2011

Moment matrices in conditional heteroskedastic models under elliptical distributions with applications in AR-ARCH models

Shuangzhe Liu

Chris Heyde

Wing Keung Wong
Hong Kong Baptist University, awong@hkbu.edu.hk

statistical papers manuscript No.
(will be inserted by the editor)

Moment matrices in conditional heteroskedastic models under elliptical distributions with applications in AR-ARCH models

Shuangzhe Liu · Chris C. Heyde · Wing-Keung Wong

Received: 10 September 2008 / Revised: 14 May 2009

Abstract It is well known that moment matrices play a very important rôle in econometrics and statistics. Liu and Heyde (2008) give exact expressions for two-moment matrices, including the Hessian for ARCH models under elliptical distributions. In this paper, we extend the theory by establishing two additional moment matrices for conditional heteroskedastic models under elliptical distributions. The moment matrices established in this paper implement the maximum likelihood estimation by some estimation algorithms like the scoring method. We illustrate the applicability of the additional moment matrices established in this paper by applying them to establish an AR-ARCH model under elliptical distribution.

Keywords Heteroskedasticity · likelihood · BHHH method · Newton-Raphson method · scoring method · AR-ARCH model

1 Introduction

The characteristics of asset returns have been of great concern to financial economists and statisticians. Are asset returns iid and normally distributed? We first discuss the distributions of asset returns. By examining the empirical data, Fama (1963; 1965a, b) conclude that the normality assumption in the distribution of a portfolio return is violated such that the distribution is 'flat-tailed' and suggest the family of stable Paretian distributions between normal and Cauchy distributions for the stock returns. Others suggest stock

S. Liu (✉)
Faculty of Information Sciences and Engineering, University of Canberra, Canberra, ACT 2601, Australia
Tel.: +61-2-62012513, Fax: +61-2-62012683, E-mail: shuangzhe.liu@canberra.edu.au
W.-K. Wong
Department of Economics and Institute for Computational Mathematics, Hong Kong Baptist University, Hong Kong, China
E-mail: awong@hkbu.edu.hk

returns follow student-t distribution (Blattberg and Gonedes, 1974), a mixture of normal distributions (Clark, 1973) or a mixture of non-normal stable distributions (Fielitz and Rozelle, 1983). The class of elliptical distributions has gained prominence to delineate to portfolio theory and risk management in recent years. Introduced by Kelker (1970) and further discussed by, for example, Fang et al. (1987) and Fang and Zhang (1990), elliptical probability distributions allow for the presence of heavy tails and asymptotic tail dependence. It contains the multivariate normal distribution as a special case; as well as many non-normal multivariate distributions including the multivariate Cauchy, multivariate exponential, multivariate elliptical analog of student-t distribution, logistic, Bessel distribution, and non-normal variance mixtures of multinormal distributions. Thus, elliptical distributions should be an excellent candidate to model the distributions of portfolio returns.

Now, we turn to discuss the iid issue for asset returns. It is well known that stock returns typically possess heavy tails and time-varying volatility features that could be heteroskedastic in their variances, see, for example, Tse (1991), Rachev and Mittnik (2000), Tsay (2002) and McAleer (2005). To circumvent this limitation, Engle (1982) first introduces an autoregressive conditional heteroskedastic (ARCH) model with normality assumption to data.[1] Extension of the ARCH model includes a generalized autoregressive conditional heteroscedasticity (GARCH) innovation to allow both autoregressive and moving average components in the heteroskedastic variance of the returns to display a high degree of persistence.[2]

To overcome both "non-iid" and "non-normality" issues for stock returns, different studies have produced a range of different models. For example, to capture the characteristics of heavy tails and non-iid nature for stock returns, some non-normal and non-iid models have been proposed to replace the normality and iid assumptions. For example, Bollerslev (1987) and Spanos (1994) use GARCH models under student-t distributions assumption while Rachev and Mittnik (2000) advocate models under stable distributions. Recent development includes models under the Laplace distributions investigated by Heyde and Kou (2004). For further distributions and studies reported, readers may refer to Hamilton (1994), Tiku et al. (2000), Knight and Satchell (2001) and Wong and Bian (2005) for more information. To improve the modeling technique further, the powerful elliptical distribution family is introduced in the system; see, for example, Bingham and Kiesel (2002), Liu (2004), McNeil et al. (2005), and Liu and Heyde (2008).

