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Ka Chun Cheung

*Hong Kong Baptist University*, kccheung@astri.org

Leevan Ling

*Hong Kong Baptist University*, lling@hkbu.edu.hk

Robert Schaback

*Georg-August-Universität*, schaback@math.uni-gottingen.de

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# $H^2$ -CONVERGENCE OF LEAST-SQUARES KERNEL COLLOCATION METHODS

KA CHUN CHEUNG\*, LEEVAN LING\*<sup>†</sup>, AND ROBERT SCHABACK<sup>‡</sup>

**Abstract.** The strong-form asymmetric kernel-based collocation method, commonly referred to as the Kansa method, is easy to implement and hence is widely used for solving engineering problems and partial differential equations despite the lack of theoretical support. The simple least-squares (LS) formulation, on the other hand, makes the study of its solvability and convergence rather nontrivial. In this paper, we focus on general second order linear elliptic differential equations in  $\Omega \subset \mathbb{R}^d$  under Dirichlet boundary conditions. With kernels that reproduce  $H^m(\Omega)$  and some smoothness assumptions on the solution, we provide denseness conditions for a constrained least-squares method and a class of weighted least-squares algorithms to be convergent. Theoretically, we identify some  $H^2(\Omega)$  convergent LS formulations that have an optimal error behavior like  $h^{m-2}$ . We also demonstrate the effects of various collocation settings on the respective convergence rates, as well as how these formulations perform with high order kernels and when coupled with the stable evaluation technique for the Gaussian kernel.

**Key words.** Meshfree method, Kansa method, radial basis function, overdetermined collocation.

**AMS subject classifications.** 65D15, 65N35, 41A63.

**1. Introduction.** Mathematical models or differential equations are meaningful only if they can somehow mirror the overly complicated real world. Similarly, numerical methods are useful only if they can produce approximations guaranteed to converge to the outcome that the mathematical model predicts. It could take tens of years for some good numerical strategies to mature and become a well-established class of numerical methods with a complete and rigid theoretical framework. Take the finite element method as an example. It waited for a quarter of a century to get its rigorous mathematical foundation. This paper aims to continue our theoretical contributions to the unsymmetric radial basis function collocation method, which is also known as the Kansa method in the community and we shall use this name throughout this paper for brevity.

To quickly overview the development of the Kansa method and its connection to the radial basis function (RBF) scattered data interpolation problem, let us look at some of its cornerstones [5, 7, 40]. An RBF is a smooth scalar function  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ , which usually is induced from a kernel function  $\Phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  in today's applications, such that the interpolant of an interpolation problem is given as a linear combination

$$u = \sum_{j=1}^{n_Z} \lambda_j \phi(\|\cdot - z_j\|_2) = \sum_{j=1}^{n_Z} \lambda_j \Phi(\cdot, z_j), \quad (1.1)$$

of shifted RBFs in which the set  $Z = \{z_1, \dots, z_{n_Z}\}$  contains *trial centers* that specify the shifts of the kernel function in the expansion. Dealing with scaling has been another huge topic in Kansa methods [12, 19, 38] for a decade, but we will ignore this point for the sake of simplicity.

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\*Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong.

<sup>†</sup>Correspondence to L. Ling (E-mail: [lling@hkbu.edu.hk](mailto:lling@hkbu.edu.hk))

<sup>‡</sup>Institut für Numerische und Angewandte Mathematik, Georg-August-Universität Göttingen, Germany.

Impressed by the meshfree nature, simplicity to program, dimension independence, and arbitrarily high convergence rates interpolations, E.J. Kansa [17, 18] proposed to modify the RBF interpolation method to solve partial differential equations (PDEs) in the early 90s. Using the same RBF expansion (1.1), Kansa imposed strong-form collocation conditions instead of interpolation conditions for identifying the unknown coefficients. Consider a PDE given by  $\mathcal{L}u = f$  in  $\Omega$  and  $\mathcal{B}u = g$  on  $\Gamma = \partial\Omega$ . The Kansa method collocates the PDE at the trial centers  $Z$  to yield exactly  $n_Z$  conditions:

$$\begin{aligned}\mathcal{L}u(z_i) &= \sum \lambda_j \mathcal{L}\phi(\|z_i - z_j\|), & \text{for } z_i \in Z \cap \Omega, \\ \mathcal{B}u(z_i) &= \sum \lambda_j \mathcal{B}\phi(\|z_i - z_j\|), & \text{for } z_i \in Z \cap \Gamma,\end{aligned}\tag{1.2}$$

for identifying the unknown  $\lambda_j$  or equivalently, a numerical approximation to  $u$  from the *trial space*

$$\mathcal{U}_Z = \mathcal{U}_{Z,\Omega,\Phi} := \text{span}\{\Phi(\cdot - z_j) : z_j \in Z\}.\tag{1.3}$$

This approach requires no re-formulation of the PDE and no triangularization. As long as one knows how to program for an interpolation problem, it only takes minutes to understand and code up something for the Kansa method. Since invented, the Kansa method has been widely used in vast numbers of applications in physics and engineering [3, 20, 24, 33].

Since the differential and boundary operators of a PDE are independently applied to yield different rows of the final linear system of equations, it is easy to see why any Kansa system matrix is unsymmetric. While this has some implications for the choice of linear solvers, the unsymmetric matrix places the Kansa method far away from the approximation theories from which RBFs interpolation theories were built. Though the technique introduced by Kansa is very successful in a large variety of applications in Engineerings and Science, there were no proven results about it for over 10 years. After many unsuccessful attempts to establish such a foundation, Hon and Schaback [14] showed in 2001 that there are extremely rare cases where the original approach can fail because the underlying linear system can be singular. This puts an end to all attempts to prove stability of the Kansa method in general. One workaround is to apply *symmetric collocation* [4, 9] that mimics scattered Hermite interpolation. While the Kansa trial space basis in (1.1) is independent of the collocation, the symmetric method takes a basis that is itself dependent on the collocation. This approach yields positive definite symmetric system matrices at the expense of higher smoothness requirements and less stability. On the positive side, symmetric collocation can be proven [36] to be error-optimal, because it is a pointwise optimal recovery of the solution from discrete input data.

The situation for the Kansa method remained the same until 2006, when we provided the first solvability results for an extended Kansa method. In order to ensure solvability, *overtesting* is applied. Keeping the trial space (1.3) based on a set  $Z$  of *trial centers*, the standard Kansa system (1.2) is modified by taking another, but usually larger discrete set  $X$  of collocation points that is sufficiently fine relative to the set  $Z$  of trial centers. Readers are referred to the original articles [28] and an extension [35] to the corresponding weak problems for details. In 2008, we had a partial answer to the convergence of an overdetermined Kansa formulation [29]. Our analysis was carried out based on the continuous and discrete maximum norms. We showed that the  $\ell^\infty$ -minimizer of a residual functional converges to the exact solution at the optimal speed, i.e. with the same convergence rate as the interpolant converges

to the exact solution. From then on, we attempted to extend the theories to the least-squares (LS) minimizer [21] and numerically verified in extended precision arithmetic that the LS-minimizer also converges at the optimal rate [23]. Recently, in [37], we gave an  $L_\infty$  convergence rate of  $m - 2 - d/2 = m - 3$  for an overdetermined Kansa method in  $H^m$  for  $m > 3$ . In this study, we continue to work on the overdetermined Kansa method and concentrate on the popular LS solution. In Section 2, we will provide all the necessary assumptions and prove error estimates for a constrained least-squares (CLS) and a class of *weighted least-squares* (WLS) formulations. The convergence for the CLS formulation will then be given in Section 3. In Section 4 and 5, the theory for WLS formulations in two trial spaces will be given. Lastly, we will numerically verify the accuracy and convergence rates of some proven convergent formulations in Section 6.

**2. Notations, assumptions and main theorems.** Throughout the paper, the notation  $C$  will be reserved for generic constants whose subscripts indicate the dependencies of the constant.

We consider a general second order elliptic differential equation in some bounded domain  $\Omega \subset \mathbb{R}^d$  subject to the Dirichlet boundary condition on  $\Gamma = \partial\Omega$ :

$$\begin{aligned} \mathcal{L}u &= f & \text{in } \Omega, \\ u &= g & \text{on } \Gamma, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \mathcal{L}u &:= \sum_{i,j=1}^d \frac{\partial}{\partial x^j} \left( a^{ij}(x) \frac{\partial}{\partial x^i} u(x) \right) + \sum_{j=1}^d \frac{\partial}{\partial x^j} (b^j(x)u(x)) \\ &+ \sum_{i=1}^d c^i(x) \frac{\partial}{\partial x^i} u(x) + d(x)u(x) = f(x). \end{aligned} \tag{2.2}$$

The Sobolev regularity of the true solution will be denoted by  $m$ , and we will work with standard Hilbert spaces  $H^k(\Omega)$  and  $H^{k-1/2}(\Gamma)$  with norms  $\|u\|_{k,\Omega}$  and  $\|u\|_{k-1/2,\Gamma}$ , respectively, for  $k \leq m$ .

**ASSUMPTION 2.1** (Smoothness of domain and solution). *We assume that the bounded domain  $\Omega$  has a piecewise  $C^m$ -boundary  $\Gamma$  so that  $\Omega$  is Lipschitz continuous and satisfies an interior cone condition. Also, we assume that the functions  $f$  and  $g$  are smooth enough to admit a classical solution  $u^* \in H^m(\Omega)$ .  $\square$*

Now the trace theorem [41] can be applied and we can define a trace operator:

$$\mathcal{T} : H^m(\Omega) \rightarrow H^{m-1/2}(\Gamma) \text{ such that } \mathcal{T}u = u|_\Gamma \text{ for all } u \in C^m(\bar{\Omega}),$$

for  $m > 1/2$ , with a continuous right-inverse linear extension operator  $\mathcal{E}$  such that

$$\mathcal{T} \circ \mathcal{E}g = g \text{ for all } g \in H^{m-1/2}(\Gamma).$$

The smoothness assumption also allows a partition of unity of the boundary, each part of which can be mapped to the unit ball in  $\mathbb{R}^{d-1}$  by a  $C^m$ -diffeomorphism. This allows us to define Sobolev norms on  $\Gamma$  and apply some Sobolev inequalities (i.e., kernel independent ones).

Let  $\chi$  be any discrete set of  $n_\chi$  points in  $\Omega$ . For any  $u \in H^m(\Omega)$ , we define discrete norms on  $\chi$  by

$$\|u\|_\chi = \|u\|_{0,\chi} = \|u\|_{\ell_2(\chi)}, \quad \|u\|_{k,\chi} := \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_\chi^2 \right)^{1/2}, \quad 0 \leq k < m - d/2,$$

where  $\alpha$  is some multi-index and  $D^\alpha u \in C(\Omega)$  are weak derivatives of  $u$ . The same notations will also be used to denote discrete norms on boundary for any discrete set  $\chi \subset \Gamma$ .

**ASSUMPTION 2.2** (Differential operator). *Assume that  $\mathcal{L}$  as in (2.2) is a strongly elliptic operator with coefficients belonging to  $W_\infty^m(\Omega)$ .*  $\square$   
Then, by results in [11],  $\mathcal{L}$  is a bounded operator from  $H^m(\Omega)$  to  $H^{m-2}(\Omega)$  with

$$\|\mathcal{L}u\|_{m-k-2,\Omega} \leq C_{\Omega,\mathcal{L}} \|u\|_{m-k,\Omega}, \quad 0 \leq k \leq m-2, \quad k \in \mathbb{N}, \quad (2.3)$$

for all  $u \in H^m(\Omega)$ . Moreover, the following boundary regularity estimate [16] holds:

$$\|u\|_{k+2,\Omega} \leq C_{\Omega,\mathcal{L},k} (\|\mathcal{L}u\|_{k,\Omega} + \|u\|_{k+1+1/2,\Gamma}), \quad 0 \leq k \leq m-2, \quad (2.4)$$

for all  $u \in H^m(\Omega)$  with  $C_{\Omega,\mathcal{L},k}$  depending on  $\Omega$ , the ellipticity constant of  $\mathcal{L}$ , and  $k \geq 0$ .

**ASSUMPTION 2.3** (Kernel). *Assume  $\Phi$  is a reproducing kernel of  $H^m(\Omega)$  for some integer  $m \geq 2 + \lceil \frac{1}{2}(d+1) \rceil$ . More precisely, we use a symmetric positive definite kernel  $\Phi$  on  $\mathbb{R}^d$  with smoothness  $m$  that satisfies*

$$c_{\Phi_m} (1 + \|\omega\|_2^2)^{-m} \leq \widehat{\Phi}_m(\omega) \leq C_{\Phi_m} (1 + \|\omega\|_2^2)^{-m} \quad \text{for all } \omega \in \mathbb{R}^d, \quad (2.5)$$

for two constants  $0 < c_{\Phi_m} \leq C_{\Phi_m}$ .  $\square$

For any  $m > d/2$ , its native space  $\mathcal{N}_{\Omega,\Phi}$  on  $\mathbb{R}^d$  [2, 40] is norm-equivalent to  $H^m(\mathbb{R}^d)$ . This includes the standard *Whittle-Matérn-Sobolev* kernel with exact Fourier transform  $(1 + \|\omega\|_2^2)^{-m}$  that takes the form

$$\Phi(x) := \|x\|_2^{m-d/2} \mathcal{K}_{m-d/2}(\|x\|_2) \quad \text{for all } x \in \mathbb{R}^d,$$

where  $\mathcal{K}_\nu$  is the Bessel functions of the second kind. The compactly supported piecewise polynomial Wendland functions [39] are another examples of kernels satisfying (2.5).

