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Multisymplectic Preissman scheme for the Time-Domain Maxwell's Equations *

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Abstract

From the Bridges' multisymplectic form of Maxwell's equations, we derive a multisymplectic Preissman scheme which couples two time levels for 2+1 dimensional Maxwell's equations. The scheme is proved to preserve the discrete local energy exactly. Numerical results are reported to illustrate that the scheme is effective and it can get more precise numerical solutions than Yee's scheme. Our numerical results can also indicate that the scheme keeps the discrete local energy and the global energy very well.

Keywords: Maxwell's equations; multisymplectic structure; Preissman scheme; energy conservation law ; Yee's scheme

1. Introduction

Maxwell's equations in an isotropic, homogeneous, nondispersive medium are

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \qquad &(Faraday's \quad Law),\\
\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = 0 \qquad &(Ampere's \quad Law),\\
\mathbf{B} = \mu \mathbf{H},&\\
\mathbf{D} = \varepsilon \mathbf{E}.&
\end{aligned} \tag{1.1}$$

In the absence of impressed electric charge, the magnetic induction and electric displacement fields satisfy the constraints (Gauss's Law)

$$\begin{aligned}
\nabla \cdot \mathbf{B} = 0,\\
\nabla \cdot \mathbf{D} = 0.
\end{aligned} \tag{1.2}$$

Scattering obstacle will be modeled by a spatial variation of ε and μ. In free space, ε and μ are constant, equal to their minimum values ε_0 and μ_0. The speed of light in free space is $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$.

Maxwell's equations describe the evolutions of electromagnetic fields in space and time. They have been applied to a wide range of different physical situations. In many cases, numerical methods are required to solve Maxwell's equations.

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The computations of electromagnetics have developed for a long time and many efficient methods have been given. The famous time-domain technique in computational electromagnetic (CEM) was developed by K.S.Yee [1]. The method, generally referred as the finite-difference time-domain method [2], is based on staggered central difference in space and staggered leapfrog integration in time for Cartesian coordinates.

In two dimensions, the Eqs. (1.1) decouples into two independent sets of equations, each representing a distinct polarization. We shall use our model system of equations those of the transverse magnetic polarization, where the electric field is a scalar while the magnetic field is a plane vector,

$$\begin{aligned}\frac{\partial E_z}{\partial t} &= \frac{1}{\varepsilon}(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}),\\ \frac{\partial H_x}{\partial t} &= -\frac{1}{\mu}\frac{\partial E_z}{\partial y},\\ \frac{\partial H_y}{\partial t} &= \frac{1}{\mu}\frac{\partial E_z}{\partial x}.\end{aligned} \tag{1.3}$$

The discretization of (1.3) with the staggered Yee's scheme is

$$\begin{aligned}E^{n+1}_{zi,j} &= E^{n}_{zi,j} + \frac{\Delta t}{\varepsilon \Delta x}\delta_x H^{n+1/2}_{yi,j} - \frac{\Delta t}{\varepsilon \Delta y}\delta_y H^{n+1/2}_{xi,j},\\ H^{n+1/2}_{xi,j-1/2} &= H^{n-1/2}_{xi,j-1/2} - \frac{\Delta t}{\mu \Delta y}\delta_y E^{n}_{zi,j-1/2},\\ H^{n+1/2}_{yi-1/2,j} &= H^{n-1/2}_{yi-1/2,j} + \frac{\Delta t}{\mu \Delta x}\delta_x E^{n}_{zi-1/2,j},\end{aligned} \tag{1.4}$$

where

$$\begin{aligned}\delta_x U_{i,j} &= U_{i+1/2,j} - U_{i-1/2,j},\\ \delta_y U_{i,j} &= U_{i,j+1/2} - U_{i,j-1/2}.\end{aligned}$$

Many authors have enriched Yee's method and applied it to various problems. The Yee's method is a second-order explicit scheme, so its time step must satisfy the Courant condition and must be small to reach a precise solution.

Finite volumes were introduced to CEM by Shankar *et al.* [3] by exporting methods from computational fluid dynamics. Their early work used structured grids, but lately he has turned to unstructured grids. Their main reason for doing so is the difficulty of creating a global body-conforming grid for realistic geometries, such as a complete aircraft. The Spectral-domain split-operator technique proposed in [4] is one of the many forms that results from the use of the Lie-Trotter-Suzuki product-formulas. This technique makes use of fast Fourier Transforms to compute the matrix exponentials of the displacement operators. Due to the practical interests, more and more researchers have done their important contributions on the numerical approximations to the Maxwell's equations and also on the analysis of the numerical scheme.

In 1990s, the symplectic schemes were introduced and systematically developed for the Hamiltonian systems within the framework of symplectic geometry [5-7]. The Hamiltonian nature of Maxwell's equations was revealed by Morrison [8] and Marsden and Weinstein [9] in their work on the Maxwell-Vlasov equation. The symplectic schemes which preserve the Hamiltonian nature of Maxwell's equations were reported to have higher performance over the non-symplecitic schemes [10]. Recently, Marsden, Patrick, Shkoller [11], Bridges and Reich [12] proposed the concept of multisymplectic partial differential equations (PDEs) and multisymplectic schemes which can be viewed as the generalization of symplectic schemes. The multisymplectic schemes have been applied successfully to lots of important equations such as the nonlinear wave equation [12, 13], the nonlinear schrödinger equation [14, 15], the Korteweg-de Vries equation [16, 17], the Zakharov-Kuznetsov equation [18] and Kadomtsev-Petviashvili equation [19] and so on. With regard to details, please refer to the published survey by Bridges and Reich [20] and the references therein. By many computational experiments and theoretical analysis, the multisymplectic schemes were shown to be much superior to other standard methods in the performance of numerical stability in long time computation.

The main purpose of this paper is to check whether the multisymplectic Preissman scheme could be applied to integrate Maxwell's equations and still has good numerical performance. Through numerical experiments, it would be no difficult task to find the virtues of the multisymplectic Preissman scheme.

This paper is organized as follows: In section 2, we review multisymplectic Bridges' form for Maxwell's equations. Multisymplectic Preissman scheme which couples two time levels is derived based on Bridges' form in section 3. In section 4, we prove our multisymplectic Preissman scheme satisfies the conservation law of the discrete local energy exactly. Based on this, we obtain the discrete local energy of our scheme in the same section. In section 5, numerical results are presented to indicate the merits of the multisymplectic Preissman scheme and we finish the paper with conclusion remarks in section 6.

2. Multisymplectic formulation for Maxwell's equations

Introduce two vector functions $\mathbf{U}$ and $\mathbf{V}$ satisfying $\mathbf{U}_t = \mathbf{E}$ and $\mathbf{V}_t = \mathbf{H}$, respectively, then the Lagrangian density for Eqs. (1.1) can be written as

$$L = \frac{1}{2}\mu\langle \mathbf{V}_t, \mathbf{V}_t\rangle + \frac{1}{2}\langle \mathbf{V}_t, \nabla\times\mathbf{U}\rangle + \frac{1}{2}\varepsilon\langle \mathbf{U}_t, \mathbf{U}_t\rangle - \frac{1}{2}\langle \mathbf{U}_t, \nabla\times\mathbf{V}\rangle, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ represents the standard inner production of vector space. For details, we can refer to [21]. The generalized conjugate momentums can be derived by covariant Legendre transform correspondingly,

$$\begin{aligned}
\mathbf{P} &= \frac{\partial \mathbf{L}}{\partial \mathbf{V}_t} = \mu\mathbf{V}_t + \frac{1}{2}\nabla\times\mathbf{U}, \quad \frac{\partial L}{\partial \nabla\times\mathbf{V}} = -\frac{1}{2}\mathbf{U}_t, \\
\mathbf{Q} &= \frac{\partial L}{\partial \mathbf{U}_t} = \varepsilon\mathbf{U}_t - \frac{1}{2}\nabla\times\mathbf{V}, \quad \frac{\partial L}{\partial \nabla\times\mathbf{U}} = \frac{1}{2}\mathbf{V}_t,
\end{aligned} \tag{2.2}$$

further the covariant Hamiltonian by

$$\begin{aligned}
S &= \langle \mathbf{P}, \mathbf{V}_t\rangle + \langle \mathbf{Q}, \mathbf{U}_t\rangle + \langle\frac{\partial L}{\partial \nabla\times\mathbf{V}}, \nabla\times\mathbf{V}\rangle + \langle\frac{\partial L}{\partial \nabla\times\mathbf{U}}, \nabla\times\mathbf{U}\rangle - L \\
&= \langle \mathbf{P}, \mathbf{H}\rangle + \langle \mathbf{Q}, \mathbf{E}\rangle - \frac{1}{2}\mu\langle \mathbf{H}, \mathbf{H}\rangle - \frac{1}{2}\varepsilon\langle \mathbf{E}, \mathbf{E}\rangle.
\end{aligned} \tag{2.3}$$

Set $Z = [\mathbf{H}, \mathbf{E}, \mathbf{V}, \mathbf{U}, \mathbf{P}, \mathbf{Q}]^T$, then Eqs. (1.1) are transformed into the following form

$$\begin{aligned}
\frac{1}{2}\nabla \times \mathbf{U} &= \mathbf{P} - \mu\mathbf{H}, \\
-\frac{1}{2}\nabla \times \mathbf{V} &= \mathbf{Q} - \varepsilon\mathbf{E}, \\
-\mathbf{P}_t - \frac{1}{2}\nabla \times \mathbf{E} &= 0, \\
-\mathbf{Q}_t + \frac{1}{2}\nabla \times \mathbf{H} &= 0, \\
\mathbf{V}_t &= \mathbf{H}, \\
\mathbf{U}_t &= \mathbf{E}.
\end{aligned} \tag{2.4}$$