Estimation of (G)ARCH type models can be achieved by adopting the maximum likelihood estimation (MLE) method, a benchmark of the estimation. Solving the MLE by iterative methods is a formidable task and can be very problematic. One may encounter multiple roots, slow convergence, or convergence to wrong values or even divergence. Readers may refer to, for ex-

[1] Readers may refer to, for example, Gouriéroux (1997), Chan (2002), Lütkepohl (2005), and Tsay (2005) for more discussion.

[2] See, for example, Bollerslev (1986) and McAleer (2005) for more discussion.

ample, Engle (1982) and Fomby et al. (1984), for early discussion, estimation, and applications. Recent work include McAleer (2005), McNeil et al. (2005) and references therein. To obtain the MLE, it is common to use the Berndt, Hall, Hall and Hausman (BHHH)[3] and Newton-Raphson methods based on numerical derivatives; see e.g. Berndt et al. (1974), Bollerslev (1987), Hamilton (1994), Mak et al. (1997) and McNeil et al. (2005). Another important algorithm to get numerical optimisation is the scoring method.

Some econometricians prefer to use the BHHH algorithm because it requires only the first order of derivatives. Nonetheless, its weakness is that it may encounter a variety of numerical difficulties under certain situations, as pointed out by e.g. Li (2004, p. 98). On the other hand, Mak et al. (1997) recommend to use an iteratively reweighted least squares scheme which provides better convergence properties than the BHHH. In addition, studies demonstrate that the Newton-Raphson method performs better than the BHHH in the estimation. Thus, the Newton-Raphson has been recommended to use in a wide range of areas, including econometrics, statistics, engineering, etc.

However, it is found that the Newton-Raphson method is working well provided that the log-likelihood is smooth and convex around the maximum and provided that the initial value is reasonably close to the true value of the parameter. When the true parameters are less well behaved, estimation could go wrong by using the Newton-Raphson method. In this situation, the scoring method is found to yield superior results, especially in generalised linear modeling. It is also found to be equivalent to iteratively reweighted least squares, see, for example, Faraway (2006, pp 281-282). It is reasonable to expect that the scoring method performs very well because it uses the exact information matrix instead of numerical derivatives. In particular, in estimating the parameters in ARCH type models the scoring method has been long employed; see, for example, Engle (1982), a very important seminal paper. Readers may also refer to Liu and Polasek (1999) the uses of the theory in different cases of ARCH models.

In order to study ARCH type models with pseudo-maximum likelihood estimation, Gouriéroux (1997) and Liu and Neudecker (2009) develop results of moment matrices in these models. On the other hand, Liu and Heyde (2008) give exact expressions for two-moment matrices, including the Hessian for ARCH models, under elliptical distributions with MLE. The two-moment matrices can be used for BHHH and Newton-Raphson methods in the computations. Now, in order to employ the scoring method for MLE in conditional heteroskedastic models under elliptical distributions, the exact conditional moment matrices are required.

To achieve the target, in the present paper, we extend the theory further by establishing two additional moment matrices. These are the conditional expectations of the moment matrices developed by Liu and Heyde (2008). The newly found matrices can be used in applying the scoring method to obtain the maximum likelihood estimates for conditional heteroskedastic models un-

[3] BHHH is a MLE method proposed by Berndt, Hall, Hall, and Hausman (1974).

der elliptical distributions. In addition, we illustrate the applicability of the additional moment matrices established in this paper by applying them to establish an AR-ARCH model under elliptical distribution.

The remainder of the paper is organised as follows: In Section 2, we discuss a general univariate conditional heteroskedastic time series model under an elliptical distribution and present its log-likelihood function. In Section 3, we establish our proposed moment matrices. We give an example to illustrate the applicability of our established moment matrices in an AR-ARCH model in Section 4. Section 5 rounds up the paper by providing several well-grounded observations.

2 The model and its log-likelihood function

In this section, we first discuss a general univariate conditional heteroskedastic time series model under an elliptical distribution, and thereafter present the form of its log-likelihood function.

We consider a general univariate conditional heteroskedastic time series model under an elliptical distribution. It is treated in a unified approach: the model needs only the specification of the conditional mean and conditional variance which is equivalent to specifying the scale structure. This model contains a class of GARCH models as a special case.