**ASSUMPTION 2.4** (Trial space). *Let  $Z = \{z_1, \dots, z_{n_Z}\}$  be a discrete set of trial centers in  $\Omega$ . In analogy to (1.3), but now with translation-invariance, we define the finite-dimensional trial space  $\mathcal{U}_Z$  as*

$$\mathcal{U}_Z = \mathcal{U}_{Z,\Omega,\Phi} := \text{span}\{\Phi(\cdot - z_j) : z_j \in Z\} \subset \mathcal{N}_{\Omega,\Phi}. \quad \square$$

For describing the denseness of  $Z \subset \Omega$ , its *fill distance* for fixed  $\Omega$  and *separation distance* are defined as

$$h_Z := \sup_{z \in \Omega} \min_{z \in Z} \|z - \zeta\|_{\ell_2(\mathbb{R}^d)} \quad \text{and} \quad q_Z := \frac{1}{2} \min_{\substack{z_i, z_j \in Z \\ z_i \neq z_j}} \|z_i - z_j\|_{\ell_2(\mathbb{R}^d)},$$

respectively, and the quantity  $h_Z/q_Z =: \rho_Z$  is commonly referred as the *mesh ratio* of  $Z$ . For any  $u$  in the native space  $\mathcal{N}_{\Omega, \Phi}$  of  $\Phi$ , we denote  $I_Z u$  to be the interpolant of  $u$  on  $Z$  from the trial space  $\mathcal{U}_Z$ .

**ASSUMPTION 2.5** (Collocation points). *Let  $X = \{x_1, \dots, x_{n_X}\}$  be a discrete set of PDE collocation points in  $\Omega$  and  $Y = \{y_1, \dots, y_{n_Y}\}$  be a set of boundary collocation points on  $\Gamma$ . We assume the set  $Z$  of discrete trial centers to be sufficiently dense with respect to  $\Omega$ ,  $\Phi$ , and  $\mathcal{L}$  but independent of the solution, and the sets of points  $X$  and  $Z$  to be asymptotically quasi-uniform. That is, there exist constants  $\gamma_\chi > 1$  such that*

$$\gamma_\chi^{-1} q_\chi \leq h_\chi \leq \gamma_\chi q_\chi \quad \text{for } \chi \in \{X, Z\}. \quad (2.6)$$

*Note that the sets  $X$  and  $Y$  of collocation points together have to be as dense as the trial centers in  $Z$  to ensure stability. This paper will provide rigid sufficient conditions for this.  $\square$*

Imposing strong testing on (2.1) at collocation points in  $X$  and  $Y$  yields  $n_X + n_Y > n_Z$  conditions, from which one can hopefully identify a numerical approximation from some trial spaces. The following theorems summarize our convergence results for three possible least-squares alternatives. The first concerns the case where we enlarge the set  $Z$  of trial points by adding the set  $Y$  of boundary collocation points to it. Then, we can keep the numerical solution to be exact on  $Y$ , and we add this as a constraint.

**THEOREM 2.6** (Constrained least squares (CLS)). *Suppose the Assumptions 2.1 to 2.5 hold. Let  $u^* \in H^m(\Omega)$  denote the exact solution of the elliptic PDE (2.1). In addition, the relative fill distances  $h_X/h_{Z \cup Y}$  and  $h_Z/h_{Z \cup Y}$  are sufficiently small and satisfy condition (3.3). Let  $u_{X,Y}^{CLS} \in \mathcal{U}_{Z \cup Y}$  be the constrained least-squares solution defined as*

$$u_{X,Y}^{CLS} := \arg \inf_{u \in \mathcal{U}_{Z \cup Y}} \|\mathcal{L}u - f\|_X^2 \quad \text{subject to } u|_Y = g|_Y. \quad (2.7)$$

*Then the error estimates*

$$\|u_{X,Y}^{CLS} - u^*\|_{2,\Omega} \leq C_{\Omega, \Phi, \mathcal{L}, \gamma_X} h_{Z \cup Y}^{m-d/2-2} \|u^*\|_{m,\Omega} \quad \text{for } m \geq 2 + \lceil \frac{d+1}{2} \rceil,$$

*and*

$$\|u_{X,Y}^{CLS} - u^*\|_{2,\Omega} \leq C_{\Omega, \Phi, \mathcal{L}, \gamma_X} h_{Z \cup Y}^{m-2} \|u^*\|_{m,\Omega} \quad \text{for } m > 3 + \frac{d}{2},$$

*hold for some constant  $C_{\Omega, \Phi, \mathcal{L}, \gamma_X}$  that depends only on  $\Omega$ ,  $\Phi$ ,  $\mathcal{L}$ , and the uniformity constant  $\gamma_X$  of  $X$ .*

The next case does not require exactness on  $Y$  but still keeps  $Z \cup Y$  as the set of trial centers.

**THEOREM 2.7** (Weighted least squares (WLS)). *Suppose all the assumptions in Theorem 2.6 hold. Let  $u_{X,Y,Z \cup Y}^{WLS,\theta} \in \mathcal{U}_{Z \cup Y}$  be the weighted least-squares solution defined as*

$$u_{X,Y,Z \cup Y}^{WLS,\theta} := \arg \inf_{u \in \mathcal{U}_{Z \cup Y}} \|\mathcal{L}u - f\|_X^2 + \left( \frac{h_Y}{h_X} \right)^{d\theta/2} h_Y^{-2\theta} \|u - g\|_Y^2 \quad \text{for } \theta \geq 0. \quad (2.8)$$

Then the error estimate

$$\|u_{X,Y,Z \cup Y}^{WLS,\theta} - u^*\|_{2,\Omega} \leq C_{\Omega,\Phi,\mathcal{L},\gamma_X} \left( h^{d/2} + h_X^{\frac{(\theta-2)d}{4}} h_Y^{\frac{(\theta-2)(d-4)}{4}} \right) h_{Z \cup Y}^{m-d/2-2} \|u^*\|_{m,\Omega}$$

for  $m \geq 2 + \lceil \frac{d+1}{2} \rceil$ , and

$$\|u_{X,Y,Z \cup Y}^{WLS,\theta} - u^*\|_{2,\Omega} \leq C_{\Omega,\Phi,\mathcal{L},\gamma_X} \left( 1 + h_X^{\frac{(\theta-2)d}{4}} h_Y^{\frac{(\theta-2)(d-4)}{4}} \right) h_{Z \cup Y}^{m-2} \|u^*\|_{m,\Omega}$$

for  $m > 3 + \frac{d}{2}$ , hold for  $h_X \leq h_Y < 1$ ,  $0 \leq \theta \leq 2$  and some constant  $C_{\Omega,\Phi,\mathcal{L}}$  that depends only on  $\Omega$ ,  $\Phi$ , and  $\mathcal{L}$ . For  $2 \leq \theta \leq \infty$ , the estimates in Theorem 2.6 remain valid.

Finally, we go back to the case where  $Z$  is the set of trial nodes, independent of  $X$  and  $Y$ .

**THEOREM 2.8** (WLS in a smaller trial space). *Suppose the trial space of the weighted least-squares approximation in Theorem 2.7 is restricted to  $u_{X,Y,Z}^{WLS,\theta} \in \mathcal{U}_Z$  instead of  $\mathcal{U}_{Z \cup Y}$ . Moreover, the relative fill distances  $h_X/h_{Z \cup Y}$  and  $h_Z/h_{Z \cup Y}$  are sufficiently small and satisfy condition (3.4) instead of (3.3). Further assume that the sets  $Y$  are asymptotically quasi-uniform with constant  $\gamma_Y > 1$  as in (2.6) and  $h_Y \leq h_Z$ . Then the error estimates*

$$\|u_{X,Y,Z}^{WLS,\theta} - u^*\|_{2,\Omega} \leq C_{\Omega,\Phi,\mathcal{L},\vec{\gamma}} \left( h^{d/2} + h_X^{\frac{(\theta-2)d}{4}} h_Y^{\frac{(\theta-2)(d-4)}{4}} + h_Y^{-3/2} h_Z^2 \right) h_Z^{m-d/2-2} \|u^*\|_{m,\Omega}$$

for  $m \geq 2 + \lceil \frac{d+1}{2} \rceil$ , and

$$\|u_{X,Y,Z}^{WLS,\theta} - u^*\|_{2,\Omega} \leq C_{\Omega,\Phi,\mathcal{L},\vec{\gamma}} \left( 1 + h_X^{\frac{(\theta-2)d}{4}} h_Y^{\frac{(\theta-2)(d-4)}{4}} + h_Y^{-2} h_Z^2 \right) h_Z^{m-2} \|u^*\|_{m,\Omega}$$

for  $m > 3 + d/2$ , hold for  $0 \leq \theta \leq 2$  and some constant  $C_{\Omega,\Phi,\mathcal{L},\vec{\gamma}}$  that depends only on  $\Omega$ ,  $\Phi$ ,  $\mathcal{L}$ , and uniformity constants  $\vec{\gamma} = [\gamma_X, \gamma_Y, \gamma_Z]$ .

**3. Optimal convergence rates for CLS.** We first prove some necessary inequalities essential to our proofs.

**LEMMA 3.1** (Sampling Inequality of fractional order). *Suppose  $\Omega \subset \mathbb{R}^d$  is a bounded Lipschitz domain with a piecewise  $C^m$ -boundary. Then there is a constant  $C_{\Omega,m,s}$  depending only on  $\Omega$ ,  $m$  and  $s$  such that the followings hold:*

$$\|u\|_{s,\Omega} \leq C_{\Omega,m,s} \left( h_X^{m-s} \|u\|_{m,\Omega} + h_X^{d/2-s} \|u\|_X \right) \quad \text{for } 0 \leq s \leq m,$$

and

$$\|u\|_{s-1/2,\Gamma} \leq C_{\Omega,m,s} \left( h_Y^{m-s} \|u\|_{m,\Omega} + h_Y^{d/2-s} \|u\|_Y \right) \quad \text{for } 1/2 \leq s \leq m,$$

for any  $u \in H^m(\Omega)$  with  $m > d/2$  and any discrete sets  $X \subset \Omega$  and  $Y \subset \Gamma$  with sufficiently small mesh norm  $h_X$  and  $h_Y$ .

**PROOF.** The interior sampling inequality for  $X \subset \Omega$ , which only requires  $\Omega$  be a bounded Lipschitz domain, is a special case of a sampling inequality in [1]. Applying the interior sampling inequality to the union of unit balls in  $\mathbb{R}^{d-1}$ , which are images of the partition of unity of  $\Gamma$  under the  $C^m$ -diffeomorphism in Assumption 2.1, yields

$$\|u\|_{s-1/2,\Gamma} \leq C \left( h_Y^{(m-1/2)-(s-1/2)} \|u\|_{m-1/2,\Gamma} + h_Y^{(d-1)/2-(s-1/2)} \|u\|_Y \right),$$

for all  $1/2 \leq s \leq m$ . Finally, by applying the trace theorem, the desired boundary sampling inequality is obtained.  $\square$

LEMMA 3.2 (Inverse Inequality). *Let a kernel  $\Phi_m : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying (2.5) with smoothness  $m > d/2$  be given. Suppose  $\Omega \subset \mathbb{R}^d$  is a bounded Lipschitz domain satisfying an interior cone condition. Assume  $0 \leq \nu \leq m - d/2$  and  $d/2 < \sigma \leq m - 2\nu$  for some integers  $\nu$  and positive  $\sigma$ . Then there is a constant  $C_{\Omega, \Phi_m, \sigma, \nu}$  depending only on  $\Omega, \Phi_m, \sigma$ , and  $\nu$  such that*

$$\|u\|_{\sigma+2\nu, \Omega} \leq C_{\Omega, \Phi_m, \sigma, \nu} \rho_Z^{m-\nu} h_Z^{-\sigma} \|u\|_{2\nu, \Omega} \quad \text{for all } u \in \mathcal{U}_Z \quad (3.1)$$

holds in the trial space of  $\Phi_m$  on all sufficiently dense and quasi-uniform sets  $Z \subset \Omega$  with fill distance  $h_Z$  and mesh ratio  $\rho_Z$ .

PROOF. The basic proof idea is to use an inverse inequality from [13, Eqn. 3.19]. It has the  $L_2(\Omega)$  norm on the right-hand side, but for (3.1) we have to go over to derivatives there. The idea is to push the derivatives into a new kernel.