The above equations can be organized into Bridges' multisymplectic form as

$$\mathbf{M}Z_t + N\nabla \times Z = \nabla_Z S(Z), \tag{2.5}$$

where

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{pmatrix},$$

I is the identity element belonging to $R^{3\times 3}$. The rotation action $\nabla \times Z$ denotes $[\nabla \times \mathbf{H}, \nabla \times \mathbf{E}, \nabla \times \mathbf{V}, \nabla \times \mathbf{U}, \nabla \times \mathbf{P}, \nabla \times \mathbf{Q}]^T$ and

$$N\nabla \times Z = \mathbf{F}Z_x + \mathbf{L}Z_y + \mathbf{W}Z_z, \tag{2.6}$$

where

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}R_1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}R_1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}R_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}R_2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}R_2 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}R_2 & 0 & 0 & 0 & 0 \\ \frac{1}{2}R_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{W} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}R_3 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}R_3 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}R_3 & 0 & 0 & 0 & 0 \\ \frac{1}{2}R_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus we easily find that matrices $\mathbf{M}$, $\mathbf{F}$, $\mathbf{L}$ and $\mathbf{W}$ ($\in R^{18\times 18}$) are skew-symmetric. Additionally, the rotation operator may be simplified as $\nabla\times = R_1\frac{\partial}{\partial x} + R_2\frac{\partial}{\partial y} + R_3\frac{\partial}{\partial z}$.

The representation Eq. (2.5) is simplified expression in vector form, its complete extension is the multisymplectic Hamiltonian system which was first introduced by Bridges and Derks in [22]. In two dimensions, Eq. (2.5) can be written as follows

$$\mathbf{M}Z_t + \mathbf{F}Z_x + \mathbf{L}Z_y = \nabla_Z S(Z). \tag{2.7}$$

Because matrices $\mathbf{M}$, $\mathbf{F}$, $\mathbf{L}$ are skew-symmetric and $S : R^n \longrightarrow R$ is a smooth function of the state variable $Z(x, y, t)$, it can be shown that the multisymplectic PDEs [Eq. (2.7)] satisfies the following multisymplectic conservation law according to Bridges' theory

$$\frac{\partial \omega}{\partial t} + \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} = 0, \tag{2.8}$$

where

$$\omega = \frac{1}{2}(dZ \wedge \mathbf{M}dZ), \qquad \varphi = \frac{1}{2}(dZ \wedge \mathbf{F}dZ), \qquad \psi = \frac{1}{2}(dZ \wedge \mathbf{L}dZ). \tag{2.9}$$

3. The Preissman scheme for Maxwell's equations

Because Eq. (2.7) preserves the multisymplectic conservation law, naturally, when discreting Hamiltonian PDEs [Eq. (2.7)] by a numerical scheme, we also expect that the multisymplectic conservation law should be preserved. Bridges and Reich defined a numerical scheme as a multisymplectic scheme if the scheme preserves a discrete multisymplectic conservation law[12]. Especially, if we discrete the Hamiltonian PDEs [Eq. (2.7)] as follows

$$\mathbf{M}\partial_t^{i,j,k} Z_{i,j}^k + \mathbf{F}\partial_x^{i,j,k} Z_{ij}^k + \mathbf{L}\partial_y^{i,j,k} Z_{ij}^k = \nabla_Z S(Z_{ij}^k), \tag{3.1}$$

where $Z_{ij}^k = Z(x_i, y_j, t_k)$, $\partial_t^{i,j,k}$, $\partial_x^{i,j,k}$ and $\partial_y^{i,j,k}$ are the discretizations of the derivatives ∂_t, ∂_x and ∂_y respectively, the scheme is multisymplectic provided that it can preserve the following discrete conservation law

$$\partial_t^{i,j,k} \omega_{i,j}^k + \partial_x^{i,j,k} \varphi_{i,j}^k + \partial_y^{i,j,k} \psi_{i,j}^k = 0, \tag{3.2}$$

where $\omega_{i,j}^k = dZ_{i,j}^k \wedge \mathbf{M}dZ_{i,j}^k$, $\varphi_{i,j}^k = dZ_{ij}^k \wedge \mathbf{F}dZ_{i,j}^k$, $\psi_{i,j}^k = dZ_{i,j}^k \wedge \mathbf{L}dZ_{i,j}^k$. In the conventional schemes, Euler-box scheme and Preissman scheme are shown to be multisymplectic [12, 23].

Set t_k, $k = 1, 2, ...;$ T, $i = 1, 2, ..., M;$ and y_j, $j = 1, 2, ..., N$ be the regular grids of the integral domain. $Z_{i,j}^k$ is an approximation to $Z(x_i, y_j, t_k)$, $\Delta t = t_{k+1} - t_k$ is the time step, $\Delta x = x_{i+1} - x_i$ is the x-direction step, and $\Delta y = y_{j+1} - y_j$ is the y-direction step.

Applying the symplectic midpoint rule to both the time and space derivatives in Eq. (2.7) yields the Preissman scheme

$$\mathbf{M}\frac{Z_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1} - Z_{i+\frac{1}{2},j+\frac{1}{2}}^{k}}{\Delta t} + \mathbf{F}\frac{Z_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}} - Z_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}}{\Delta x} + \mathbf{L}\frac{Z_{i+\frac{1}{2},j+1}^{k+\frac{1}{2}} - Z_{i+\frac{1}{2},j}^{k+\frac{1}{2}}}{\Delta y} = \nabla_Z S(Z_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}), \tag{3.3}$$

where

$$\begin{aligned} Z_{i,j+\frac{1}{2}}^{k+\frac{1}{2}} &= \frac{1}{4}(Z_{i,j}^k + Z_{i,j}^{k+1} + Z_{i,j+1}^k + Z_{i,j+1}^{k+1}), \quad Z_{i+\frac{1}{2},j}^{k+\frac{1}{2}} = \frac{1}{4}(Z_{i,j}^k + Z_{i,j}^{k+1} + Z_{i+1,j}^k + Z_{i+1,j}^{k+1}), \\ Z_{i+\frac{1}{2},j+\frac{1}{2}}^k &= \frac{1}{4}(Z_{i,j}^k + Z_{i+1,j}^k + Z_{i,j+1}^k + Z_{i+1,j+1}^k). \end{aligned} \tag{3.4}$$

Theorem 3.1. *Pressiman scheme Eq. (3.3) is multisymplectic with the following discrete multisymplectic conservation law*

$$\frac{\omega^{k+1}_{i+\frac{1}{2},j+\frac{1}{2}} - \omega^{k}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t} + \frac{\varphi^{k+\frac{1}{2}}_{i+1,j+\frac{1}{2}} - \varphi^{k+\frac{1}{2}}_{i,j+\frac{1}{2}}}{\Delta x} + \frac{\psi^{k+\frac{1}{2}}_{i+\frac{1}{2},j+1} - \psi^{k+\frac{1}{2}}_{i+\frac{1}{2},j}}{\Delta y} = 0, \tag{3.5}$$

where

$$\begin{aligned} \varphi^{k+\frac{1}{2}}_{i,j+\frac{1}{2}} &= \frac{1}{2}(dZ^{k+\frac{1}{2}}_{i,j+\frac{1}{2}} \wedge \mathbf{F} dZ^{k+\frac{1}{2}}_{i,j+\frac{1}{2}}), \qquad \psi^{k+\frac{1}{2}}_{i+\frac{1}{2},j} = \frac{1}{2}(dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j} \wedge \mathbf{L} dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j}), \\ \omega^{k}_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{2}(dZ^{k}_{i+\frac{1}{2},j+\frac{1}{2}} \wedge \mathbf{M} dZ^{k}_{i+\frac{1}{2},j+\frac{1}{2}}). \end{aligned}$$

Proof. We can obtain the discrete variational equation associated with Eq. (3.3) given by

$$\mathbf{M}\frac{dZ^{k+1}_{i+\frac{1}{2},j+\frac{1}{2}} - dZ^{k}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t} + \mathbf{F}\frac{dZ^{k+\frac{1}{2}}_{i+1,j+\frac{1}{2}} - dZ^{k+\frac{1}{2}}_{i,j+\frac{1}{2}}}{\Delta x} + \mathbf{L}\frac{dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j+1} - dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j}}{\Delta y} = S_{ZZ} dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{3.6}$$