Firstly, we assume that the second conditional moment of the model is finite. Thereafter, we make specifications on the two equations involving both mean and scale parameters of the model. In this paper, we follow Liu and Heyde (2008) to consider the following model:

$$y_t = m_t + u_t, \qquad t = 1, \ldots, T, \tag{1}$$

where y_t is an observation of the time series, $m_t = E(y_t \mid C_{t-1})$ is its corresponding conditional mean (in which C_{t-1} is the information set containing all available information at time $t-1$) and u_t is a disturbance term with conditional mean and conditional variance satisfying $E(u_t \mid C_{t-1}) = 0$ and $E(u_t^2 \mid C_{t-1}) = \sigma_t^2 > 0$, respectively.

We further assume that u_t follows a conditional elliptical distribution (but, in general, it does not follow a conditional normal distribution), and assume that σ_t^2 is linked to the scale parameter v_t such that $\sigma_t^2 = av_t$ in which a is a constant. We also make the following assumptions on the third and fourth conditional moments, respectively, for the disturbance term, u_t:

$$E(u_t^3 \mid C_{t-1}) = s_t = 0 \text{ and } E(u_t^4 \mid C_{t-1}) = k_t. \tag{2}$$

Let $\theta = (\theta_1, \ldots, \theta_p)'$ be a $p \times 1$ vector of fundamental parameters. In this paper, we focus on developing the properties for $m_t = m_t(\theta)$ and $v_t = v_t(\theta)$, both are scalar functions of θ.

We turn to build up the log-likelihood function of the model being studied in our paper as follows:

$$L = L(y, \theta) = \sum_{t=1}^{T} L_t, \tag{3}$$

where

$$L_t = -\frac{1}{2} \log v_t + \log g(c_t), \text{ and} \tag{4}$$

$$c_t = u_t^2 v_t^{-1}. \tag{5}$$

L_t is the conditional log-likelihood function associated with y_t, $u_t = y_t - m_t$ is the disturbance term, and g is the density generator for $t = 1, \ldots, T$. The function forms of g contains members of the elliptical distribution family, see, for example, Liu (2004) for more information. Obviously, g, L_t, and L are scalar functions of θ.

Next, we let d be the differential operator such that both $dm_t = dm_t(\theta)$ and $dv_t = dv_t(\theta)$ stand for the differentials with respect to θ, and $\dot{g} = \dot{g}(c_t)$ stands for the derivative of $g = g(c_t)$. In addition, we define $w = w(c_t) = g^{-1}\dot{g}$. Thereby, taking the differential of L_t in (4) with respect to θ, via m_t and v_t, we obtain dL_t to be a scalar function of θ satisfying:

$$dL_t = -2wu_t v_t^{-1} dm_t - wu_t^2 v_t^{-2} dv_t - \frac{1}{2} v_t^{-1} dv_t\,. \tag{6}$$

This leads to establish the score vector as follows, see Liu and Heyde (2008):

$$\begin{aligned} f(\theta) &= \sum_{t=1}^{T} \frac{\partial L_t}{\partial \theta} \\ &= -\sum_{t=1}^{T} 2wu_t v_t^{-1} \frac{\partial m_t}{\partial \theta} - \sum_{t=1}^{T} wu_t^2 v_t^{-2} \frac{\partial v_t}{\partial \theta} - \sum_{t=1}^{T} \frac{1}{2} v_t^{-1} \frac{\partial v_t}{\partial \theta}. \end{aligned} \tag{7}$$

If we set $f(\theta) = 0$ and if its solution could be solved explicitly, then one will obtain an analytical expression for the MLE's and it is not necessary to employ any algorithm in this situation. However, in most of the situations, it is impossible to obtain the explicit solution for the MLE's. In this situation, an algorithm has to be used to compute the MLE's. One may recommend to employ the BHHH, Newton-Raphson, and scoring methods for this purpose. In next section, we will derive the second differential of L with respect to θ and then provide its analytical derivatives which will be needed in the iterations for the scoring method, to obtain the MLE's.