Let  $\Phi_m$  denote the given kernel satisfying (2.5) with parameter  $m$ . For all  $0 \leq \nu < m - d/2$ , we define symmetric positive definite kernels  $\Psi_{m-\nu} := (I - \Delta)^\nu \Phi_m$ , whose Fourier transforms satisfy

$$c_1(1 + \|\omega\|_2^2)^{-(m-\nu)} \leq \widehat{\Psi}_{m-\nu}(\omega) = (1 + \|\omega\|_2^2)^\nu \widehat{\Phi}_m(\omega) \leq C_1(1 + \|\omega\|_2^2)^{-(m-\nu)},$$

and hence,  $\Psi_{m-\nu}$  has behavior like  $\Phi_{m-\nu}$ . This is like applying the operator  $(1 - \Delta)^{\nu/2}$  to both arguments of  $\Phi_m$ , if  $\Phi_m$  is written in difference form.

We use the notation  $u_{\beta, Z, \Phi_m} := \sum_{z_j \in Z} \beta_j \Phi_m(\cdot - z_j)$  to denote the functions in the trial space  $\mathcal{U}_{Z, \Phi_m}$  spanned by translates of the kernel  $\Phi_m$  on the trial centers in  $Z$  with coefficients forming a vector  $\beta \in \mathbb{R}^{|Z|}$ . Then

$$(I - \Delta)^\nu u_{\beta, Z, \Phi_m} = u_{\beta, Z, \Psi_{m-\nu}}$$

holds, and these are the functions that we use in [13, Eqn. 3.19]. This yields

$$\|u_{\beta, Z, \Psi_{m-\nu}}\|_{\sigma, \Omega} \leq C_{\Omega, \Phi_m, \sigma, \nu} \rho_Z^{m-\nu} h_Z^{-\sigma} \|u_{\beta, Z, \Phi_m}\|_{0, \Omega},$$

for all  $\beta \in \mathbb{R}^{|Z|}$ ,  $0 \leq \nu < m - d/2$ , and  $0 \leq \sigma \leq m - \nu$ . The right hand side is what we want, because of

$$\begin{aligned} \|u_{\beta, Z, \Psi_{m-\nu}}\|_{0, \Omega}^2 &= \int_{\Omega} |(I - \Delta)^\nu u_{\beta, Z, \Phi_m}|^2 dx \\ &\leq C_{\Omega, \nu} \sum_{|\alpha| \leq 2\nu} \int_{\Omega} |D^\alpha u_{\beta, Z, \Phi_m}|^2 dx \\ &= C_{\Omega, \nu} \|u_{\beta, Z, \Phi_m}\|_{2\nu, \Omega}^2, \end{aligned}$$

for  $2\nu \leq m$ . We are now left with

$$\|u_{\beta, Z, \Psi_{m-\nu}}\|_{\sigma, \Omega}^2 \leq C_{\Omega, \Phi_m, \sigma, \nu} \rho_Z^{m-\nu} h_Z^{-\sigma} \|u_{\beta, Z, \Phi_m}\|_{2\nu, \Omega}^2,$$

and our goal is to bound this from below by  $\|u_{\beta, Z, \Phi_m}\|_{\sigma+2\nu, \Omega}^2$ . By definition via Fourier transforms,

$$H_2^k(\mathbb{R}^d) = \left\{ v : (1 + \|\omega\|_2^2)^{k/2} \widehat{v}(\omega) \in L^2(\mathbb{R}^d) \right\}.$$



We use the sloppy notation  $\Phi_k$  to denote kernels in the family of  $\Phi_m$  with smoothness  $d/2 < k \leq m$ . Then,  $H_2^k(\mathbb{R}^d)$  is norm equivalent to the native space  $N_{\Phi_k}(\mathbb{R}^d)$  of  $\Phi_k$  on  $\mathbb{R}^d$  and equal as sets; i.e.,

$$c_{\Phi_m, k} \|u\|_{N_{\Phi_k}(\mathbb{R}^d)} \leq \|u\|_{H_2^k(\mathbb{R}^d)} \leq C_{\Phi_m, k} \|u\|_{N_{\Phi_k}(\mathbb{R}^d)}$$

for some constants  $0 < c_{\Phi_m, k} \leq C_{\Phi_m, k}$ . Before we go over to subdomains, we note that the Fourier transform of the operator  $I - \Delta$  is  $1 + \|\omega\|_2^2$ , and this extends to arbitrary non-integer powers. Thus,

$$(I - \Delta)^\nu H_2^{k+2\nu}(\mathbb{R}^d) = H_2^k(\mathbb{R}^d) \quad \text{for } 0 \leq 2\nu \leq m - k,$$

and

$$\|(I - \Delta)^\nu u\|_{H_2^k(\mathbb{R}^d)} = \|u\|_{H_2^{k+2\nu}(\mathbb{R}^d)} \quad \text{for all } u \in H_2^{k+2\nu}(\mathbb{R}^d), \quad 0 \leq 2\nu \leq m - k.$$

For a domain  $\Omega \subset \mathbb{R}^d$  with Lipschitz boundary, we also have that  $N_{\Phi_k}(\Omega)$  and  $H_2^k(\Omega)$  are equal as sets and the norms are equivalent [40, Cor 10.48], where  $H_2^k(\Omega)$  has the standard definition via weak derivatives and  $N_{\Phi_k}(\Omega)$  has the standard definition via a closure of  $\Phi_k$ -translates. We use the sloppy notation

$$c_{\Omega, \Phi_m, k} \|u\|_{N_{\Phi_k}(\Omega)} \leq \|u\|_{H_2^k(\Omega)} \leq C_{\Omega, \Phi_m, k} \|u\|_{N_{\Phi_k}(\Omega)}$$

for some constants  $0 < c_{\Omega, \Phi_m, k} \leq C_{\Omega, \Phi_m, k}$ . By Theorem 10.47 there, the restriction operator  $R_{\Phi_k, \Omega} : N_{\Phi_k}(\mathbb{R}^d) \rightarrow N_{\Phi_k}(\Omega)$  is well-defined and satisfies

$$\|R_{\Phi_k, \Omega} v\|_{N_{\Phi_k}(\Omega)} \leq \|v\|_{N_{\Phi_k}(\mathbb{R}^d)} \quad \text{for all } v \in N_{\Phi_k}(\mathbb{R}^d).$$

Going the other way, there is an isometric extension operator  $E_{\Phi_k, \Omega} : N_{\Phi_k}(\Omega) \rightarrow N_{\Phi_k}(\mathbb{R}^d)$  [40, Th. 10.46]. Most of this can already be found in [34]. Another extension operator is  $E_{k, \Omega} : H_2^k(\Omega) \rightarrow H_2^k(\mathbb{R}^d)$ , and it is bounded. This takes into account that the global space is defined via Fourier transforms, while the local one has  $L_2$  integrals over weak derivatives.

It is known that the Sobolev extension operators do not commute with general derivatives. On the trial functions, we do have  $E_{\Phi_\sigma, \Omega}(I - \Delta)^\nu = (I - \Delta)^\nu E_{\Phi_{\sigma+2\nu}, \Omega}$  for  $m - \nu > d/2$  and  $0 \leq \sigma + 2\nu \leq m$  since

$$\begin{aligned} E_{\Phi_\sigma, \Omega}(I - \Delta)^\nu u_{\beta, Z, \Phi_m} &= E_{\Phi_\sigma, \Omega} u_{\beta, Z, \Psi_{m-\nu}} \\ &= u_{\beta, Z, \Psi_{m-\nu}} \\ &= (I - \Delta)^\nu u_{\beta, Z, \Phi_m} \\ &= (I - \Delta)^\nu E_{\Phi_{\sigma+2\nu}, \Omega} u_{\beta, Z, \Phi_m} \end{aligned} \tag{3.2}$$

hold if the functions lie in the correct spaces, i.e.,

$$E_{\Phi_\sigma, \Omega} u_{\beta, Z, \Psi_{m-\nu}} = u_{\beta, Z, \Psi_{m-\nu}}, \quad \text{and} \quad E_{\Phi_{\sigma+2\nu}, \Omega} u_{\beta, Z, \Phi_m} = u_{\beta, Z, \Phi_m}.$$

The functions are globally defined anyway, and thus they coincide with their extension if the global norms are bounded. Thus, we need that

$$u_{\beta, Z, \Psi_{m-\nu}} \in N_{\Phi_\sigma}(\mathbb{R}^d), \quad \text{and} \quad u_{\beta, Z, \Phi_m} \in N_{\Phi_{\sigma+2\nu}}(\mathbb{R}^d).$$

The condition for the first case is  $2(m - \nu) - \sigma > d/2$ . The second case requires a finite  $d$ -variate integral over

$$|\widehat{\Phi}_m(\omega)|^2 (1 + \|\omega\|_2^2)^{\sigma+2\nu} = (1 + \|\omega\|_2^2)^{-2m+\sigma+2\nu},$$

that yields the same condition. Using the native space extension operators, we get

$$\begin{aligned} \|(I - \Delta)^\nu u_{\beta, Z, \Phi_m}\|_{H_2^\sigma(\Omega)} &\geq C_{\Omega, \Phi_m, \sigma} \|(I - \Delta)^\nu u_{\beta, Z, \Phi_m}\|_{N_{\Phi_\sigma}(\Omega)} \\ &= C_{\Omega, \Phi_m, \sigma} \|E_{\Phi_\sigma, \Omega} (I - \Delta)^\nu u_{\beta, Z, \Phi_m}\|_{N_{\Phi_\sigma}(\mathbb{R}^d)} \\ &\geq C'_{\Omega, \Phi_m, \sigma} \|E_{\Phi_\sigma, \Omega} (I - \Delta)^\nu u_{\beta, Z, \Phi_m}\|_{H_2^\sigma(\mathbb{R}^d)}, \end{aligned}$$

and, by the extension identity (3.2),

$$\begin{aligned} \|(I - \Delta)^\nu u_{\beta, Z, \Phi_m}\|_{H_2^\sigma(\Omega)} &\geq C'_{\Omega, \Phi_m, \sigma} \|(I - \Delta)^\nu E_{\Phi_{\sigma+2\nu}, \Omega} u_{\beta, Z, \Phi_m}\|_{H_2^\sigma(\mathbb{R}^d)} \\ &= C'_{\Omega, \Phi_m, \sigma} \|E_{\Phi_{\sigma+2\nu}, \Omega} u_{\beta, Z, \Phi_m}\|_{H_2^{\sigma+2\nu}(\mathbb{R}^d)}. \end{aligned}$$

We now go local by

$$\begin{aligned} \|(I - \Delta)^\nu u_{\beta, Z, \Phi_m}\|_{H_2^\sigma(\Omega)} &\geq C_{\Omega, \Phi_m, \sigma, \nu} \|E_{\Phi_{\sigma+2\nu}, \Omega} u_{\beta, Z, \Phi_m}\|_{N_{\Phi_{\sigma+2\nu}}(\mathbb{R}^d)} \\ &= C_{\Omega, \Phi_m, \sigma, \nu} \|u_{\beta, Z, \Phi_m}\|_{N_{\Phi_{\sigma+2\nu}}(\Omega)} \\ &\geq C'_{\Omega, \Phi_m, \sigma, \nu} \|u_{\beta, Z, \Phi_m}\|_{H_2^{\sigma+2\nu}(\Omega)}. \end{aligned}$$

We completed proving a local Bernstein inequality; note that the weaker norm on the right hand side must take an even order, whereas the global counterpart [32] allows any nonnegative integer orders.  $\square$

**LEMMA 3.3** ( $H^2$ -Stability). *Let a kernel  $\Phi$  as in (2.5) with smoothness  $m \geq 2$  and  $m > d/2$  be given. Suppose  $\Omega \subset \mathbb{R}^d$  is a bounded Lipschitz domain satisfying an interior cone condition. If the elliptic operator  $\mathcal{L}$  satisfies all assumptions to allow regularity (2.4), then there exists a constant  $C_{\Omega, \mathcal{L}}$ , depending only on  $\Omega$ ,  $\Phi$ , and  $\mathcal{L}$  such that*

$$\|u\|_{2, \Omega} \leq C_{\Omega, \mathcal{L}} (h_X^{d/2} \|\mathcal{L}u\|_X + h_Y^{d/2-2} \|u\|_Y)$$

holds in two circumstances:

- for all  $u \in \mathcal{U}_{Z \cup Y}$  under the condition

$$C_{\Omega, \mathcal{L}, m} (h_X^{m-2} + h_Y^{m-2}) h_{Z \cup Y}^{-m+2} \rho_{Z \cup Y}^{m-1} < 1/2, \quad (3.3)$$

- or for all  $u \in \mathcal{U}_Z$  under the condition

$$C_{\Omega, \mathcal{L}, m} (h_X^{m-2} + h_Y^{m-2}) h_Z^{-m+2} \rho_Z^{m-1} < 1/2. \quad (3.4)$$

Note that the factor at the second term in the assertion is not  $h_Y^{(d-1)/2}$  as one would expect. This might be connected to the fact that the natural norm on the boundary is the  $L_\infty$  norm, due to the Maximum Principle.