Taking the wedge product of Eq. (3.6) with $dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}}$, then we note that the right-hand term vanishes, since S_{ZZ} is symmetric. We know

$$\begin{aligned} & dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \wedge \mathbf{M}(dZ^{k+1}_{i+\frac{1}{2},j+\frac{1}{2}} - dZ^{k}_{i+\frac{1}{2},j+\frac{1}{2}}) \\ =& \frac{1}{2}(dZ^{k+1}_{i+\frac{1}{2},j+\frac{1}{2}} + dZ^{k}_{i+\frac{1}{2},j+\frac{1}{2}}) \wedge \mathbf{M}(dZ^{k+1}_{i+\frac{1}{2},j+\frac{1}{2}} - dZ^{k}_{i+\frac{1}{2},j+\frac{1}{2}}) \\ =& \frac{1}{2}(dZ^{k+1}_{i+\frac{1}{2},j+\frac{1}{2}} \wedge \mathbf{M} dZ^{k+1}_{i+\frac{1}{2},j+\frac{1}{2}} - dZ^{k}_{i+\frac{1}{2},j+\frac{1}{2}} \wedge \mathbf{M} dZ^{k}_{i+\frac{1}{2},j+\frac{1}{2}}) \\ =& \omega^{k+1}_{i+\frac{1}{2},j+\frac{1}{2}} - \omega^{k}_{i+\frac{1}{2},j+\frac{1}{2}}, \end{aligned}$$

similarly, we can derive

$$\begin{aligned} dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \wedge \mathbf{F}(dZ^{k+\frac{1}{2}}_{i+1,j+\frac{1}{2}} - dZ^{k+\frac{1}{2}}_{i,j+\frac{1}{2}}) &= \varphi^{k+\frac{1}{2}}_{i+1,j+\frac{1}{2}} - \varphi^{k+\frac{1}{2}}_{i,j+\frac{1}{2}}, \\ dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2}} \wedge \mathbf{L}(dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j+1} - dZ^{k+\frac{1}{2}}_{i+\frac{1}{2},j}) &= \psi^{k+\frac{1}{2}}_{i+\frac{1}{2},j+1} - \psi^{k+\frac{1}{2}}_{i+\frac{1}{2},j}. \end{aligned}$$

These imply

$$\frac{\omega^{k+1}_{i+\frac{1}{2},j+\frac{1}{2}} - \omega^{k}_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t} + \frac{\varphi^{k+\frac{1}{2}}_{i+1,j+\frac{1}{2}} - \varphi^{k+\frac{1}{2}}_{i,j+\frac{1}{2}}}{\Delta x} + \frac{\psi^{k+\frac{1}{2}}_{i+\frac{1}{2},j+1} - \psi^{k+\frac{1}{2}}_{i+\frac{1}{2},j}}{\Delta y} = 0. \quad \square$$

In terms of the finite difference operators

$$\begin{aligned} M_t Z^k_{i,j} &= \frac{1}{2}(Z^{k+1}_{i,j} + Z^k_{i,j}), \quad M_x Z^k_{i,j} = \frac{1}{2}(Z^k_{i+1,j} + Z^k_{i,j}), \quad M_y Z^k_{i,j} = \frac{1}{2}(Z^k_{i,j+1} + Z^k_{i,j}), \\ \delta_t Z^k_{i,j} &= \frac{1}{\Delta t}(Z^{k+1}_{i,j} - Z^k_{i,j}), \quad \delta_x Z^k_{i,j} = \frac{1}{\Delta x}(Z^k_{i+1,j} - Z^k_{i,j}), \quad \delta_y Z^k_{i,j} = \frac{1}{\Delta y}(Z^k_{i,j+1} - Z^k_{i,j}), \end{aligned} \tag{3.7}$$

Eq. (3.3) is

$$\mathbf{M}(\delta_t M_x M_y Z^k_{i,j}) + \mathbf{F}(\delta_x M_y M_t Z^k_{i,j}) + \mathbf{L}(\delta_y M_x M_t Z^k_{i,j}) = \nabla_Z S(M_t M_x M_y Z^k_{i,j}). \tag{3.8}$$

Substituting matrices **M**, **F**, **L** and vector Z into Eq. (3.8), we can derive

$$\begin{aligned}
\frac{1}{2}R_1(\delta_x M_y M_t \mathbf{U}_{i,j}^k) + \frac{1}{2}R_2(\delta_y M_x M_t \mathbf{U}_{i,j}^k) &= M_x M_y M_t(\mathbf{P}_{i,j}^k - \mu \mathbf{H}_{i,j}^k),\\
-\frac{1}{2}R_1(\delta_x M_y M_t \mathbf{V}_{i,j}^k) - \frac{1}{2}R_2(\delta_y M_x M_t \mathbf{V}_{i,j}^k) &= M_x M_y M_t(\mathbf{Q}_{i,j}^k - \varepsilon \mathbf{E}_{i,j}^k),\\
-\delta_t M_x M_y \mathbf{P}_{i,j}^k - \frac{1}{2}R_1(\delta_x M_y M_t \mathbf{E}_{i,j}^k) - \frac{1}{2}R_2(\delta_y M_x M_t \mathbf{E}_{i,j}^k) &= 0,\\
-\delta_t M_x M_y \mathbf{Q}_{i,j}^k + \frac{1}{2}R_1(\delta_x M_y M_t \mathbf{H}_{i,j}^k) + \frac{1}{2}R_2(\delta_y M_x M_t \mathbf{H}_{i,j}^k) &= 0,\\
\delta_t M_x M_y \mathbf{V}_{i,j}^k &= M_x M_y M_t \mathbf{H}_{i,j}^k,\\
\delta_t M_x M_y \mathbf{U}_{i,j}^k &= M_x M_y M_t \mathbf{E}_{i,j}^k.
\end{aligned} \tag{3.9}$$

Inspired by the technique used in the Ref.[16], the first equation in Eqs. (3.9) contains no time derivatives, so it can be more accurately discretized by omitting the time average M_t. Similarly, one can omit M_t of the second equation inEqs. (3.9). Then, Eqs. (3.9) can be written as

$$\begin{aligned}
\frac{1}{2}R_1(\delta_x M_y \mathbf{U}_{i,j}^k) + \frac{1}{2}R_2(\delta_y M_x \mathbf{U}_{i,j}^k) &= M_x M_y(\mathbf{P}_{i,j}^k - \mu \mathbf{H}_{i,j}^k),\\
-\frac{1}{2}R_1(\delta_x M_y \mathbf{V}_{i,j}^k) - \frac{1}{2}R_2(\delta_y M_x \mathbf{V}_{i,j}^k) &= M_x M_y(\mathbf{Q}_{i,j}^k - \varepsilon \mathbf{E}_{i,j}^k),\\
-\delta_t M_x M_y \mathbf{P}_{i,j}^k - \frac{1}{2}R_1(\delta_x M_y M_t \mathbf{E}_{i,j}^k) - \frac{1}{2}R_2(\delta_y M_x M_t \mathbf{E}_{i,j}^k) &= 0,\\
-\delta_t M_x M_y \mathbf{Q}_{i,j}^k + \frac{1}{2}R_1(\delta_x M_y M_t \mathbf{H}_{i,j}^k) + \frac{1}{2}R_2(\delta_y M_x M_t \mathbf{H}_{i,j}^k) &= 0,\\
\delta_t M_x M_y \mathbf{V}_{i,j}^k &= M_x M_y M_t \mathbf{H}_{i,j}^k,\\
\delta_t M_x M_y \mathbf{U}_{i,j}^k &= M_x M_y M_t \mathbf{E}_{i,j}^k.
\end{aligned} \tag{3.10}$$

Eqs. (3.10) is the concrete form of the Preissman scheme. In above formulas, applying $\delta_t M_x M_y$ to the first and the second equations, $M_x M_y$ to the third and fourth equations, and noting that the mutual commutability of the operators, gives

$$\begin{aligned}
\frac{1}{2}R_1(\delta_t \delta_x M_x M_y^2 \mathbf{U}_{i,j}^k) + \frac{1}{2}R_2(\delta_t \delta_y M_x^2 M_y \mathbf{U}_{i,j}^k) &= \delta_t M_x^2 M_y^2(\mathbf{P}_{i,j}^k - \mu \mathbf{H}_{i,j}^k),\\
-\frac{1}{2}R_1(\delta_t \delta_x M_x M_y^2 \mathbf{V}_{i,j}^k) - \frac{1}{2}R_2(\delta_t \delta_y M_x^2 M_y \mathbf{V}_{i,j}^k) &= \delta_t M_x^2 M_y^2(\mathbf{Q}_{i,j}^k - \varepsilon \mathbf{E}_{i,j}^k),\\
-\delta_t M_x^2 M_y^2 \mathbf{P}_{i,j}^k - \frac{1}{2}R_1(\delta_x M_x M_y^2 M_t \mathbf{E}_{i,j}^k) - \frac{1}{2}R_2(\delta_y M_y M_x^2 M_t \mathbf{E}_{i,j}^k) &= 0,\\
-\delta_t M_x^2 M_y^2 \mathbf{Q}_{i,j}^k + \frac{1}{2}R_1(\delta_x M_x M_y^2 M_t \mathbf{H}_{i,j}^k) + \frac{1}{2}R_2(\delta_y M_y M_x^2 M_t \mathbf{H}_{i,j}^k) &= 0,\\
\delta_t M_x M_y \mathbf{V}_{i,j}^k &= M_x M_y M_t \mathbf{H}_{i,j}^k,\\
\delta_t M_x M_y \mathbf{U}_{i,j}^k &= M_x M_y M_t \mathbf{E}_{i,j}^k.
\end{aligned} \tag{3.11}$$