3 Moment matrices

In this paper, we consider four matrices of derivatives with respect to θ as follows:

$$\begin{aligned} G &= \sum_{t=1}^{T} \frac{\partial L_t}{\partial \theta} \frac{\partial L_t}{\partial \theta'}, \\ F_1 &= E(G \mid C_{t-1}), \qquad (8) \\ H &= \sum_{t=1}^{T} \frac{\partial^2 L_t}{\partial \theta \partial \theta'}, \quad \text{and} \\ F_2 &= -E(H \mid C_{t-1}), \qquad (9) \end{aligned}$$

in which $G = G(\theta)$, $F_1 = F_1(\theta)$, $H = H(\theta)$, and $F_2 = F_2(\theta)$ are $p \times p$ matrices. In particular, H is the Hessian matrix, and F_1 and F_2 are conditional expectation matrices on C_{t-1} such that $E(F_1) = E(G)$ and $E(F_2) = E(H)$. Thereby, we establish the following equations which are equivalent to (8) and (9), respectively:

$$\begin{aligned} (d\theta)' F_1 \, d\theta &= (d\theta)' E(G \mid C_{t-1}) d\theta \quad \text{and} \\ (d\theta)' F_2 \, d\theta &= -(d\theta)' E(H \mid C_{t-1}) d\theta. \qquad (10) \end{aligned}$$

Readers may refer to Magnus and Neudecker (1999) for the equivalence of the above equations and for the theory of the standard matrix differential calculus, and refer to Liu and Heyde (2008) for the detailed expressions of G and H.

F_1 and F_2 can be used to implement the scoring method in the following iteration procedure; see, for example, Liu and Heyde (2008) and references therein:

$$\hat{\theta}_{i+1} = \hat{\theta}_i + F^{-1}(\hat{\theta}_i) f(\hat{\theta}_i) \,, \qquad (11)$$

where $\hat{\theta}_i$ and $\hat{\theta}_{i+1}$ are the estimates at the i^{th} and $i+1^{th}$ steps, respectively, of the numerical iteration, F is the moment matrix say to be F_1 or F_2, f is the score vector, and both F and f are evaluated at $\hat{\theta}_i$.

We are now ready to find the terms at the right-hand side of these equations, which will, then, be used to derive the expression of F_1 and F_2. We will discuss their expressions in the following subsections.

3.1 Matrix F_1

We first derive the expression of F_1. Multiplying dL_t in (7) by itself, we obtain

$$\begin{aligned} dL_t dL_t = {} & 4w^2 u_t^2 v_t^{-2} dm_t dm_t \\ & + (2w^2 u_t^3 v_t^{-3} + w u_t v_t^{-2}) dm_t dv_t \\ & + (w^2 u_t^4 v_t^{-4} + w u_t^2 v_t^{-3} + \frac{1}{4} v_t^{-2}) dv_t dv_t. \qquad (12) \end{aligned}$$

Taking the expectation of $dL_t dL_t$ in (12) conditional on C_{t-1}, and thereafter incorporating the results from (2), we have

$$E(dL_t dL_t \mid C_{t-1}) = 4w^2 a v_t^{-1} dm_t dm_t + (w^2 k_t v_t^{-4} + w a v_t^{-2} + \frac{1}{4} v_t^{-2}) dv_t dv_t. \tag{13}$$

From (10) and (13), we establish the expression of F_1 as follows:

$$\begin{aligned} F_1 &= E\left(\sum_{t=1}^{T} \frac{\partial L_t}{\partial \theta} \frac{\partial L_t}{\partial \theta'} \mid C_{t-1}\right) \\ &= \sum_{t=1}^{T} 4w^2 a v_t^{-1} \left(\frac{\partial m_t}{\partial \theta'}\right)' \frac{\partial m_t}{\partial \theta'} \\ &\quad + \sum_{t=1}^{T} (w^2 k_t v_t^{-2} + wa + \frac{1}{4}) v_t^{-2} \left(\frac{\partial v_t}{\partial \theta'}\right)' \frac{\partial v_t}{\partial \theta'}. \end{aligned} \tag{14}$$

3.2 Matrix F_2

We turn to derive the expression of F_2. In order to establish the Hessian matrix, we first develop the second differential of L. Let $\dot{w}$ denote the derivative of w. Taking the differential of dL_t in (7) with respect to θ, we get

$$\begin{aligned} d^2 L_t = &(4\dot{w} u_t^2 v_t^{-2} + 2w v_t^{-1}) dm_t dm_t \\ &+ (4\dot{w} u_t^3 v_t^{-3} + 4w u_t v_t^{-2}) dm_t dv_t \\ &+ (\dot{w} u_t^4 v_t^{-4} + 2w u_t^2 v_t^{-3} + \frac{1}{2} v_t^{-2}) dv_t dv_t. \end{aligned} \tag{15}$$