**PROOF.** We apply the first inequality of Lemma 3.1 (for  $u = \mathcal{L}u$ ,  $s = 0$ ,  $m = m - 2$ ) to get

$$\|\mathcal{L}u\|_{0, \Omega} \leq C_{\Omega, m-2, 0} \left( h_X^{m-2} \|\mathcal{L}u\|_{m-2, \Omega} + h_X^{d/2} \|\mathcal{L}u\|_X \right) \quad \text{for all } u \in H^m(\Omega)$$

and, by (2.3),

$$\|\mathcal{L}u\|_{0, \Omega} \leq C_{\Omega, m-2, 0} \left( h_X^{m-2} \|u\|_{m, \Omega} + h_X^{d/2} \|\mathcal{L}u\|_X \right) \quad \text{for all } u \in H^m(\Omega).$$

Using the second inequality of Lemma 3.1 (for  $s = 2, m$ ), we get

$$\|u\|_{1+1/2, \Gamma} \leq C_{\Omega, m, 2} \left( h_Y^{m-2} \|u\|_{m, \Omega} + h_Y^{d/2-2} \|u\|_Y \right) \quad \text{for all } u \in H^m(\Omega)$$

and this combines with the  $H^2$  regularity estimate (2.4):

$$\|u\|_{2,\Omega} \leq C_{\Omega,\mathcal{L}}(\|\mathcal{L}u\|_{0,\Omega} + \|u\|_{1+1/2,\Gamma})$$

into

$$\|u\|_{2,\Omega} \leq C_{\Omega,\mathcal{L},m} \left( (h_X^{m-2} + h_Y^{m-2}) \|u\|_{m,\Omega} + h_X^{d/2} \|\mathcal{L}u\|_X + h_Y^{d/2-2} \|u\|_Y \right).$$

Up to here, we are still in full Sobolev space. Now we use the inverse inequality, whatever the trial space is. If we only take  $Z$  nodes like in the lemma, then

$$\|u\|_{2,\Omega} \leq C_{\Omega,\mathcal{L},m} \left( (h_X^{m-2} + h_Y^{m-2}) h_Z^{-m+2} \rho_Z^{m-1} \|u\|_{2,\Omega} + h_X^{d/2} \|\mathcal{L}u\|_X + h_Y^{d/2-2} \|u\|_Y \right)$$

for all  $u \in \mathcal{U}_Z$  and the  $H^2$  stability is

$$\|u\|_{2,\Omega} \leq C_{\Omega,\mathcal{L},m} \left( h_X^{d/2} \|\mathcal{L}u\|_X + h_Y^{d/2-2} \|u\|_Y \right)$$

for all  $u \in \mathcal{U}_Z$  under the condition

$$C_{\Omega,\mathcal{L},m} (h_X^{m-2} + h_Y^{m-2}) h_Z^{-m+2} \rho_Z^{m-1} < 1/2.$$

If we now take  $Z \cup Y$  nodes, then

$$\|u\|_{2,\Omega} \leq C_{\Omega,\mathcal{L},m} \left( (h_X^{m-2} + h_Y^{m-2}) h_{Z \cup Y}^{-m+2} \rho_{Z \cup Y}^{m-1} \|u\|_{2,\Omega} + h_X^{d/2} \|\mathcal{L}u\|_X + h_Y^{d/2-2} \|u\|_Y \right)$$

for all  $u \in \mathcal{U}_{Z \cup Y}$  and the  $H^2$  stability is

$$\|u\|_{2,\Omega} \leq C_{\Omega,\mathcal{L},m} \left( h_X^{d/2} \|\mathcal{L}u\|_X + h_Y^{d/2-2} \|u\|_Y \right)$$

for all  $u \in \mathcal{U}_{Z \cup Y}$  under the condition

$$C_{\Omega,\mathcal{L},m} (h_X^{m-2} + h_Y^{m-2}) h_{Z \cup Y}^{-m+2} \rho_{Z \cup Y}^{m-1} < 1/2.$$

□

LEMMA 3.4 (Consistency). *If the elliptic operator  $\mathcal{L}$  satisfies Assumption 2.2 and if the kernel satisfies Assumption 2.3, we have*

$$\min_{\substack{v \in \mathcal{U}_{Z \cup Y} \\ v|_Y = u^*|_Y}} \|\mathcal{L}v - \mathcal{L}u^*\|_X \leq C_{\Omega,\Phi,\mathcal{L}} n_X^{1/2} h_{Z \cup Y}^{m-2-d/2} \|u^*\|_{m,\Omega} \quad \text{for } m \geq 2 + \lceil \frac{d+1}{2} \rceil,$$

and

$$\min_{\substack{v \in \mathcal{U}_{Z \cup Y} \\ v|_Y = u^*|_Y}} \|\mathcal{L}v - \mathcal{L}u^*\|_X \leq C_{\Omega,\Phi,\mathcal{L}} n_X^{1/2} \rho_X^{d/2} h_{Z \cup Y}^{m-2} \|u^*\|_{m,\Omega} \quad \text{for } m > 3 + d/2,$$

for any  $u^* \in H^m(\Omega)$ .

PROOF. By comparing the minimizer  $v^* \in \mathcal{U}_{Z \cup Y}$  of the optimization problem with the interpolant  $I_{Z \cup Y} u^* \in \mathcal{U}_{Z \cup Y}$  that also satisfies the constraints at  $Y$ , we turn the problem into an error estimate for radial basis function interpolation:

$$\|\mathcal{L}v^* - \mathcal{L}u^*\|_X \leq \|\mathcal{L}I_{Z \cup Y} u^* - \mathcal{L}u^*\|_X.$$

The first error estimate can be derived based on native space error estimates [40, Thm.11.9] and upper bounds of power functions [5, Sec.15.1.2]. For  $m \geq 2 + \lceil \frac{d+1}{2} \rceil$ , we have

$$\begin{aligned} \|\mathcal{L}I_{Z \cup Y} u^* - \mathcal{L}u^*\|_X &\leq n_X^{1/2} \|\mathcal{L}I_{Z \cup Y} u^* - \mathcal{L}u^*\|_{L^\infty(\Omega)} \\ &\leq C_{\Omega, \mathcal{L}} n_X^{1/2} \max_{|\alpha| \leq 2} |D^\alpha I_{Z \cup Y} u^* - D^\alpha u^*| \\ &\leq C_{\Omega, \Phi, \mathcal{L}} n_X^{1/2} h_{Z \cup Y}^{m-d/2-2} \|u^*\|_{m, \Omega}. \end{aligned}$$

If we employ kernels with a higher smoothness parameter  $m > 3 + d/2$ , we can use the estimates for functions with scattered zeros. Applying [31, Prop.3.3] to our Hilbert space setting and taking care of the definitions of discrete norms yield the desired error bound.  $\square$

To prove Theorem 2.6, suppose Assumptions 2.1– 2.5 hold so that all lemmas in this section can be applied. Let  $u_{X, Y}^{CLS} \in \mathcal{U}_{Z \cup Y}$  be the CLS approximation of (2.1), defined as in (2.7). Moreover, let  $I_{Z \cup Y} u^*$  denote the unique interpolant of the exact solution  $u^* \in H^m(\Omega)$  from the trial space  $\mathcal{U}_{Z \cup Y} \subset \mathcal{N}_{\Omega, \Phi} = H^m(\Omega)$ . Assume the condition (3.3) holds; we shall show that the CLS solution converges to the interpolant in  $\mathcal{U}_{Z \cup Y}$ .

$$\begin{aligned} \|u_{X, Y}^{CLS} - u^*\|_{2, \Omega} &\leq \|u_{X, Y}^{CLS} - I_{Z \cup Y} u^*\|_{2, \Omega} + \|I_{Z \cup Y} u^* - u^*\|_{2, \Omega} \\ &\leq \|u_{X, Y}^{CLS} - I_{Z \cup Y} u^*\|_{2, \Omega} + C_{\Omega, \Phi, \mathcal{L}} h_{Z \cup Y}^{m-2} \|u^*\|_{m, \Omega}, \end{aligned}$$

since the stability result in Lemma 3.3 only applies to functions in the trial space. The last inequality (by Lemma 3.1) suggests that we can focus on the difference  $u_{X, Y}^{CLS} - I_{Z \cup Y} u^* \in \mathcal{U}_{Z \cup Y}$ , which has zeros at nodes  $Y$ . Using the  $H^2$ -stability in Lemma 3.3, we have

$$\|u_{X, Y}^{CLS} - I_{Z \cup Y} u^*\|_{2, \Omega} \leq C_{\Omega, \Phi, \mathcal{L}} (h_X^{d/2} \|\mathcal{L}u_{X, Y}^{CLS} - \mathcal{L}I_{Z \cup Y} u^*\|_X + 0).$$

Applying Lemma 3.4 yields

$$\|u_{X, Y}^{CLS} - u^*\|_{2, \Omega} \leq C_{\Omega, \Phi, \mathcal{L}} (h_X^{d/2} n_X^{1/2} + 1) h_{Z \cup Y}^{m-2-d/2} \|u^*\|_{m, \Omega},$$

and

$$\|u_{X, Y}^{CLS} - u^*\|_{2, \Omega} \leq C_{\Omega, \Phi, \mathcal{L}} (h_X^{d/2} n_X^{1/2} \rho_X^{d/2} + 1) h_{Z \cup Y}^{m-2} \|u^*\|_{m, \Omega},$$

for  $m \geq 2 + \lceil \frac{d+1}{2} \rceil$  and  $m > 3 + d/2$  respectively. The bracketed factor is bounded if  $X$  is uniformly distributed.

**4. Convergence for WLS.** Instead of a specific weight, we will consider a class of weighted least-squares formulations by a simple inequality.

LEMMA 4.1. *Let  $a, b > 0$ ,  $0 < \epsilon < 1$ , and  $0 \leq \theta \leq 2$ . Then the following inequalities hold:*

$$(\epsilon a + b)^2 \leq 2(\epsilon^\theta a^2 + b^2).$$

PROOF. Consider  $0 \leq \theta/2 \leq 1$ . From  $(\epsilon a + b)^2 \leq 2(\epsilon^2 a^2 + b^2)$  and  $\epsilon \leq \epsilon^{\theta/2}$ , we have  $(\epsilon a + b)^2 \leq 2(\epsilon^\theta a^2 + b^2)$ .  $\square$

LEMMA 4.2 ( $H^2$ -Stability). *Suppose the assumptions in Lemma 3.3 hold under the condition (3.3). If  $h_X \leq h_Y < 1$ , then there exists a constant  $C_{\Omega, \Phi, \mathcal{L}}$ , depending only on  $\Omega$ ,  $\Phi$ , and  $\mathcal{L}$  such that*

$$\|u\|_{2, \Omega} \leq C_{\Omega, \Phi, \mathcal{L}} h_Y^{d/2-2} \left( \left( \frac{h_Y}{h_X} \right)^{d\theta/2} h_Y^{-2\theta} \right)^{-1/2} \left( \|\mathcal{L}u\|_X^2 + \left( \frac{h_Y}{h_X} \right)^{d\theta/2} h_Y^{-2\theta} \|u\|_Y^2 \right)^{1/2}$$

holds for all  $0 \leq \theta \leq 2$  and all  $u \in \mathcal{U}_{Z \cup Y}$  for any finite sets  $X \subset \Omega$  and  $Y \subset \Gamma$ .