Further eliminating the auxiliary variables $\mathbf{P}_{i,j}^k$, $\mathbf{Q}_{i,j}^k$, $\mathbf{U}_{i,j}^k$ and $\mathbf{V}_{i,j}^k$ in Eqs. (3.11), we derive formulas that only contain $E_{i,j}^k$ and $H_{i,j}^k$,

$$\begin{aligned}
R_1(\delta_x M_y^2 M_t M_x \mathbf{E}_{i,j}^k) + R_2(\delta_y M_x^2 M_t M_y \mathbf{E}_{i,j}^k) &= -\delta_t M_x^2 M_y^2(\mu \mathbf{H}_{i,j}^k),\\
R_1(\delta_x M_x M_y^2 M_t \mathbf{H}_{i,j}^k) + R_2(\delta_y M_y M_x^2 M_t \mathbf{H}_{i,j}^k) &= \delta_t M_x^2 M_y^2(\varepsilon \mathbf{E}_{i,j}^k).
\end{aligned} \tag{3.12}$$

Substituting $\mathbf{E}^k_{i,j} = [0,0,E^k_{zi,j}]^T$ and $\mathbf{H}^k_{i,j} = [H^k_{xi,j}, H^k_{yi,j}, 0]^T$ into Eqs. (3.12), we have

$$\begin{aligned} \delta_x M_y^2 M_t M_x E^k_{zi,j} &= \delta_t M_x^2 M_y^2 (\mu H^k_{yi,j}), \\ \delta_y M_x^2 M_t M_y E^k_{zi,j} &= -\delta_t M_x^2 M_y^2 (\mu H^k_{xi,j}), \\ -\delta_y M_x^2 M_t M_y H^k_{xi,j} + \delta_x M_y^2 M_t M_x H^k_{yi,j} &= \delta_t M_x^2 M_y^2 (\varepsilon E^k_{zi,j}). \end{aligned} \tag{3.13}$$

Thus we obtain, by recasting Eqs. (3.13) into grid points, schemes coupled two time level for Maxwell's equations as

$$\begin{aligned} &\frac{\Delta t}{\Delta x}(-E^{k+1}_{zi+2,j+2} - 2E^{k+1}_{zi+2,j+1} - E^{k+1}_{zi+2,j} + E^{k+1}_{zi,j+2} + 2E^{k+1}_{zi,j+1} + E^{k+1}_{zi,j}) + \mu(H^{k+1}_{yi+2,j+2} + \\ &2H^{k+1}_{yi+2,j+1} + 2H^{k+1}_{yi+1,j+2} + 4H^{k+1}_{yi+1,j+1} + H^{k+1}_{yi,j+2} + 2H^{k+1}_{yi,j+1} + H^{k+1}_{yi+2,j} + 2H^{k+1}_{yi+1,j} + H^{k+1}_{yi,j}) \\ =&\frac{\Delta t}{\Delta x}(E^k_{zi+2,j+2} + 2E^k_{zi+2,j+1} + E^k_{zi+2,j} - E^k_{zi,j+2} - 2E^k_{zi,j+1} - E^k_{zi,j}) + \mu(H^k_{yi+2,j+2} + 2H^k_{yi+2,j+1} \\ &+ 2H^k_{yi+1,j+2} + 4H^k_{yi+1,j+1} + H^k_{yi,j+2} + 2H^k_{yi,j+1} + H^k_{yi+2,j} + 2H^k_{yi+1,j} + H^k_{yi,j}), \end{aligned} \tag{3.14a}$$

$$\begin{aligned} &\frac{\Delta t}{\Delta y}(E^{k+1}_{zi+2,j+2} + 2E^{k+1}_{zi+1,j+2} + E^{k+1}_{zi,j+2} - E^{k+1}_{zi+2,j} - 2E^{k+1}_{zi+1,j} - E^{k+1}_{zi,j}) + \mu(H^{k+1}_{xi+2,j+2} + \\ &2H^{k+1}_{xi+2,j+1} + 2H^{k+1}_{xi+1,j+2} + 4H^{k+1}_{xi+1,j+1} + H^{k+1}_{xi,j+2} + 2H^{k+1}_{xi,j+1} + H^{k+1}_{xi+2,j} + 2H^{k+1}_{xi+1,j} + H^{k+1}_{xi,j}) \\ =&\frac{\Delta t}{\Delta y}(-E^k_{zi+2,j+2} - 2E^k_{zi+1,j+2} - E^k_{zi,j+2} + E^k_{zi+2,j} + 2E^k_{zi+1,j} + E^k_{zi,j}) + \mu(H^k_{xi+2,j+2} + 2H^k_{xi+2,j+1} \\ &+ 2H^k_{xi+1,j+2} + 4H^k_{xi+1,j+1} + H^k_{xi,j+2} + 2H^k_{xi,j+1} + H^k_{xi+2,j} + 2H^k_{xi+1,j} + H^k_{xi,j}), \end{aligned} \tag{3.14b}$$

$$\begin{aligned} &\frac{\Delta t}{\Delta y}(-H^{k+1}_{xi+2,j+2} - 2H^{k+1}_{xi+1,j+2} - H^{k+1}_{xi,j+2} + H^{k+1}_{xi+2,j} + 2H^{k+1}_{xi+1,j} + H^{k+1}_{xi,j}) + \frac{\Delta t}{\Delta x}(H^{k+1}_{yi+2,j+2} \\ &+ 2H^{k+1}_{yi+2,j+1} + H^{k+1}_{yi+2,j} - H^{k+1}_{yi,j+2} - 2H^{k+1}_{yi,j+1} - H^{k+1}_{yi,j}) - \varepsilon(E^{k+1}_{zi+2,j+2} + 2E^{k+1}_{zi+2,j+1} \\ &+ 2E^{k+1}_{zi+1,j+2} + 4E^{k+1}_{zi+1,j+1} + E^{k+1}_{zi,j+2} + 2E^{k+1}_{zi,j+1} + E^{k+1}_{zi+2,j} + 2E^{k+1}_{zi+1,j} + E^{k+1}_{zi,j}) \\ =&\frac{\Delta t}{\Delta y}(H^k_{xi+2,j+2} + 2H^k_{xi+1,j+2} + H^k_{xi,j+2} - H^k_{xi+2,j} - 2H^k_{xi+1,j} - H^k_{xi,j}) - \frac{\Delta t}{\Delta x}(H^k_{yi+2,j+2} \\ &+ 2H^k_{yi+2,j+1} + H^k_{yi+2,j} - H^k_{yi,j+2} - 2H^k_{yi,j+1} - H^k_{yi,j}) - \varepsilon(E^k_{zi+2,j+2} + 2E^k_{zi+2,j+1} \\ &+ 2E^k_{zi+1,j+2} + 4E^k_{zi+1,j+1} + E^k_{zi,j+2} + 2E^k_{zi,j+1} + E^k_{zi+2,j} + 2E^k_{zi+1,j} + E^k_{zi,j}). \end{aligned} \tag{3.14c}$$

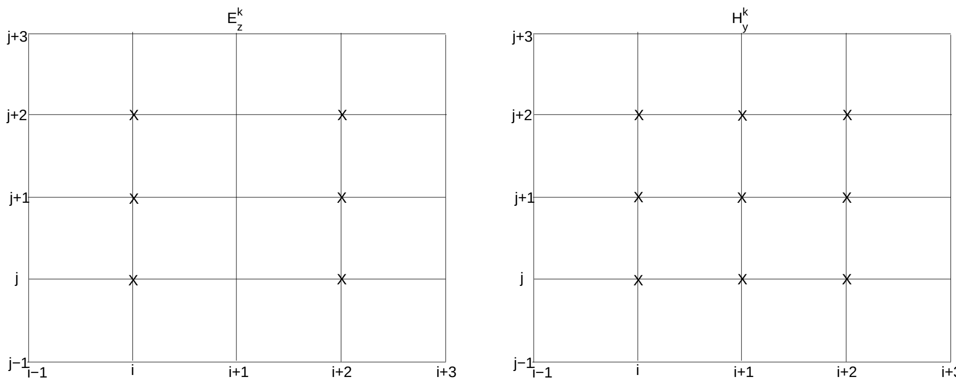

Fig.1 Left: shows the grid points of E_z used by the first time level in Eq. (3.14a). Right: shows the grid points of H_y used by the k-th time level in Eq. (3.14a).

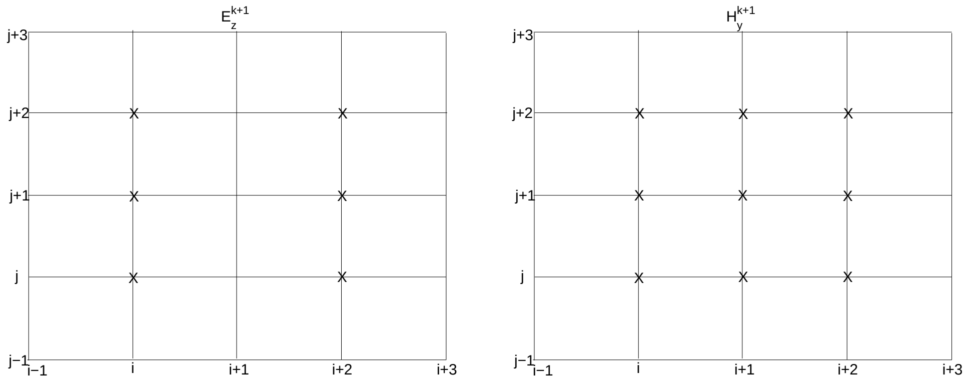

Fig.2 These two sketch maps show the mesh points of E_z and H_y used by the $(k+1)$-th time level in Eq.(3.14a), respectively.