To derive the expression of F_2, we need to impose the assumptions on the conditional moments u_t^3 and u_t^4 as shown in (2). Similar to the procedure in finding F_1, taking the expectation of $d^2 L_t$ in (15) conditional on C_{t-1}, and thereafter incorporating the results in (2), we have

$$\begin{aligned} E(d^2 L_t \mid C_{t-1}) &= E(4\dot{w} u_t^2 v_t^{-2} + 2w v_t^{-1}) dm_t dm_t \\ &\quad + E(4\dot{w} u_t^3 v_t^{-3} + 4w u_t v_t^{-2}) dm_t dv_t \\ &\quad + E(\dot{w} u_t^4 v_t^{-4} + 2w u_t^2 v_t^{-3} + \frac{1}{2} v_t^{-2}) dv_t dv_t \\ &= (d\theta)'(4\dot{w} a + 2w) v_t^{-1} \left(\frac{\partial m_t}{\partial \theta'}\right)' \frac{\partial m_t}{\partial \theta'} d\theta \\ &\quad + (d\theta)'(\dot{w} k_t v_t^{-2} + 2wa + \frac{1}{2}) v_t^{-2} \left(\frac{\partial v_t}{\partial \theta'}\right)' \frac{\partial v_t}{\partial \theta'} d\theta. \end{aligned} \tag{16}$$

Combining the results in (10) and (16), we establish F_2 as follows:

$$\begin{aligned} F_2 &= -E\left(\sum_{t=1}^{T} \frac{\partial^2 L_t}{\partial\theta\partial\theta'} \mid C_{t-1}\right) \\ &= -\sum_{t=1}^{T}(4\dot{w}a + 2w)v_t^{-1}\left(\frac{\partial m_t}{\partial\theta'}\right)'\frac{\partial m_t}{\partial\theta'} \\ &\quad -\sum_{t=1}^{T}(\dot{w}k_t v_t^{-2} + 2wa + \frac{1}{2})v_t^{-2}\left(\frac{\partial v_t}{\partial\theta'}\right)'\frac{\partial v_t}{\partial\theta'}. \end{aligned} \tag{17}$$

Before we turn to discuss the applications of F_1 and F_2, we summarize the above observations as follows.

Theorem 1: *For the model stated in (1) satisfying the assumptions stated in (2), the moment matrices F_1 and F_2 defined in (8) and (9), respectively, possess their explicit expressions as stated in (14) and (17), respectively.*

4 An example

The theory of moment matrices developed in this paper could be applied in a variety of models. In this section, we illustrate the applicability of our proposed moment matrices by applying them to build up and estimate AR-ARCH models.

We first consider a special case of model (1) in which the parameter vector θ is partitioned to be two vectors such that $\theta = (\beta', \alpha')'$, where β is the vector of parameters for $m_t = m_t(\beta)$ and α is the vector of parameters for v_t with $v_t = v_t(\beta, \alpha)$. Here, m_t and v_t have been discussed in model (1) such that $y_t = m_t + u_t$ with scale $v_t = \sigma_t^2/a$.

For this model, Liu and Heyde (2008) have presented the expression of the score vector f. Based on the two additional moment matrices we developed in the previous section, we further express the functions F_1 and F_2 as follows: From (14), we get

$$F_1 = \begin{pmatrix} F_{1bb} & F_{1ba} \\ F'_{1ba} & F_{1aa} \end{pmatrix}, \tag{18}$$

where

$$F_{1bb}(\beta) = \sum_{t=1}^{T} 4w^2 a v_t^{-1} \left(\frac{\partial m_t}{\partial \beta'}\right)' \frac{\partial m_t}{\partial \beta'} + \sum_{t=1}^{T} (w^2 k_t v_t^{-2} + wa + \frac{1}{4}) v_t^{-2} \left(\frac{\partial v_t}{\partial \beta'}\right)' \frac{\partial v_t}{\partial \beta'} \,,$$

$$F_{1ba}(\beta, \alpha) = \sum_{t=1}^{T} (w^2 k_t v_t^{-2} + wa + \frac{1}{4}) v_t^{-2} \left(\frac{\partial v_t}{\partial \beta'}\right)' \frac{\partial v_t}{\partial \alpha'} \,, \text{ and}$$

$$F_{1aa}(\alpha) = \sum_{t=1}^{T} (w^2 k_t v_t^{-2} + wa + \frac{1}{4}) v_t^{-2} \left(\frac{\partial v_t}{\partial \alpha'}\right)' \frac{\partial v_t}{\partial \alpha'} \,.$$