PROOF. The CLS stability in Lemma 3.3 has to be further modified to suit the need of WLS. With the denseness requirement (3.3), let us start with

$$\|u\|_{2, \Omega} \leq C_{\Omega, \Phi, \mathcal{L}} (h_X^{d/2} \|\mathcal{L}u\|_X + h_Y^{d/2-2} \|u\|_Y) \quad (4.1)$$

for all  $u \in \mathcal{U}_{Z \cup Y}$ . We want to obtain a stability estimate with discrete sum of squares. Rewrite (4.1) as

$$\|u\|_{2, \Omega}^2 \leq C_{\Omega, \Phi, \mathcal{L}} h_Y^{d-4} (\epsilon \|\mathcal{L}u\|_X + \|u\|_Y)^2 \quad \text{with } \epsilon = (h_X/h_Y)^{d/2} h_Y^2.$$

Note that having  $\epsilon < 1$  is a very mild requirement, for example  $h_X \leq h_Y < 1$ , and will not be an obstacle between theories and practice. By Lemma 4.1, we have

$$\begin{aligned} \|u\|_{2, \Omega} &\leq (C_{\Omega, \Phi, \mathcal{L}} h_Y^{d-4} (\epsilon^\theta \|\mathcal{L}u\|_X^2 + \|u\|_Y^2))^{1/2} \\ &\leq C_{\Omega, \Phi, \mathcal{L}} h_Y^{d/2-2} \epsilon^{\theta/2} (\|\mathcal{L}u\|_X^2 + \epsilon^{-\theta} \|u\|_Y^2)^{1/2}, \end{aligned}$$

for any  $0 \leq \theta \leq 2$ . Substituting  $\epsilon$  back yields

$$\|u\|_{2, \Omega} \leq C_{\Omega, \Phi, \mathcal{L}} h_Y^{d/2-2} ((h_X/h_Y)^{d/2} h_Y^2)^{\theta/2} (\|\mathcal{L}u\|_X^2 + ((h_X/h_Y)^{d/2} h_Y^2)^{-\theta} \|u\|_Y^2)^{1/2},$$

and we obtain the desired WLS stability after simplification.  $\square$

LEMMA 4.3 (Consistency). *For any  $W > 0$ , define a functional  $J_W : H^m(\Omega) \rightarrow \mathbb{R}$  by  $J_W(u) := (\|\mathcal{L}u\|_X^2 + W\|u\|_Y^2)^{1/2}$ . Suppose the assumptions in Lemma 3.4 hold. Then the error estimates in Lemma 3.4 also hold if the left-handed sides are replaced by  $\min_{v \in \mathcal{U}_{Z \cup Y}} J_W(v - u^*)$  for any  $W > 0$ .*

PROOF. Again, we compare the minimizer  $v^*$  with the interpolant  $I_{Z \cup Y} u^*$  in  $\mathcal{U}_{Z \cup Y}$ :

$$\begin{aligned} J_W^2(v^* - u^*) &\leq J_W^2(I_{Z \cup Y} u^* - u^*) \\ &= \|\mathcal{L}I_{Z \cup Y} u^* - \mathcal{L}u^*\|_X^2 + W\|I_{Z \cup Y} u^* - u^*\|_Y^2, \end{aligned}$$

where the last term vanishes due to the zeros of  $I_{Z \cup Y} u^* - u^*$  at  $Y$ .  $\square$

With both consistency and stability results, we can now prove the convergence of a class of WLS solutions defined by (2.8). By similar arguments used in Section 3, we only need to show that the WLS solution converges to the interpolant  $I_{Z \cup Y} u^*$  of the exact solution  $u^*$  from the trial space  $\mathcal{U}_{Z \cup Y}$ . For  $0 \leq \theta \leq 2$ , consider the functional

$$J_{W(\theta)}(u) := (\|\mathcal{L}u\|_X^2 + W(\theta)\|u\|_Y^2)^{1/2} \quad \text{with } W(\theta) := (h_Y/h_X)^{d\theta/2} h_Y^{-2\theta}. \quad (4.2)$$

Applying the results of Lemmas 4.2 and 4.3, we have the WLS solution convergence within the trial space; for simplicity, let  $\tau$  be  $d/2$  if  $m \geq 2 + \lceil \frac{d+1}{2} \rceil$  and zero if

$m > 3 + d/2$ . Then,

$$\begin{aligned} \|u_{X,Y,Z_{UY}}^{WLS,\theta} - I_{Z_{UY}}u^*\|_{2,\Omega} &\leq C_{\Omega,\Phi,\mathcal{L}}h_Y^{d/2-2}W^{-1/2}J_{W(\theta)}(u_{X,Y,Z_{UY}}^{WLS,\theta} - I_{Z_{UY}}u^*) \\ &\leq C_{\Omega,\Phi,\mathcal{L},\gamma_X} \left(\frac{h_X}{h_Y}\right)^{d\theta/4} h_X^{-d/2}h_Y^{d/2-(2-\theta)}h_{Z_{UY}}^{m-2-\tau}\|u^*\|_{m,\Omega}. \end{aligned}$$

The last holds because  $h_X^{d/2}n_X^{1/2}\rho_X^{d/2}$  can be bounded by some  $C_{\gamma_X}$ . Now we can compare the difference between the WLS solution and the exact solution,

$$\begin{aligned} \|u_{X,Y,Z_{UY}}^{WLS,\theta} - u^*\|_{2,\Omega} &\leq \|I_{Z_{UY}}u^* - u^*\|_{2,\Omega} + \|u_{X,Y,Z_{UY}}^{WLS,\theta} - I_{Z_{UY}}u^*\|_{2,\Omega} \\ &\leq C_{\Omega,\Phi,\mathcal{L},\gamma_X}(h_{Z_{UY}}^{m-2}\|u^*\|_{m,\Omega} + h_Y^{-(2-\theta)(4-d)/4}h_X^{d\theta/4-d/2}h_{Z_{UY}}^{m-2-\tau}\|u^*\|_{m,\Omega}) \\ &\leq C_{\Omega,\Phi,\mathcal{L},\gamma_X}(h^\tau + h_X^{-d(2-\theta)/4}h_Y^{(2-\theta)(d-4)/4})h_{Z_{UY}}^{m-2-\tau}\|u^*\|_{m,\Omega}, \end{aligned}$$

for any  $0 \leq \theta \leq 2$ . The constant  $1 = h^0$ , coming from  $\|I_{Z_{UY}}u^* - u^*\|_{2,\Omega}$ , is absolutely necessary or else the error bound will allow arbitrarily fast convergence with respect to  $h_X \rightarrow 0$  for some  $\theta$  and  $d$ . It is obvious that  $\theta = 2$  maximizes the convergence rate:

$$\|u_{X,Y,Z_{UY}}^{WLS,2} - I_{Z_{UY}}u^*\|_{2,\Omega} \leq C_{\Omega,\Phi,\mathcal{L},\gamma_X}h_{Z_{UY}}^{m-2-\tau}\|u^*\|_{m,\Omega},$$

where  $\tau = d/2$  for  $m \geq 2 + \lceil \frac{d+1}{2} \rceil$  or  $\tau = 0$  for  $m > 3 + d/2$ . The CLS and the optimal WLS( $\theta = 2$ ) formulation share convergence estimates of the same form. They both match the convergence estimate of the interpolant exactly for  $m > 3 + d/2$ , that in turn confirms their optimality. To complete proving Theorem 2.7, we consider the stability for  $\theta = 2$  and Lemma 4.2 gives

$$\begin{aligned} \|u\|_{2,\Omega} &\leq C_{\Omega,\Phi,\mathcal{L}}h_X^{d/2}(\|\mathcal{L}u\|_X^2 + (h_Y/h_X)^d h_Y^{-4}\|u\|_Y^2)^{1/2} \\ &\leq C_{\Omega,\Phi,\mathcal{L}}h_X^{d/2}(\|\mathcal{L}u\|_X^2 + (h_Y/h_X)^d h_Y^{-2\theta}\|u\|_Y^2)^{1/2}, \end{aligned}$$

for any  $\theta \geq 2$  as long as  $h_Y < 1$ . We extend the definition of functional  $J_{W(\theta)}$  to  $\theta \geq 2$  by the same definition as in (4.2). Then, for any  $u \in H^m(\Omega)$ , we have

$$J_{\theta_1}(u) \leq J_{\theta_2}(u), \quad \text{for } 2 \leq \theta_1 \leq \theta_2 \leq \infty.$$

Since the CLS formulation is equivalent to the WLS with  $\theta = \infty$ , for any  $\theta \geq 2$ , we have

$$\min_{v \in \mathcal{U}_{Z_{UY}}} J_{W(\theta)}(v - u^*) \leq J_{W(\theta)}(u_{X,Y}^{CLS} - u^*) \leq J_{W(\infty)}(u_{X,Y}^{CLS} - u^*),$$

where the last term is minimal by the definition of CLS solution. Theorem 2.7 can now be concluded based on Theorem 2.6. We have to take (3.3) into account in both theorems.

**5. Optimal WLS weighting revisited.** Hu, Chen and et al. [15] showed by scaling analysis that the optimal weighting for overdetermined Kansa methods is  $n_Z^2$  for bounded  $\Omega \subset \mathbb{R}^2$ . Since the fill distances  $h_X$ ,  $h_Y$ , and  $h_Z$  are of the same magnitude in [15], this weighting corresponds to  $\theta = 1$  in our notation and the bracketed factors in the WLS estimates (Theorem 2.7) is  $\mathcal{O}(1)$ . To ensure our proven theories are consistent with the previous findings, we must extend our theories to a smaller

trial space  $\mathcal{U}_Z$  used in [15]. In the rest of this section, we will focus on the WLS convergence in this smaller trial space and prove Theorem 2.8.

To begin, let us return to the proof for WLS consistency (Lemma 4.3) but restrict the approximation in the smaller trial space  $\mathcal{U}_Z$ , within which the stability result in Lemma 4.2 remains valid. However, we can only compare the minimizer  $v^* \in \mathcal{U}_Z$  with the interpolant  $I_Z u^* \in \mathcal{U}_Z$  to the exact solution  $u^* \in H^m(\Omega)$ :

$$\begin{aligned} \min_{v \in \mathcal{U}_Z} J_W^2(v - u^*) &\leq J_W^2(I_Z u^* - u^*) \\ &= \|\mathcal{L}I_Z u^* - \mathcal{L}u^*\|_X^2 + W\|I_Z u^* - u^*\|_Y^2. \end{aligned}$$

The PDE residual on  $X$  is exactly the same as that in the previous section. Without  $Y$  in the trial centers to annihilate the boundary collocation, we simply need to identify the extra terms associated with the boundary error on  $Y$ . Following the ideas in the proof of Lemma 3.4, for  $m \geq 2 + \lceil \frac{d+1}{2} \rceil$ , we can bound the boundary term by

$$\begin{aligned} \|I_Z u^* - u^*\|_Y^2 &\leq n_Y^{1/2} \|I_Z u^* - u^*\|_{L^\infty(\Gamma)} \\ &\leq n_Y^{1/2} \|I_Z u^* - u^*\|_{L^\infty(\Omega)} \\ &\leq C_{\Omega, \Phi, \mathcal{L}} n_Y^{1/2} h_Z^{m-d/2} \|u^*\|_{m, \Omega}. \end{aligned}$$

Hence, the error estimate for WLS on  $\mathcal{U}_Z$  contains an extra term

$$\begin{aligned} &h_Y^{d/2-2} W^{-1/2} \left( W^{1/2} C_{\Omega, \Phi, \mathcal{L}} n_Y^{1/2} h_Z^{m-d/2} \|u^*\|_{m, \Omega} \right) \\ &\leq C_{\Omega, \Phi, \mathcal{L}, \gamma_Y} h_Y^{d/2-2} h_Y^{-(d-1)/2} h_Z^{m-d/2} \|u^*\|_{m, \Omega} \\ &\leq C_{\Omega, \Phi, \mathcal{L}, \gamma_Y} (h_Y^{-3/2} h_Z^2) h_Z^{m-d/2-2} \|u^*\|_{m, \Omega}. \end{aligned} \tag{5.1}$$

For the other case when  $m > 3 + d/2$ , we want to bound the  $\ell_2(Y)$  norm on the boundary by some  $\ell_2(\tilde{Z} \cup Y)$  norm in the domain (like the trace theorem does). For any subset  $\tilde{Z} \subseteq Z$ , we have

$$\begin{aligned} \|I_Z u^* - u^*\|_Y &= \|I_Z u^* - u^*\|_{\tilde{Z} \cup Y} \\ &\leq C_{\Omega, \Phi} n_{\tilde{Z} \cup Y}^{1/2} \rho_{\tilde{Z} \cup Y}^{d/2} h_Z^m \|u^*\|_{m, \Omega}. \end{aligned}$$

We want to select  $\tilde{Z}$  so that  $\rho_{\tilde{Z} \cup Y}$  can be bounded by the denseness measures of  $Y$  and  $Z$ . We already assumed  $Z$  is quasi-uniform in Assumption 2.5. Let us further assume that  $Y$  is also quasi-uniform on  $\Gamma$  and satisfies (2.6) with some constant  $\gamma_Y > 1$ . Moreover, the set  $Y$  is sufficiently dense with respect to  $Z$  and  $\Omega$  so that  $h_Y \leq h_Z$  and  $q_{Y, \Gamma} < q_{Y, \Omega}$  (see [10, Thm.6]).