The mesh points of E_z, H_y and H_x which are used in Eq. (3.14b) and Eq. (3.14c) can be similarly drawn by figures.

4. Conservative properties of the schemes (3.14)

Taking the inner product of Eq. (2.7) with Z_t^T, then we obtain the energy conservation law

$$\frac{\partial \chi}{\partial t} + \frac{\partial \zeta}{\partial x} + \frac{\partial \mho}{\partial y} = 0, \tag{4.1}$$

where

$$\chi = S(Z) - \frac{1}{2}Z^T\mathbf{F}Z_x - \frac{1}{2}Z^T\mathbf{L}Z_y, \;\; \zeta = \frac{1}{2}Z^T\mathbf{F}Z_t, \;\; \mho = \frac{1}{2}Z^T\mathbf{L}Z_t. \tag{4.2}$$

Substitute $S(Z)$, Z, $\mathbf{F}$ and $\mathbf{L}$ into Eq. (4.1), and eliminating the auxiliary variables $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{U}$, and $\mathbf{V}$, we get conservation law of the local energy for Maxwell's equations as

$$\frac{\partial}{\partial t}[\frac{1}{2}(\varepsilon\langle \mathbf{E}, \mathbf{E}\rangle + \mu\langle \mathbf{H}, \mathbf{H}\rangle)] + \langle \nabla \times \mathbf{E}, \mathbf{H}\rangle - \langle \nabla \times \mathbf{H}, \mathbf{E}\rangle = 0 \tag{4.3}$$

i.e.

$$\frac{1}{2}\frac{\partial}{\partial t}(\varepsilon {E_z}^2 + \mu {H_x}^2 + \mu {H_y}^2) + \frac{\partial (E_z H_x)}{\partial y} - \frac{\partial (E_z H_y)}{\partial x} = 0. \tag{4.4}$$

Next, we will discuss the discrete conservative property of the local energy and the global energy of the schemes (3.14). We can write Eq. (3.8) as

$$\mathbf{M}(\delta_t M_x M_y Z_{i,j}^k) + \mathbf{F}(\delta_x M_y M_t Z_{i,j}^k) + \mathbf{L}(\delta_y M_x M_t Z_{i,j}^k) = \mathbf{A}(M_t M_x M_y Z_{i,j}^k), \tag{4.5}$$

where

$$\mathbf{A} = \begin{pmatrix} -\mu & 0 & 0 & 0 & 1 & 0 \\ 0 & -\varepsilon & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4.1. *The multisymplectic Preissman scheme [Eq.(4.5)] conserves discrete local energy law*

$$\begin{aligned}\delta_t[(M_xM_yZ^k_{i,j})^T\mathbf{A}(M_xM_yZ^k_{i,j})-(M_xM_yZ^k_{i,j})^T\mathbf{F}(M_y\delta_xZ)-(M_xM_yZ^k_{i,j})^T\mathbf{L}(M_x\delta_yZ^k_{i,j})]\\ +\delta_x[(M_tM_yZ^k_{i,j})^T\mathbf{F}(M_y\delta_tZ^k_{i,j})]+\delta_y[(M_tM_xZ^k_{i,j})^T\mathbf{L}(M_x\delta_tZ^k_{i,j})]=0.\end{aligned}\tag{4.6}$$

Proof. Taking the inner product of Eq. (4.5) with $(\delta_tM_xM_yZ^k_{i,j})^T$ yields

$$\begin{aligned}&(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{M}(\delta_tM_xM_yZ^k_{i,j})+(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{F}(\delta_xM_yM_tZ^k_{i,j})\\ &+(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{L}(\delta_yM_xM_tZ^k_{i,j})=(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{A}(M_tM_xM_yZ^k_{i,j}).\end{aligned}\tag{4.7}$$

We note that $(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{M}(\delta_tM_xM_yZ^k_{i,j})$ vanishes because matrix $\mathbf{M}$ is skew-symmetry. Further, we write Eq. (4.7) as

$$\begin{aligned}&\frac{1}{2}(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{F}(\delta_xM_yM_tZ^k_{i,j})+\frac{1}{2}(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{L}(\delta_yM_xM_tZ^k_{i,j})\\ =&(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{A}(M_tM_xM_yZ^k_{i,j})-\frac{1}{2}(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{F}(\delta_xM_yM_tZ^k_{i,j})\\ &-\frac{1}{2}(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{L}(\delta_yM_xM_tZ^k_{i,j}).\end{aligned}\tag{4.8}$$

Firstly, due to the skew-symmetric property of matrices $\mathbf{F}$ and $\mathbf{L}$, we respectively obtain

$$\begin{aligned}(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{F}(\delta_xM_yM_tZ^k_{i,j})&=-\delta_x[(M_yM_tZ^k_{i,j})^T\mathbf{F}(M_y\delta_tZ^k_{i,j})]-(\delta_x\delta_tM_yZ^k_{i,j})^T\mathbf{F}(M_xM_yM_tZ^k_{i,j}),\\ (\delta_tM_xM_yZ^k_{i,j})^T\mathbf{L}(\delta_yM_xM_tZ^k_{i,j})&=-\delta_y[(M_xM_tZ^k_{i,j})^T\mathbf{L}(M_x\delta_tZ^k_{i,j})]-(\delta_y\delta_tM_xZ^k_{i,j})^T\mathbf{L}(M_xM_yM_tZ^k_{i,j}),\\ (\delta_tM_xM_yZ^k_{i,j})^T\mathbf{F}(\delta_xM_yM_tZ^k_{i,j})&=\delta_t[(M_xM_yZ^k_{i,j})^T\mathbf{F}(M_y\delta_xZ^k_{i,j})]+(\delta_x\delta_tM_yZ^k_{i,j})^T\mathbf{F}(M_xM_yM_tZ^k_{i,j}),\\ (\delta_tM_xM_yZ^k_{i,j})^T\mathbf{L}(\delta_yM_xM_tZ^k_{i,j})&=\delta_t[(M_xM_yZ^k_{i,j})^T\mathbf{L}(M_x\delta_yZ^k_{i,j})]+(\delta_y\delta_tM_xZ^k_{i,j})^T\mathbf{L}(M_xM_yM_tZ^k_{i,j}).\end{aligned}\tag{4.9}$$

Secondly, because of the symmetric property of matrix A, we have

$$(\delta_tM_xM_yZ^k_{i,j})^T\mathbf{A}(M_tM_xM_yZ^k_{i,j})=\frac{1}{2}\delta_t[(M_xM_yZ^k_{i,j})^T\mathbf{A}(M_xM_yZ^k_{i,j})].\tag{4.10}$$

Substituting Eqs. (4.9) and Eq. (4.10) into Eq. (4.8) leads to the conclusion of the theorem. □

Substitute Z, $\mathbf{A}$, $\mathbf{F}$ and $\mathbf{L}$ into Eq. (4.6), further eliminating $\mathbf{P}^k_{i,j}$, $\mathbf{Q}^k_{i,j}$, $\mathbf{U}^k_{i,j}$, $\mathbf{V}^k_{i,j}$ yields

$$\begin{aligned}&\delta_t[\frac{1}{2}\mu\langle M_xM_y\mathbf{H}^k_{i,j},M_xM_y\mathbf{H}^k_{i,j}\rangle+\frac{1}{2}\varepsilon\langle M_xM_y\mathbf{E}^k_{i,j},M_xM_y\mathbf{E}^k_{i,j}\rangle]+\langle R_1(M_yM_t\delta_x\mathbf{E}^k_{i,j}),M_xM_yM_t\mathbf{H}^k_{i,j}\rangle\\ &-\langle R_1(M_yM_t\delta_x\mathbf{H}^k_{i,j}),M_xM_yM_t\mathbf{E}^k_{i,j}\rangle+\langle R_2(M_xM_t\delta_y\mathbf{E}^k_{i,j}),M_xM_yM_t\mathbf{H}^k_{i,j}\rangle\\ &-\langle R_2(M_xM_t\delta_y\mathbf{H}^k_{i,j}),M_xM_yM_t\mathbf{E}^k_{i,j}\rangle=0.\end{aligned}\tag{4.11}$$