From (17), we get

$$F_2 = \begin{pmatrix} F_{2bb} & F_{2ba} \\ F'_{2ba} & F_{2aa} \end{pmatrix} , \qquad (19)$$

where

$$F_{2bb}(\beta) = -\sum_{t=1}^{T} (4\dot{w}a + 2w) v_t^{-1} \left(\frac{\partial m_t}{\partial \beta'}\right)' \frac{\partial m_t}{\partial \beta'} - \sum_{t=1}^{T} (\dot{w} k_t v_t^{-2} + 2wa + \frac{1}{2}) v_t^{-2} \left(\frac{\partial v_t}{\partial \beta'}\right)' \frac{\partial v_t}{\partial \beta'} \,,$$

$$F_{2ba}(\beta, \alpha) = -\sum_{t=1}^{T} (\dot{w} k_t v_t^{-2} + 2wa + \frac{1}{2}) v_t^{-2} \left(\frac{\partial v_t}{\partial \beta'}\right)' \frac{\partial v_t}{\partial \alpha'} \,, \text{ and}$$

$$F_{2aa}(\alpha) = -\sum_{t=1}^{T} (\dot{w} k_t v_t^{-2} + 2wa + \frac{1}{2}) v_t^{-2} \left(\frac{\partial v_t}{\partial \alpha'}\right)' \frac{\partial v_t}{\partial \alpha'} \,.$$

After expressing our proposed model (1) in a 'bivariate' format as shown in the above, we are now ready to estimate an AR(1)-ARCH(1) model as follows:

$$\begin{aligned} y_t &= m_t + u_t \,, \\ m_t &= \beta_0 + \beta_1 y_{t-1} \,, \\ v_t &= \alpha_0 + \alpha_1 u_{t-1}^2 \,, \text{ and} \\ \sigma_t^2 &= a v_t \,, \end{aligned} \qquad (20)$$

where u_t, $t = 1, \ldots, T$, follows a conditional elliptical distribution. For the model in (20) with $\beta = (\beta_0, \beta_1)'$ and $\alpha = (\alpha_0, \alpha_1)'$, we have

$$\begin{aligned}
\frac{\partial m_t}{\partial \beta'} &= (1, y_{t-1})\,, \\
\frac{\partial m_t}{\partial \alpha'} &= (0, 0)\,, \\
\frac{\partial v_t}{\partial \beta'} &= -2\alpha_1 u_{t-1}(1, y_{t-1})\,, \text{ and} \\
\frac{\partial v_t}{\partial \alpha'} &= (1, u_{t-1}^2)\,.
\end{aligned} \tag{21}$$

We note that when one makes further specification of the underlying distribution for the model, one can obtain w, the expressions in (21), and then obtain the exact expressions of F_1 and F_2. These functions could then be used in the scoring method to implement the MLE.

5 Concluding remarks

In this paper, we extend the work by Liu and Heyde (2008) and others by establishing the additional moment matrices which could then be used in the scoring method to implement the MLE. Thereafter, we illustrate the applicability of the additional moment matrices to establish an AR-ARCH model under an elliptical distribution.

We note that the expressions of F_1 and F_2 are simpler than G and H given in Liu and Heyde (2008). These matrices can then be used to produce competitive or better results. When the normality assumption is imposed on our model settings, one could be able to obtain the explicit solutions for the MLE's in terms of G, H, F_1, and F_2, and thus the estimators could be easily estimated, see, for example, Engle (1982), Embrechts et al. (2002), and many others. However, in most of the situations, it is impossible to obtain the explicit solutions for the MLE's. In this situation, an algorithm has to be used to compute the MLE's. In this situation, academics and practitioners could apply the matrices developed in this paper to obtain better estimation. In addition, the moment matrices F_1 and F_2 could be used in hypothesis tests for ARCH models in econometrics and statistics. Thus, the moment matrices could help academics and practitioners to obtain better modeling and testing in many well-known portfolio optimization and other important economics and financial issues; see, for example, Fong et al. (2008) and Bai et al. (2009a, b).

