \end{aligned}$$

from Eq. (4.8), we have

$$s_1 = n, s_2 = n + 2 \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j}.$$

From Lemma 2.3, we have

$$r_1 = -n, r_2(G) = \frac{n(n-1)}{2} - \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j}.$$

Since

$$\text{tr}(D^{-1/2}AD^{-1}AD^{-1}AD^{-1/2}) = \sum_{i=1}^n \frac{1}{d_i} \sum_{j=1}^n \left( \sum_{k=1}^n \frac{a_{ik}a_{kj}}{d_k} \right) \frac{a_{ji}}{d_j} = 6 \sum_{Z \in \Theta_3} \frac{1}{d_Z},$$

from Eq. (4.9), we have

$$s_3 = n + 6 \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j} - 6 \sum_{Z \in \Theta_3} \frac{1}{d_Z}.$$

From Lemma 2.3, we have

$$r_3(G) = \frac{n(n-1)(2-n)}{6} + (n-2) \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j} + 2 \sum_{Z \in \Theta_3} \frac{1}{d_Z}.$$

Since

$$\begin{aligned} &\text{tr}(D^{-1/2}AD^{-1}AD^{-1}AD^{-1}AD^{-1/2}) \\ &= \sum_{i=1}^n \frac{1}{d_i} \sum_{j=1}^n \frac{1}{d_j} \left( \sum_{k=1}^n \frac{a_{ik}a_{kj}}{d_k} \right)^2 \\ &= 2 \sum_{i=1}^n \frac{1}{d_i^2} \left( \sum_{v_i v_j \in E(G)} \frac{1}{d_j} \right)^2 + 8 \sum_{Z \in \Theta_4} \frac{1}{d_Z} - 2 \sum_{v_i v_j \in E(G)} \frac{1}{d_i^2 d_j^2}, \end{aligned}$$

from Eq. (4.10), we have

$$s_4 = n + 12 \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j} - 24 \sum_{Z \in \Theta_3} \frac{1}{d_Z} + 2 \sum_{i=1}^n \frac{1}{d_i^2} \left( \sum_{v_i v_j \in E(G)} \frac{1}{d_j} \right)^2 + 8 \sum_{Z \in \Theta_4} \frac{1}{d_Z} - 2 \sum_{v_i v_j \in E(G)} \frac{1}{d_i^2 d_j^2}.$$

From Lemma 2.3, we have Eq. (4.7) holds.  $\square$



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