Due to R_1, R_2 defined in Eq. (2.6), $\mathbf{E}^k_{i,j}=[0,0,E^k_{zi,j}]^T$ and$\mathbf{H}^k_{i,j}=[H^k_{xi,j},H^k_{yi,j},0]^T$, we have

$$\begin{aligned}&\delta_t[\frac{1}{2}\mu\langle M_xM_y\mathbf{H}^k_{i,j},M_xM_y\mathbf{H}^k_{i,j}\rangle+\frac{1}{2}\varepsilon\langle M_xM_y\mathbf{E}^k_{i,j},M_xM_y\mathbf{E}^k_{i,j}\rangle]\\ =&\mu\langle M_xM_yM_t\mathbf{H}^k_{i,j},M_xM_y\delta_t\mathbf{H}^k_{i,j}\rangle+\varepsilon\langle M_xM_yM_t\mathbf{E}^k_{i,j},\delta_tM_xM_y\mathbf{E}^k_{i,j}\rangle\\ =&\mu(M_xM_yM_tH^k_{yi,j})(\delta_tM_xM_yH^k_{yi,j})+\mu(M_xM_yM_tH^k_{xi,j})(\delta_tM_xM_yH^k_{xi,j})\\ &+\varepsilon(M_xM_yM_tE^k_{zi,j})(\delta_tM_xM_yE^k_{zi,j})\\ =&\frac{1}{2\Delta t}[\mu(M_xM_yH^{k+1}_{yi,j})^2+\mu(M_xM_yH^{k+1}_{xi,j})^2+\varepsilon(M_xM_yE^{k+1}_{zi,j})^2]\\ &-\frac{1}{2\Delta t}[\mu(M_xM_yH^k_{yi,j})^2+\mu(M_xM_yH^k_{xi,j})^2+\varepsilon(M_xM_yE^k_{zi,j})^2]\end{aligned}\tag{4.12}$$

and

$$
\begin{aligned}
&\langle R_1(M_yM_t\delta_x\mathbf{E}^k_{i,j}), M_xM_yM_t\mathbf{H}^k_{i,j}\rangle-\langle R_1(M_yM_t\delta_x\mathbf{H}^k_{i,j}), M_xM_yM_t\mathbf{E}^k_{i,j}\rangle\\
&+\langle R_2(M_xM_t\delta_y\mathbf{E}^k_{i,j}), M_xM_yM_t\mathbf{H}^k_{i,j}\rangle-\langle R_2(M_xM_t\delta_y\mathbf{H}^k_{i,j}), M_xM_yM_t\mathbf{E}^k_{i,j}\rangle\\
=&-[(M_yM_t\delta_xH^k_{yi,j})(M_xM_yM_tE^k_{zi,j})+(M_yM_t\delta_xE^k_{zi,j})(M_xM_yM_tH^k_{yi,j})]\\
&+[(M_xM_t\delta_yH^k_{xi,j})(M_xM_yM_tE^k_{zi,j})+(M_xM_t\delta_yE^k_{zi,j})(M_xM_yM_tH^k_{xi,j})\\
=&\frac{1}{\Delta y}[(M_xM_tH^k_{xi,j+1})(M_xM_tE^k_{zi,j+1})-(M_xM_tH^k_{xi,j})(M_xM_tE^k_{zi,j})]\\
&-\frac{1}{\Delta x}[(M_yM_tH^k_{yi+1,j})(M_yM_tE^k_{zi+1,j})-(M_yM_tH^k_{yi,j})(M_yM_tE^k_{zi,j})]
\end{aligned}
\tag{4.13}
$$

Substituting Eq. (4.12) and Eq. (4.13) into Eq. (4.11) gives the concrete local energy on grid points of the schemes [Eqs. (3.14)] as

$$
\begin{aligned}
&\frac{1}{2\Delta t}[\mu(M_xM_yH^{k+1}_{yi,j})^2+\mu(M_xM_yH^{k+1}_{xi,j})^2+\varepsilon(M_xM_yE^{k+1}_{zi,j})^2]-\frac{1}{2\Delta t}[\mu(M_xM_yH^k_{yi,j})^2\\
&+\mu(M_xM_yH^k_{xi,j})^2+\varepsilon(M_xM_yE^k_{zi,j})^2]+\frac{1}{\Delta y}[(M_xM_tH^k_{xi,j+1})(M_xM_tE^k_{zi,j+1})-(M_xM_tH^k_{xi,j})\\
&\cdot(M_xM_tE^k_{zi,j})]-\frac{1}{\Delta x}[(M_yM_tH^k_{yi+1,j})(M_yM_tE^k_{zi+1,j})-(M_yM_tH^k_{yi,j})(M_yM_tE^k_{zi,j})]=0.
\end{aligned}
\tag{4.14}
$$

Under condition of the computational space domain is $\Omega=[0,T_x]\times[0,T_y]$, where T_x is periodic boundary of x-direction and T_y is periodic boundary of y-direction, we can obtain the discrete global energy on grid points.

Theorem 4.2. *The multisymplectic Preissman scheme [Eqs. (3.14)] conserves the discrete global energy*

$$
\begin{aligned}
&\sum_{\substack{i=1,2,\ldots,M\\ j=1,2,\ldots,N}}[\mu(M_xM_yH^{k+1}_{yi,j})^2+\mu(M_xM_yH^{k+1}_{xi,j})^2+\varepsilon(M_xM_yE^{k+1}_{zi,j})^2]\\
=&\sum_{\substack{i=1,2,\ldots,M\\ j=1,2,\ldots,N}}[\mu(M_xM_yH^{k}_{yi,j})^2+\mu(M_xM_yH^{k}_{xi,j})^2+\varepsilon(M_xM_yE^{k}_{zi,j})^2].
\end{aligned}
\tag{4.15}
$$

Proof. We sum the discrete local energy Eq. (4.14) over i and j. Take note of

$$
\begin{aligned}
&\sum_{\substack{i=1,2,\ldots,M\\ j=1,2,\ldots,N}}\frac{1}{\Delta y}[(M_xM_tH^k_{xi,j+1})(M_xM_tE^k_{zi,j+1})-(M_xM_tH^k_{xi,j})(M_xM_tE^k_{zi,j})]\\
=&\sum_{i=1,2,\ldots,M}\frac{1}{\Delta y}[(M_xM_tH^k_{xi,N+1})(M_xM_tE^k_{zi,N+1})-(M_xM_tH^k_{xi,1})(M_xM_tE^k_{zi,1})]=0,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\substack{i=1,2,\ldots,M\\ j=1,2,\ldots,N}}\frac{1}{\Delta x}[(M_yM_tH^k_{yi+1,j})(M_yM_tE^k_{zi+1,j})-(M_yM_tH^k_{yi,j})(M_yM_tE^k_{zi,j})]\\
=&\sum_{j=1,2,\ldots,N}\frac{1}{\Delta x}[(M_yM_tH^k_{y,M+1,j})(M_yM_tE^k_{zM+1,j})-(M_yM_tH^k_{y1,j})(M_yM_tE^k_{z1,j})]=0,
\end{aligned}
$$

we can get the conclusion. ☐

Additionally, the concrete discrete global energy of the schemes [Eqs. (3.14)] at $t = t_k$ is

$$\frac{1}{32} \cdot \sum_{\substack{i=1,2,\ldots,M \\ j=1,2,\ldots,N}} [\varepsilon(E^k_{zi,j} + E^k_{zi+1,j} + E^k_{zi,j+1} + E^k_{zi+1,j+1})^2 + \mu(H^k_{xi,j} + H^k_{xi+1,j} + H^k_{xi,j+1} + H^k_{xi+1,j+1})^2$$
$$+ \mu(H^k_{yi,j} + H^k_{yi+1,j} + H^k_{yi,j+1} + H^k_{yi+1,j+1})^2] \cdot \Delta x \cdot \Delta y = constant. \tag{4.16}$$

5. Numerical results

The multisymplectic scheme [Eqs. (3.14)] is an implicit scheme that involves solving a linear system of $E^k_{zi,j}$, $H^k_{xi,j}$, $H^k_{yi,j}$ at each time step. For comparisons, the present method and the Yee's scheme are all used to solve the same problems. In the following experiments, suppose the computational space domain $\Omega = [0, T_x] \times [0, T_y]$, where T_x is period of x-direction and T_y is period of y-direction. All computations are done with the parameters $\Delta x = \Delta y = h$. Boundary conditions are chosen periodic boundaries. Denote the maximum error at time $t = t_k$ by

$$|error| = \max_{i,j} |E_z(x_i, y_j, t_k) - E^k_{zi,j}|.$$

Firstly, the initial conditions of the experiment is

$$\begin{aligned} E_z(x, y, 0) &= 0, \\ H_y(x, y, 0) &= -\frac{3}{\sqrt{5}} \cos(3\pi x) \sin(\pi y), \\ H_x(x, y, 0) &= \frac{1}{\sqrt{5}} \sin(3\pi x) \cos(\pi y). \end{aligned}$$

If $\mu = 1$ and $\varepsilon = 2$, the exact solution is

$$\begin{aligned} E_z(x, y, t) &= \sin(3\pi x) \sin(\pi y) \sin(\sqrt{5}\pi t), \\ H_y(x, y, t) &= -\frac{3}{\sqrt{5}} \cos(3\pi x) \sin(\pi y) \cos(\sqrt{5}\pi t), \\ H_x(x, y, t) &= \frac{1}{\sqrt{5}} \sin(3\pi x) \cos(\pi y) \cos(\sqrt{5}\pi t). \end{aligned}$$

Here, we test the scheme (3.14) and Yee's scheme on space domain $[0, 2/3] \times [0, 2]$.

Figure 3 is the comparison of $|error|$ for the two schemes until $t = 20$. Although the temporal step for the scheme (3.14) is larger than Yee's scheme, the numerical solutions are more precise than Yee's scheme. Figure 4 shows the error of numerical solutions obtained by the scheme (3.14) with different spatial step, respectively. Figure 5 shows the variation of $|error|$ of the numerical solutions obtained by the scheme (3.14) in long time computation. Figure 6 shows the variation of the local energy of the present method as time evolves. The local energy is calculated at central grid point throughout. Actually, our numerical experiment indicates there is no effect on local energy when calculating at any other inner grid point. The variation of the error of global energy for the present method as time evolves is given in figure 7. Figures 6 and 7 indicate that the scheme (3.14) has good conservative properties of discrete local energy and global energy.