Consider the following subset that excludes all points in  $Z$  that are within distance  $h_Z$  to the boundary:

$$\tilde{Z} := \left\{ z \in Z \cap \{ \zeta \in \Omega : \text{dist}(\zeta - \Gamma) > h_Z \} \right\} \subseteq Z.$$

Then, we have

$$\begin{aligned} \min(q_Z, q_Y) &\leq q_{\tilde{Z} \cup Y} \leq h_{\tilde{Z} \cup Y} \leq \left( \sup_{\zeta \in \Omega_{h_Z}} + \sup_{\zeta \in \Omega \setminus \Omega_{h_Z}} \right) \min_{z \in \tilde{Z} \cup Y} \|z - \zeta\|_{\ell_2(\mathbb{R}^d)} \\ &\leq h_Z + (h_Z + h_Y). \end{aligned}$$

It is now clear that the set  $\tilde{Z} \cup Y$  is also quasi-uniform with respect to some parameter  $\gamma_{\tilde{Z} \cup Y}$  that depends on  $\gamma_Z$  and  $\gamma_Y$ . Hence, we can bound  $\rho_{\tilde{Z} \cup Y}$  by some generic constant  $C_{\gamma_Y, \gamma_Z}$ . To control the term  $n_{\tilde{Z} \cup Y}$ , consider

$$\begin{aligned} n_{\tilde{Z} \cup Y} &\leq n_Z + n_Y \\ &\leq C_{\Omega, \gamma_Z} h_Z^{-d} + C_{\Omega, \gamma_Y} h_Y^{-(d-1)} \\ &\leq C_{\Omega, \gamma_Y, \gamma_Z} (h_Y^{-d} + h_Y^{-d+1}). \end{aligned}$$

Since we assumed  $h_Y < 1$ , we have  $n_{\tilde{Z} \cup Y} \leq C_{\Omega, \gamma_Y, \gamma_Z} h_Y^{-d}$ . Together, we have

$$\|I_Z u^* - u^*\|_Y \leq C_{\Omega, \Phi, \gamma_Y, \gamma_Z} h_Y^{-d/2} h_Z^m \|u^*\|_{m, \Omega},$$

and the extra term associated with boundary error on  $Y$  is

$$\begin{aligned} &h_Y^{d/2-2} W^{-1/2} \left( W^{1/2} C_{\Omega, \Phi, \gamma_Y, \gamma_Z} h_Y^{-d/2} h_Z^m \|u^*\|_{m, \Omega} \right) \\ &\leq C_{\Omega, \Phi, \mathcal{L}, \gamma_Y, \gamma_Z} (h_Y^{-2} h_Z^2) h_Z^{m-2} \|u^*\|_{m, \Omega}. \end{aligned} \quad (5.2)$$

Adding the corresponding boundary errors in (5.1) and (5.2), for  $m \geq 2 + \lceil \frac{d+1}{2} \rceil$  and  $m > 3 + d/2$  respectively, into the estimates in Theorem 2.7 completes the proof of Theorem 2.8. And, these boundary errors do not affect the convergence rates as long as  $h_Y \lesssim h_Z$ . Moreover, since both (5.1) and (5.2) are independent of  $\theta$ , our error estimate allows the least-squares weighting suggested in [15] to be optimal, but we have to take (3.4) and other requirements on the trial centers and collocation points into account.

**6. Numerical demonstrations.** We test the proposed formulations in  $\Omega = [-1, 1]^2$ . Discretization is done by using regular  $Z$  with  $n_Z = 11^2, 16^2, \dots, 36^2$  and collocation points  $X$  are either regular or scattered. For the regular cases, collocation points  $X$  (strictly in the interior) and  $Y$  are constructed similarly with  $h_X = \delta_i h_Z$  and  $h_Y = \delta_b h_Z$  with  $\delta_i = 1, 1/2, 1/3$  and  $\delta_b = 1, 1/2$  such that  $Z \subseteq X$  and  $(Z \cap \Gamma) \subseteq Y$  respectively. All reported errors, either in  $L^2$  or  $H^2$ , are absolute error approximated by using a fixed set of  $100^2$  regular points, which is denser than the collocation  $X \cup Y$  sets in all tests.

In matrix form, collocation conditions for the PDE and boundary condition can be written as

$$K_{\mathcal{L}, X} \lambda = f|_X \quad \text{and} \quad K_{\mathcal{B}, Y} \lambda = g|_Y,$$

respectively, with entries  $[K_{\mathcal{L}, X}]_{ij} = \mathcal{L}\Phi(x_i - z_j)$  and  $[K_{\mathcal{B}, Y}]_{ij} = \Phi(y_i - z_j)$  for  $x_i \in X$  and  $y_i \in Y$ . Both resultant matrices have  $n_Z + n_Y$  (and  $n_Z$ ) columns for trial space  $\mathcal{U}_{Z \cup Y}$  (and  $\mathcal{U}_Z$ ) corresponding to each  $z_j$  from the trial space. In the CLS approach (2.7), the constraints at  $Y$  are enforced using the null space matrix of the boundary collocation matrix, denoted by  $\mathcal{N}_{\mathcal{B}, Y} := \text{null}(K_{\mathcal{B}, Y})$ , as in [27], so that the unknown coefficient is expressed in the form

$$\lambda = \mathcal{N}_{\mathcal{B}, Y} \gamma + K_{\mathcal{B}, Y}^\dagger g|_Y,$$

for some new unknown  $\gamma$ , which can be found by solving

$$K_{\mathcal{L}, X} \mathcal{N}_{\mathcal{B}, Y} \gamma = f|_X - K_{\mathcal{L}, X} K_{\mathcal{B}, Y}^\dagger g|_Y.$$



In all WLS( $\theta$ ) formulations, with LS weighting specified by  $W(\theta)$  in (2.8), the unknown coefficient  $\lambda$  is obtained by solving the following overdetermined system

$$\begin{bmatrix} K_{\mathcal{L},X} \\ W(\theta)K_{\mathcal{B},Y} \end{bmatrix} \lambda = \begin{bmatrix} f|_X \\ W(\theta)g|_Y \end{bmatrix}$$

with the Matlab function `mldivide` in the least-squares sense. For all computations in this section, we did not employ any technique to deal with the problem of ill-conditioning unless specified otherwise, i.e., when we employ the stable RBF-QR decomposition for the Gaussian basis. To deal with the numerical instability, readers are referred to our adaptive techniques [25, 26, 30].

**EXAMPLE 6.1. *How dense is dense?*** First, we consider a Poisson problem with Dirichlet boundary value generated from three different exact solutions  $u^* = \sin(\pi x/2) \cos(\pi y/2)$ ,  $\text{peaks}(3x, 3y)$ , and  $\text{franke}(2x - 1, 2y - 1)$  by the corresponding functions in MATLAB. We cast the CLS formulation (2.7) using unscaled Whittle-Matérn-Sobolev kernels that reproduce  $H^m(\Omega)$  with  $m = 3, \dots, 6$ . Note that our proven  $H^2$ -convergence theories require  $m \geq 2 + \lceil 1.5 \rceil = 4$  for  $\Omega \subset \mathbb{R}^2$ . To see the effect of “over-testing”, all sets in this example are regular and we tested  $h_X = \{h_Z, h_Z/2, h_Z/3\}$  and  $h_Y = \{h_Z, h_Z/2\}$ . Figure 6.3 compactly shows all convergence profiles in  $H^2(\Omega)$  with respect to  $h_Z$  (instead of  $h_{Z \cup Y}$  for ease of comparison to the results in the next example) at a glance.

To begin, let us focus on the  $H^2(\Omega)$  errors for  $u^* = \sin(\pi x/2) \cos(\pi y/2)$  in Figure 6.3. Generally speaking, all collocation settings demonstrate an  $m - 2$  convergence rate for all tested smoothness  $m$ ; this also includes the original Kansa formulation with  $Z = X \cup Y$ . It is obvious that the error profiles for each tested  $m$  are split into two groups. The least accurate groups (i.e., the group above) correspond to  $h_X = h_Z$ . Without over-testing the PDE, this setting would probably fail the denseness requirement (3.3) but yet allow convergence at the optimal rate. All errors reduce at a rather constant rate, except that we can see two unstable profiles in the cases of  $m = 6$ . These numerical instabilities correspond to the two cases with large numbers of boundary collocations;  $(h_X, h_Y) = (h_Z/2, h_Z/2)$  and  $(h_X, h_Y) = (h_Z/3, h_Z/2)$ . In comparison, the other two tested solutions,  $u^* = \text{peaks}(3x, 3y)$  and  $\text{franke}(2x - 1, 2y - 1)$ , are more oscillatory. We can see the CLS convergence rates slow down and approach the optimal  $m - 2$  order. We can clearly see that kernels with higher smoothness typically suffers more from the effects of ill-conditioning in all tested  $u^*$ . Error reduction reaches a valley as  $Z$  is being refined and then increases again.

We omit the  $L^2(\Omega)$  error profiles, which show exactly two extra orders as one would expect and achieve an  $m$ -order convergence before numerical instability kicks in.

**EXAMPLE 6.2. *CLS convergence in trial space  $\mathcal{U}_Z$ .*** Putting the theoretical requirement aside, we are interested in the numerical performance of casting the CLS in the smaller and more practical trial spaces  $\mathcal{U}_Z$ . Elementary linear algebra says that if  $n_Z < n_Y$ , then we may not be able to find nontrivial functions from  $\mathcal{U}_Z$  with zeros at  $Y$ . However, one can observe numerically that the CLS formulation hardly runs into trouble when it is cast in this smaller trial space. Numerically, as  $h_Y \rightarrow 0$ , the rank of the boundary collocation matrix is bounded; for example, for  $n_Z = 21^2$  with finer and finer  $Y$ , we can see from Table 6.1 that the rank of the boundary matrix is numerically bounded.

Figure 6.4 shows the  $H^2(\Omega)$  error profiles for the CLS performance in  $\mathcal{U}_Z$  with all other settings identical to those in Example 6.1. Comparing the CLS convergence

$\mathcal{U}_{Z \cup Y}$		$\mathcal{U}_Z$	
$n_Y$	$\text{rank}(K_{\mathcal{B}, Y})$	$n_Y$	$\text{rank}(K_{\mathcal{B}, Y})$
80	80	80	80
160	104	160	96
244	108	244	96
328	100	328	96

TABLE 6.1

Numerical ranks of boundary collocation matrices resulting from different  $n_Y$ .

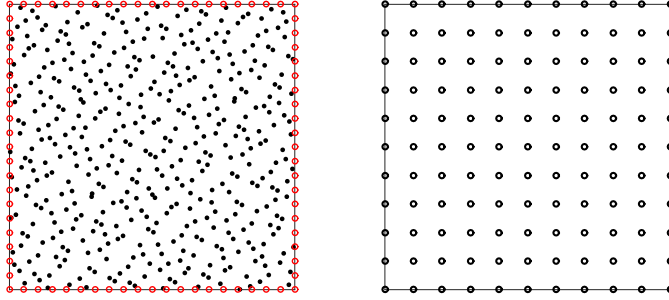


FIG. 6.1. Schematic point sets, collocations  $X$  and  $Y$  (left), and trial  $Z$  (right), used to solve various PDEs.

rates in the two trial spaces, we observe that optimal convergence is also possible in  $\mathcal{U}_Z$  but only for small enough  $h_Z$ . The CLS accuracy can “catch up” when the numerical rank of  $K_{\mathcal{B}, Y}$  is relatively insignificant compared to  $n_Z$ . Therefore, the larger the  $\text{rank}(K_{\mathcal{B}, Y})$  the longer CLS takes to achieve optimal convergence. By using a smaller trial space, we not only gain computational efficiency but suffer less ill-conditioning. In all tested cases and parameters, we see no accuracy drop due to ill-conditioning in Figure 6.4. In our next demonstration, we will see that using  $\mathcal{U}_Z$  as trial space only makes the CLS convergence lag behind, but not to any other WLS formulations.

**EXAMPLE 6.3. Numerically optimal weight for WLS.** Now we consider the  $\text{WLS}(\theta)$  formulation in (2.8) with  $\theta \in \{\infty, 0, 0.5, 1, 2\}$ . We begin with the same set up as in Example 6.1 and set  $h_X = h_Y = h_Z/2$  to solve the Poisson problem in  $\Omega = [-1, 1]^2$ . The WLS weighting in this test are  $W(\theta) = 1, n_Z, n_Z^2$  and  $n_Z^4$ , and  $\text{WLS}(\infty)$  is equivalent to the CLS. Figure 6.6 show the  $H^2(\Omega)$  error resulting from various  $\text{WLS}(\theta)$  formulations associated with  $u^* = \sin(\pi x/2, \pi y/2)$  and  $u^* = \text{peaks}(3x, 3y)$  respectively. The estimated convergence rates shown in the legends are obtained from least-squares fitting to all data; if the convergence profile is not “straight” enough, the corresponding estimate is not trustworthy.

From these figures, we immediately see that there is no benefit at all (in terms of both efficiency and accuracy) to go for the unweighted  $\text{WLS}(0)$  formulation. Unlike the CLS in  $\mathcal{U}_Z$ , all tested  $\text{WLS}(\theta)$  with  $\theta > 0$  do not have a lag in convergence rate but may suffer ill-conditioning for large  $\theta$ . By comparing all tested cases, we see that  $\theta = 1$  allows good accuracy and optimal convergence rate in both trial spaces.