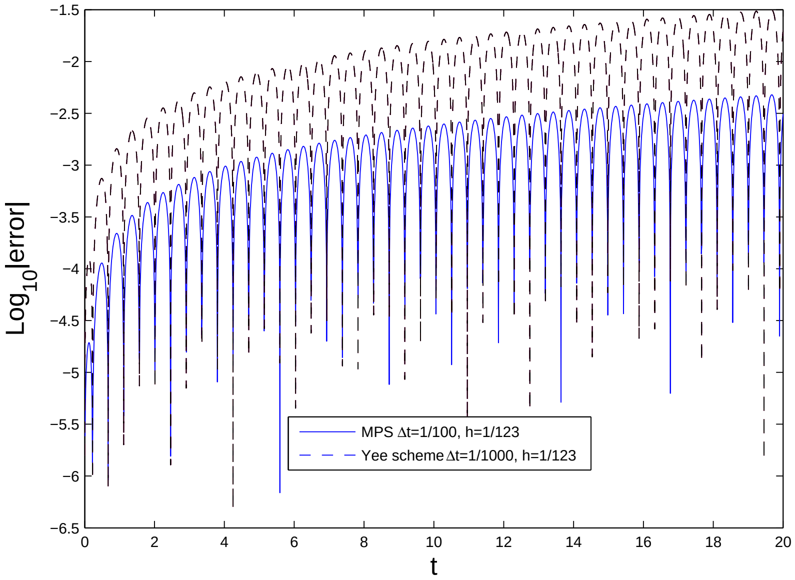

Fig.3: The comparison of $|error|$ for the scheme (3.14) and Yee's scheme as time evolves with the same spatial step $h = 1/123$, and different temporal step.

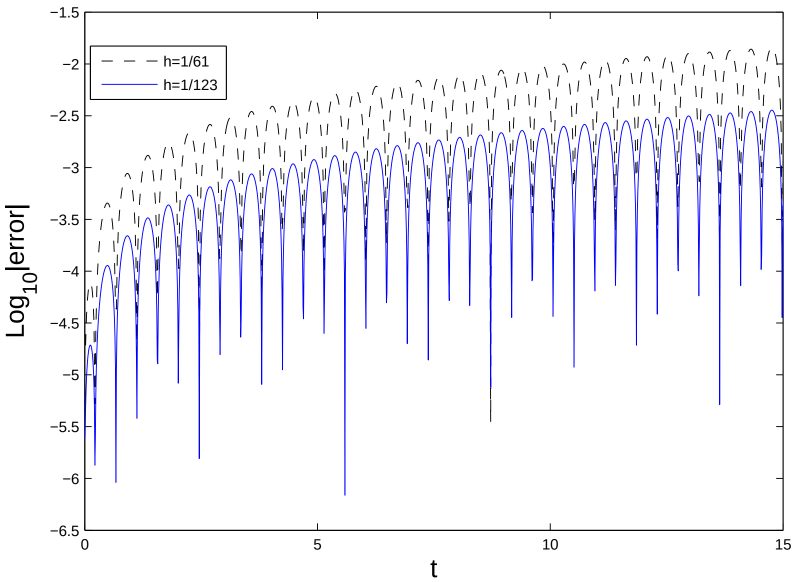

Fig.4: The comparison of $|error|$ of the scheme (3.14) as time evolves with different spatial step $h = 1/61$ and $h = 1/123$.

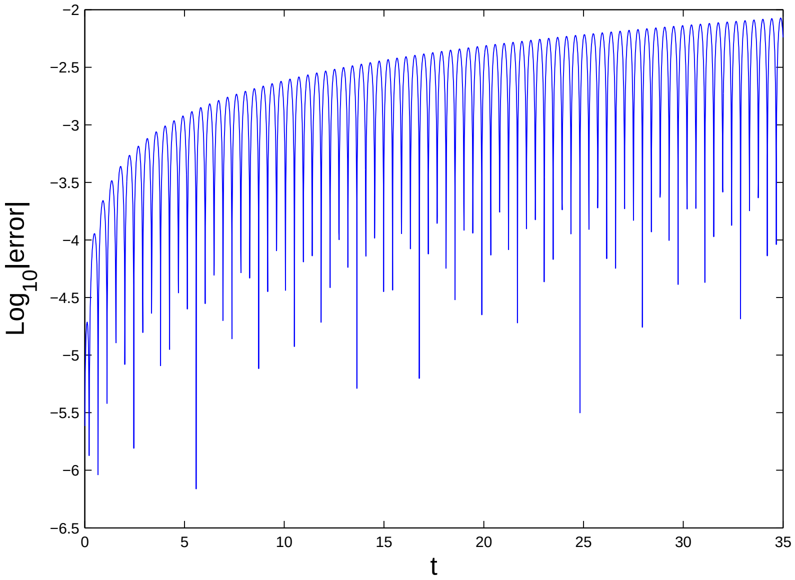

Fig.5: The variation of $|error|$ of the scheme (3.14) as time evolves with spatial step $h = 1/123$ and temporal step $\Delta t = 1/100$ until t=35.

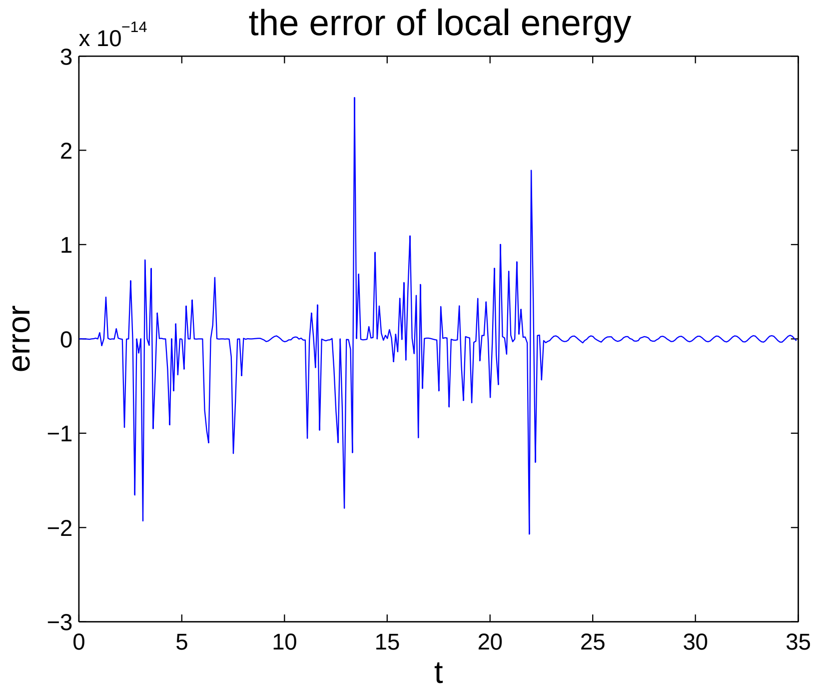

Fig.6: The trend of the local energy for the scheme (3.14) with spatial step $h = 1/123$ and temporal step $\Delta t = 1/100$ until t=35.

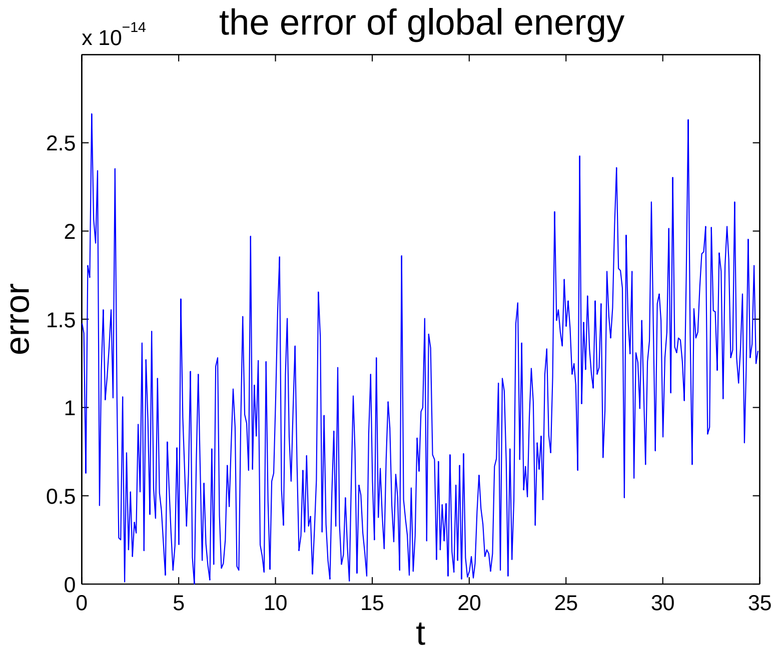

Fig.7: The trend of error of the global energy for the scheme (3.14) with $h = 1/123$, $\Delta t = 1/100$ until t=35.