To further verify these observations, we now use sets of  $n_X$  scattered collocation points (generated by the Halton sequence; see Figure 6.1) to solve three different PDEs. Boundary collocation points remain regular with  $h_Y = h_Z/2$ . We present the numerical result for  $m = 4$  in Figure 6.7. All PDEs have  $\text{peaks}(3x, 3y)$  as the exact

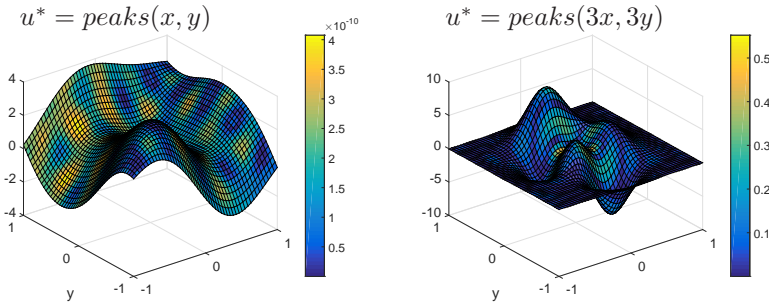


FIG. 6.2. Numerical approximations from CLS in  $\mathcal{U}_Z$  using the unscaled Gaussian kernel for solving a Poisson problem with exact solutions zoom-in peak (left) and full peak (right).

solution. These results should be compared to the results of  $m = 4$  in Figure 6.6 and the convergence patterns are very similar. Yet, there are some minor, but notable differences. Based on these test results, the WLS formulation in  $\mathcal{U}_Z$  with moderate weighting (i.e.  $\theta = 0.5$  or 1), which agrees with that in [15], is relatively stable in the numerical sense and is as accurate as the CLS formulation in  $\mathcal{U}_{Z \cup Y}$ .

**EXAMPLE 6.4. Some observations for GA and MQ.** Let us begin with two CLS approximations obtained by using the unscaled Gaussian kernel, see Figure 6.2. We generate the trial space  $\mathcal{U}_Z$  using  $n_Z = 36^2$  centers. Collocations are enforced on regular  $X$  and  $Y$  with  $h_X = h_Y = h_Z/2$ . We then solve two Poisson problems with different exact solutions  $u^* = \text{peaks}(x, y)$  and  $u^* = \text{peaks}(3x, 3y)$ . The zoom-in peaks is a nice polynomial within machine precision, while the full peaks function suffers from the enforcement of nearly zero boundary conditions. It is a nasty case like those that have a true solution that is a peak inside the domain with zero values on the boundary.

Both resultant systems share the same Kansa matrix and only differ by the right-hand vectors  $f|_X$  and  $g|_Y$ . Although both systems have exactly the same condition number, the resulting accuracy has a huge difference:  $8.1\text{E-}12$  and  $3.5\text{E-}3$   $L^2(\Omega)$  error respectively, for the zoom-in and full peaks. One could argue that the Gaussian native space is relatively small and it may not contain the full peak function that causes low accuracy. To know for sure, a change of stable basis (a.k.a. RBF-QR algorithm [6, 8]) can give us a clearer picture. Using some downloadable Matlab codes [22], we can cast the CLS formulation completely in the new stable basis and yield new  $L^2(\Omega)$  errors,  $5.7\text{E-}8$  and  $6.3\text{E-}9$  respectively for the two exact solutions. For the zoom-in peak, accuracy drops, which can be explained by the truncation error within the RBF-QR algorithm. Whereas, the huge accuracy improvement suggests there is a good candidate in the native space of Gaussian to approximate the full peak function. In Figure 6.8, we show the error profiles for direct Gaussian (GA) and stable basis (RBF-QR) for both solutions. When solving the zoom-in peak, GA can provide highly accurate approximation and its error stagnates for large  $n_Z$ ; adding RBF-QR (as is without modification) to the algorithm will introduce numerical instability. The situation is very different in the full peak when GA fails to converge; having RBF-QR brings convergence back in the game. We remark that the RBF-QR algorithm prefers a flatter (than our unscaled) Gaussian basis for both accuracy and efficiency. Moreover, some of the highly accurate approximations in RBF-QR suggest that it is highly possible to properly truncate the Gaussian expansion in order to couple with the CLS and WLS approaches. This is out of the scope of this work, but worth some

further investigation.

Our demonstration will end with the results of the unscaled multiquadrics (MQ) kernel in Figure 6.9. While the full peak function is giving the Gaussian kernel trouble, the multiquadrics is doing very well. What we see in this example again confirms that the condition number alone cannot be used to predetermine the accuracy or convergence rate of Kansa related methods using the GA or MQ basis.

**Conclusion.** We prove some error estimates for a constraint least-squares and a class of weighted least-squares strong-form RBF collocation formulations for solving general second order elliptic problem with nonhomogenous Dirichlet boundary condition. All analysis is carried out in Hilbert spaces so that both PDE and RBF theories apply. We show that the CLS and WLS formulations using sufficiently smooth kernels, which reproduce  $H^m(\Omega)$ , can converge at the optimal  $m - 2$  rate in  $H^2(\Omega)$ . Besides some standard smoothness assumptions for high order convergence, the sets of collocation points have to satisfy some denseness conditions for the convergence theories to hold.

We verify by numerical examples that there are many convergent formulations for  $\Omega \subset \mathbb{R}^2$  that enjoy the optimal convergence rate. We thoroughly study the numerical performance of Whittle-Matérn-Sobolev kernels in two trial spaces. The larger space that includes all boundary collocation points as trial centers is more theoretically sound (in the sense of the range of optimal weighting), whereas the small one is computationally more efficient. Taking both accuracy and efficiency into consideration, casting WLS in the small trial space with a moderate weight consistently yields competitive accuracy and numerical stability. This recommendation extends to the commonly used Gaussian and multiquadrics kernels, which do not reproduce Sobolev space as the theories require. We also provide a brief demonstration of using the RBF-QR algorithm on our formulations to hint at possibilities for future research.

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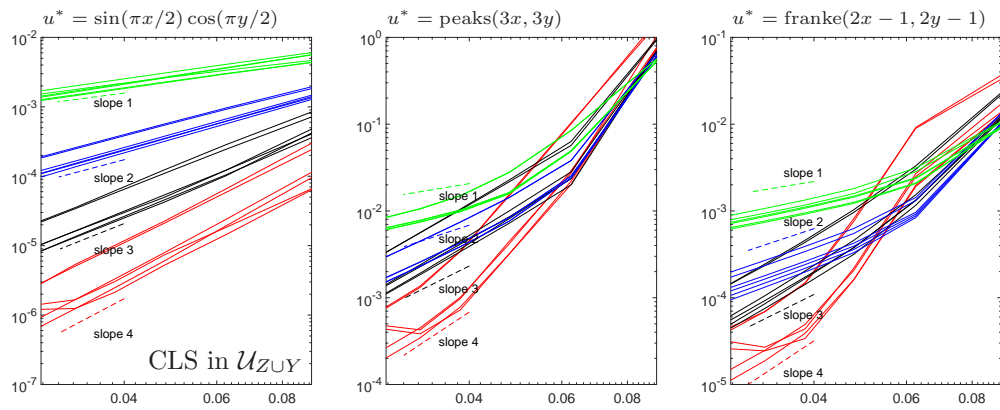


FIG. 6.3. Example 6.1:  $H^2(\Omega)$  error profiles for casting the CLS formulation in  $\mathcal{U}_{Z \cup Y}$  with Whittle-Matérn-Sobolev kernels of order  $m = 3, \dots, 6$  (green, blue, black, and red) to solve  $\Delta u = f$  with different exact solution  $u^*$ .

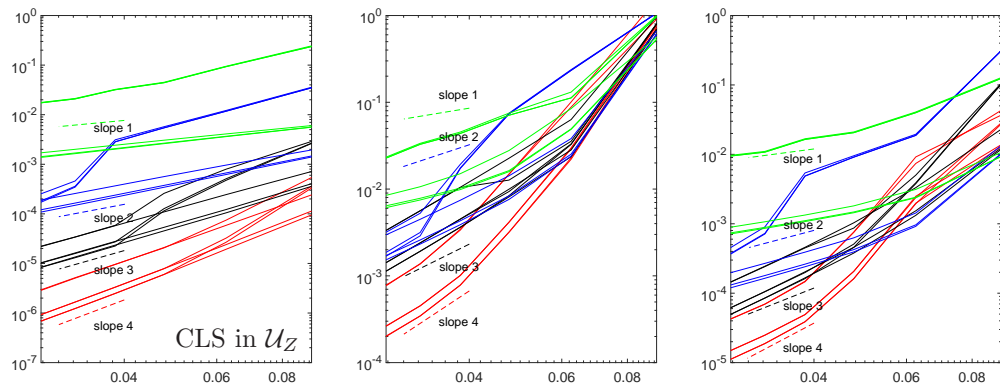


FIG. 6.4. Example 6.2:  $H^2(\Omega)$  error profiles for casting the CLS formulation in  $\mathcal{U}_Z$  to the same settings as in Figure 6.3.

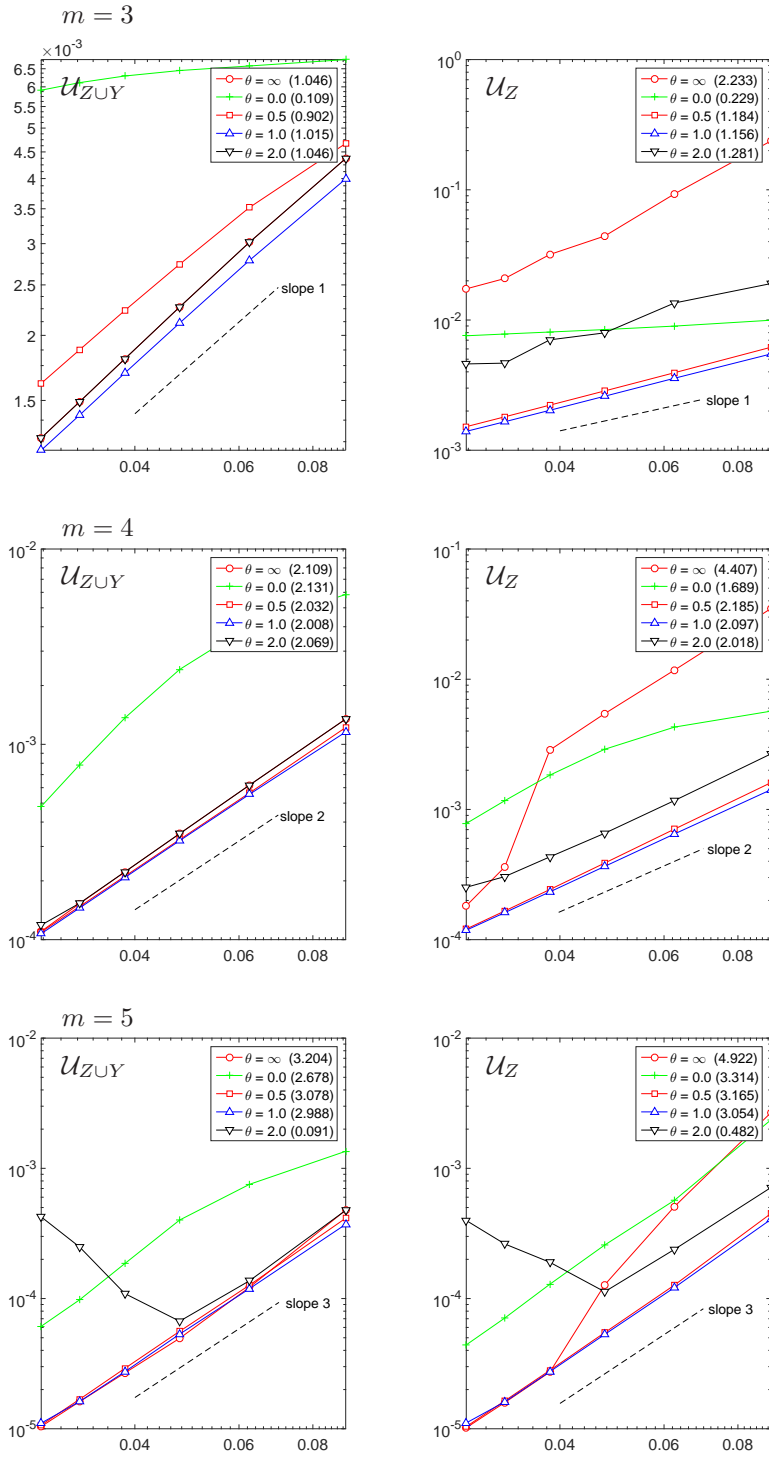


FIG. 6.5. Example 6.3:  $H^2(\Omega)$  error profiles for casting the  $WLS(\theta)$  formulation in  $U_{ZUY}$  and  $U_Z$  with Whittle-Matérn-Sobolev kernels of order  $m = 3, 4, 5$  to solve  $\Delta u = f$  with exact solution  $u^* = \sin(\pi x/2, \pi y/2)$ .