Secondly, let's pay attention to another example. We prescribe initial conditions as

$$\begin{aligned}
E_z(x, y, 0) &= \sin(3\pi x)\sin(4\pi y),\\
H_y(x, y, 0) &= -\frac{3}{5}\sin(3\pi x)\sin(4\pi y),\\
H_x(x, y, 0) &= -\frac{4}{5}\cos(3\pi x)\cos(4\pi y).
\end{aligned}$$

If $\mu = 1$ and $\varepsilon = 1$, the exact solution is

$$\begin{aligned}
E_z(x, y, t) &= \sin(3\pi x - 5\pi t)\sin(4\pi y),\\
H_y(x, y, t) &= -\frac{3}{5}\sin(3\pi x - 5\pi t)\sin(4\pi y),\\
H_x(x, y, t) &= -\frac{4}{5}\cos(3\pi x - 5\pi t)\cos(4\pi y).
\end{aligned}$$

We test the two schemes on the space domain $[0, 2/3] \times [0, 1/2]$. Computations parameters for present method is $h = 1/150$, $\Delta t = 0.005$, but $h = 1/150$, $\Delta t = 0.0005$ for Yee's scheme. Figure 8 indicates that the numerical solutions obtained by the present method are more accurate than by Yee's scheme throughout. Figure 9 illustrates the $|error|$ obtained by present method is satisfied as time evolves. As it is seen from figures 10 and 11, the scheme (3.14) also keeps the discrete local energy and the global energy very well as time evolves.

It is evident that the scheme (3.14) has good properties by our test examples. The scheme (3.14) not only obtains better numerical solutions for our problems, but also keeps the discrete local energy and the discrete global energy very well in long time computation.

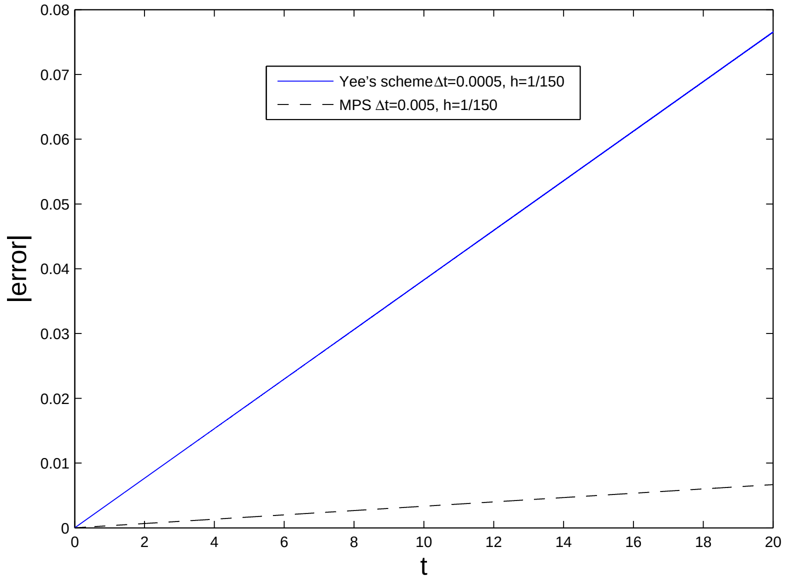

Fig.8: The comparison of $|error|$ for the scheme (3.14) and Yee's scheme as times evolves with the same spatial step $h = 1/150$, and different temporal step.

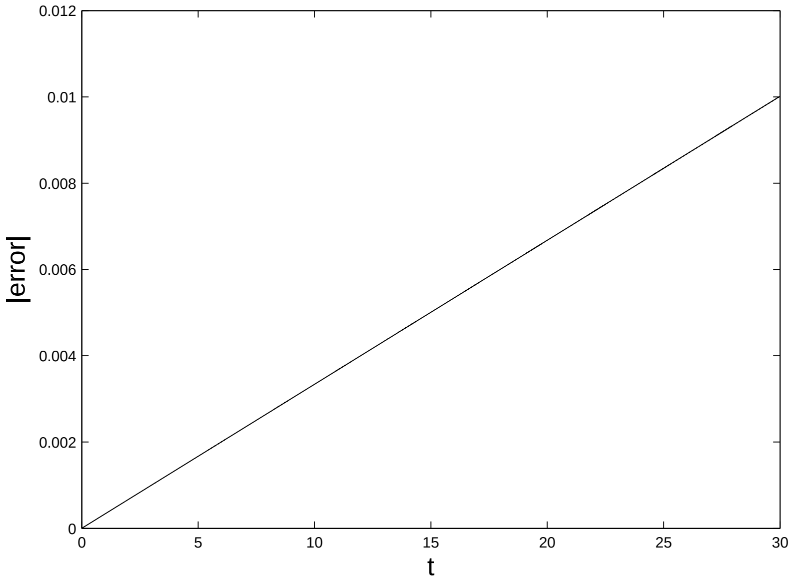

Fig.9: The trend of $|error|$ of the scheme (3.14) as time evolves with $h = 1/150$, and $\Delta t = 1/200$.

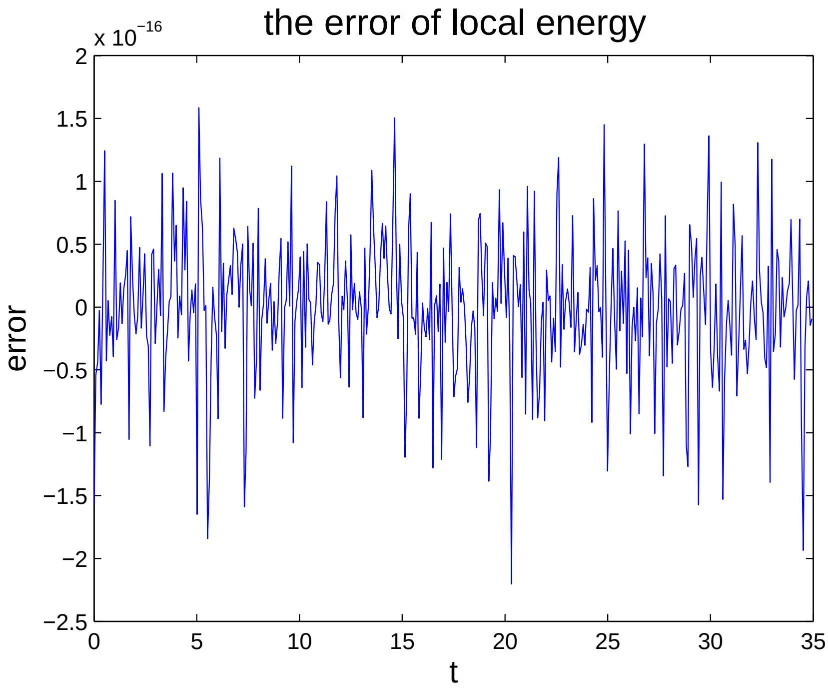

Fig.10: The trend of error of the local energy for the scheme (3.14) with $h = 1/150$, $t = 1/200$ until t=35.

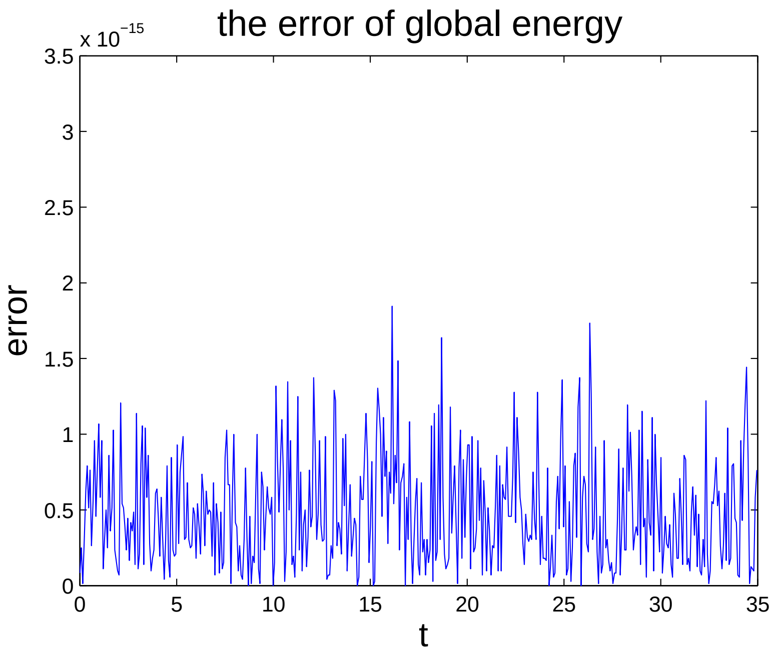

Fig.11: The trend of error of the global energy for the scheme (3.14) with $h = 1/150$, $t = 1/200$ until t=35.

6. Concluding remarks

In this paper, based on the multisymplectic Bridges' form for Maxwell's equations, we derive a two time levels multisymplectic Preissman scheme (3.14), which preserves the discrete multi-symplectic conservation law. We prove the scheme preserves the discrete local energy and the discrete global energy exactly. By numerical experiments, we find that solutions obtained by the scheme (3.14) are better than by Yee's scheme. Numerical results also show that the scheme keeps the discrete local energy and global energy very well. In our paper, eliminating M_t in Eq. (3.9) leads to a two time levels multisymplectic Preissman scheme. If we don't eliminate M_t in Eq. (3.9), we will obtain a three time levels multisymplectic Preissman scheme. Whether the two schemes are equal is worthy of studying.

Though we discuss Maxwell's equations in two dimensions, the method of this paper can be extended into 3+1 dimensional problems with variable coefficients ε and μ. Actually, the multisymplectic structure Eq. (2.5) are for 3+1 dimensional Maxwell's equations. In present paper, we discuss the numerical periodic boundary conditions, but we can also take numerical natural boundary conditions. About this numerical boundary conditions, we have a lot of works to do. We will continue to research it in the future.

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