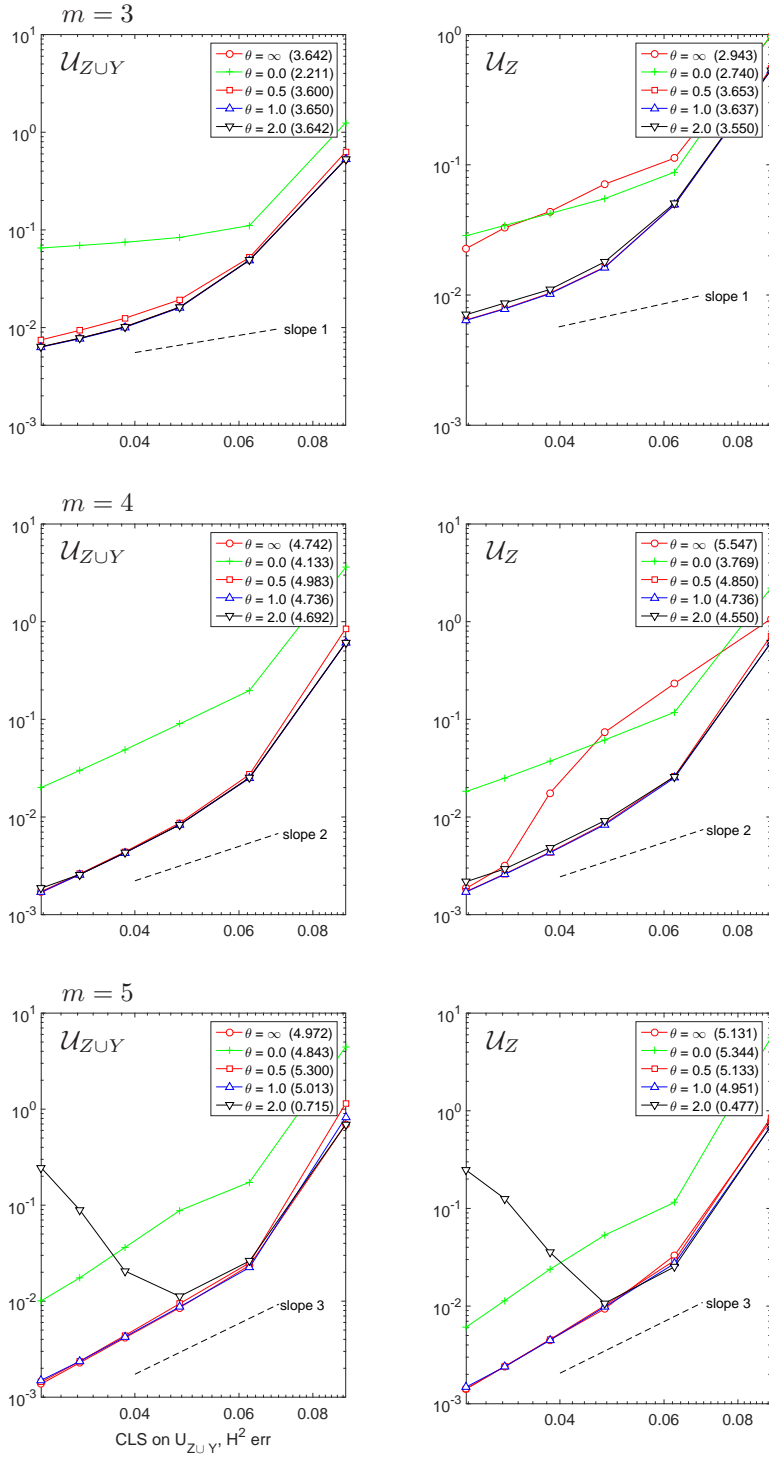


FIG. 6.6. Example 6.3:  $H^2(\Omega)$  error profiles for casting the  $WLS(\theta)$  formulation in  $U_{ZUY}$  and  $U_Z$  with Whittle-Matérn-Sobolev kernels of order  $m = 3, 4, 5$  to solve  $\Delta u = f$  with exact solution  $u^* = \text{peaks}(3x, 3y)$ .

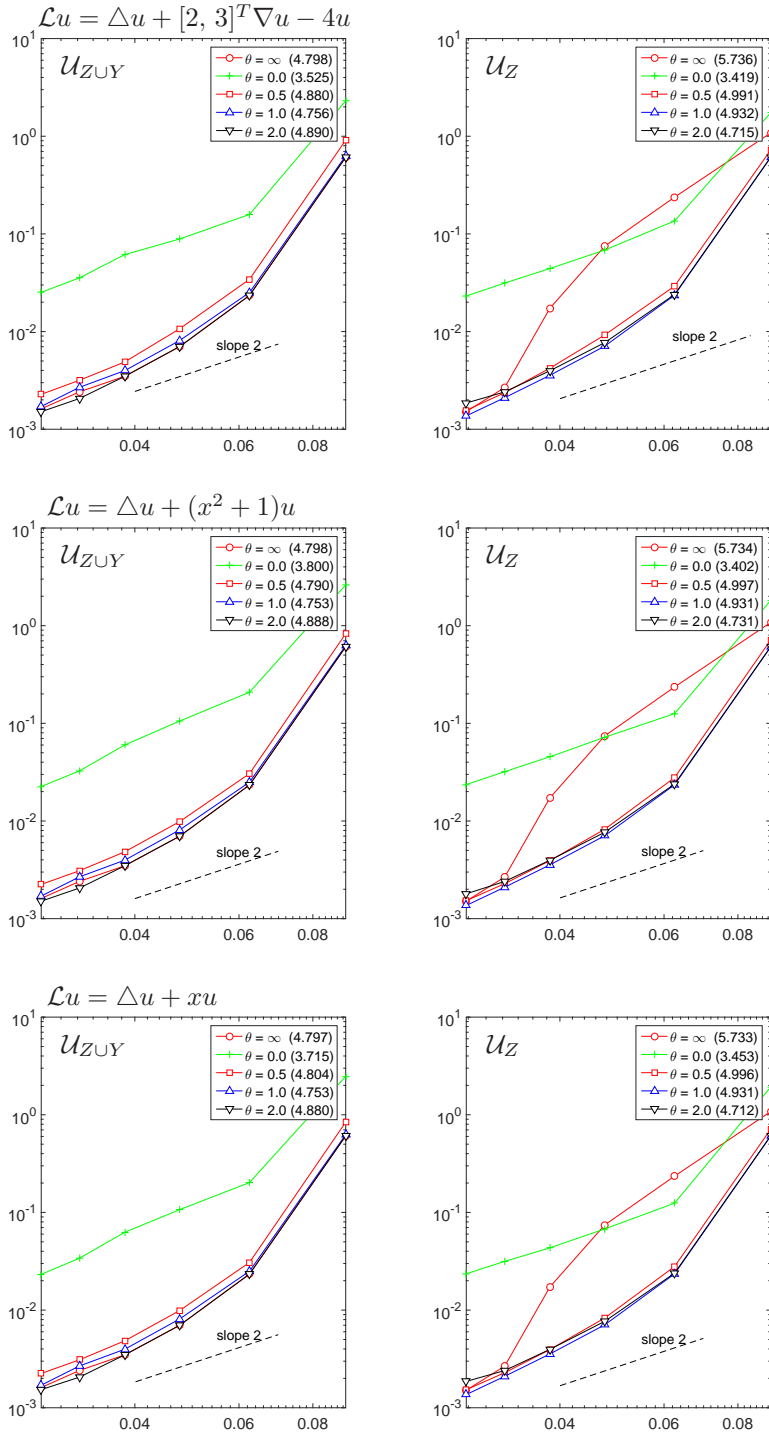


FIG. 6.7. Example 6.3:  $H^2(\Omega)$  error profiles for casting the  $WLS(\theta)$  formulation in  $\mathcal{U}_{ZUY}$  and  $\mathcal{U}_Z$  with Whittle-Matérn-Sobolev kernels of order  $m = 4$  to solve various PDEs with exact solution  $u^* = \text{peaks}(3x, 3y)$ .

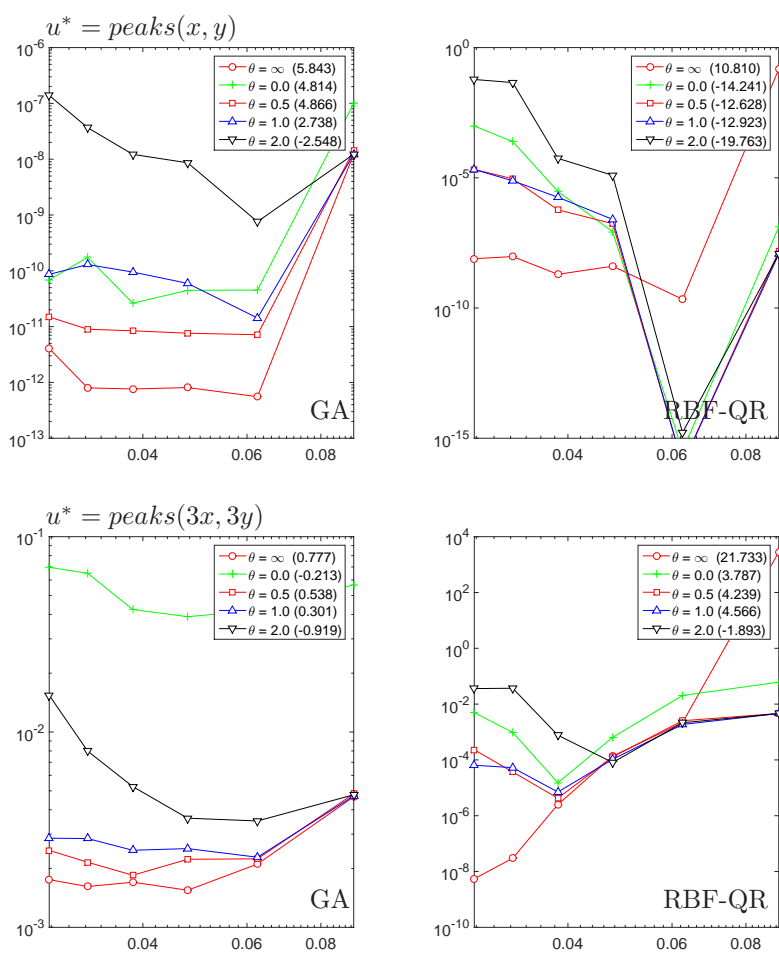


FIG. 6.8. Example 6.4:  $L^2(\Omega)$  error profiles for casting the CLS formulation in  $\mathcal{U}_Z$  with unscaled GA kernels to solve  $\Delta u = f$  with different exact solution.

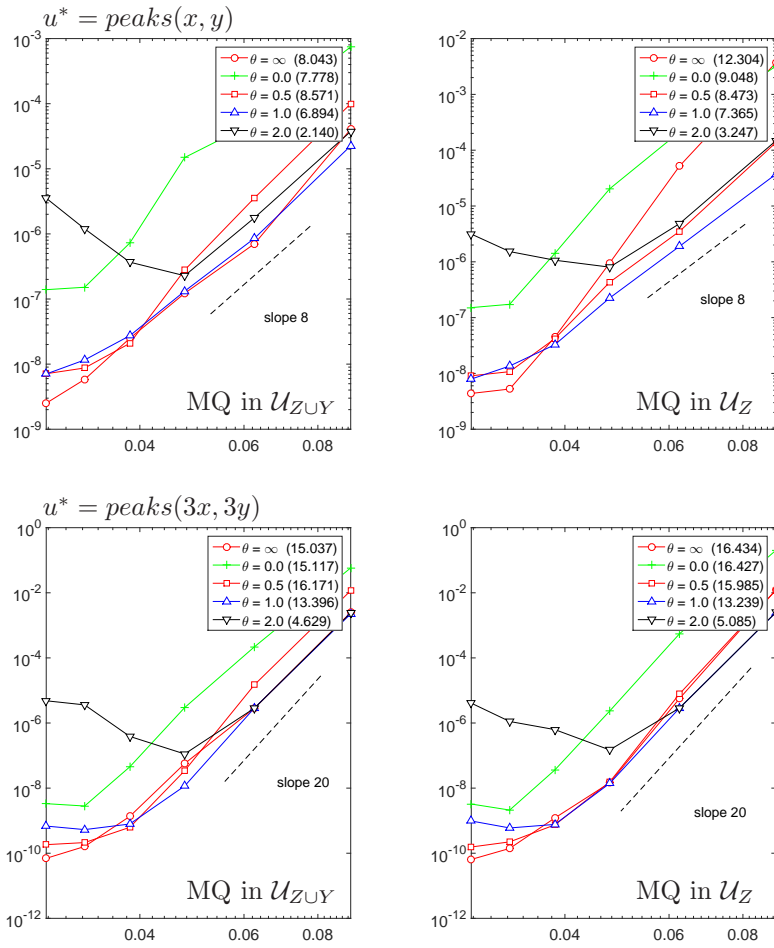


FIG. 6.9. Example 6.4: :  $L^2(\Omega)$  error profiles for casting the CLS formulation in  $\mathcal{U}_Z$  with unscaled MQ kernels to the same settings as in Figure 6.8.