Acknowledgements

The authors would like to dedicate this paper with great affection and appreciation to Professor Chris Heyde, who passed away on 6 March 2008. The research for this paper was started with our joint efforts. A talk based on this paper was presented at the Chris Heyde Memorial session, Australian Statistics Conference 2008, Melbourne, 30 June-3 July 2008. The authors are grateful to the Editor and the anonymous reviewer for substantive comments and suggestions that have significantly improved the manuscript.

References

1. Bai, Z.D., Liu, H.X., Wong, W.K. (2009a) On the Markowitz mean-variance analysis of self-financing portfolios, *Risk and Decision Analysis*, 1(1), 35–42.
2. Bai, Z.D., Liu, H.X., Wong, W.K. (2009b) Enhancement of the applicability of Markowitz's portfolio optimization by utilizing random matrix theory, *Mathematical Finance*, 19(4), 639-667.
3. Berndt, E., Hall, B., Hall, R., Hausman, J. (1974) Estimation and inference in nonlinear structural models, *Annals of Economic and Social Measurement*, 3, 653–665.
4. Bingham, N.H., Kiesel, R. (2002) Semi-parametric modelling in finance: theoretical foundations, *Quantitative Finance*, 2, 368–385.
5. Blattberg, R.C., Gonedes, N.J. (1974) A comparison of stable and student distribution as statistical models for stock prices, *Journal of Business*, 47, 244–280.
6. Bollerslev, T. (1986) Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31, 307–327, 1986.
7. Bollerslev, T. (1987) A conditionally heteroskedastic time series model for speculative prices and rates of return. *Review of Economics and Statistics*, 69, 542–547.
8. Chan, N.H. (2002) *Time Series: Applications to Finance*, John Wiley and Sons, New York.
9. Clark, P.K. (1973) A subordinated stochastic process model with finite variance for speculative prices, *Econometrica*, 37, 135–155.
10. Embrechts, P., McNeil, A., Straumann, D. (2002) Correlation and dependence in risk management: Properties and pitfalls, in *Risk Management: Value at Risk and Beyond*, edited by M.A.H. Dempster, Cambridge University Press, Cambridge, pp 176–223.
11. Engle, R.F. (1982) Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation, *Econometrica*, 50, 987–1006.
12. Fama, E.F. (1963) Mandelbrot and the stable Paretian hypothesis, *Journal of Business*, 36, 420–429.
13. Fama, E.F. (1965a) The behaviour of stock market prices, *Journal of Business*, 38, 34–105.
14. Fama, E.F. (1965b) Portfolio analysis in a stable Paretian market, *Management Science*, 11, 401–419.
15. Fang, K.T., Kotz, S., Ng, K.W. (1987) *Symmetric Multivariate and Related Distributions*, Chapman & Hall, London.
16. Fang, K.T., Zhang, Y. (1990) *Generalized Multivariate Analysis*, Science Press, Beijing and Springer-Verlag, Berlin.
17. Faraway, J.J. 2006 Extending the linear model with R. CRC. Chapman & Hall/CRC. New York.
18. Fielitz, B.D., Rozelle, J.P. (1983) Stable distributions and mixtures of distributions hypotheses for common stock return, *Journal of American Statistical Association*, 78, 28–36.
19. Fomby, T., Hill, R.C., Johnson, S.R. (1984) *Advanced Econometric Methods*, Springer-Verlag, New York.
20. Fong, W.M., Lean, H.H., Wong, W.K. (2008) Stochastic dominance and behavior towards risk: The market for internet stocks, *Journal of Economic Behavior and Organization*, 68(1), 194–208.
21. Gouriéroux, C. (1997) *ARCH Models and Financial Applications*, Springer-Verlag, New York.
22. Hamilton, J.D. (1994) *Time Series Analysis*, Princeton University Press, Princeton, New Jersey.

23. Heyde, C.C., Kou, S.G. (2004) On the controversy over tailweight of distributions. *Operations Research Letters*, 32, 399–408.
24. Kelker, D. (1970) Distribution theory of spherical distributions and location-scale parameter generalization, Ŝankhya, 32, 419–430.
25. Knight, J., Satchell, S. (2001) *Returns Distributions in Finance* (eds.), Butterworth Heinemann, Oxford.
26. Li, W.K. (2004) *Diagnostic Checks in Time Series*, Chapman & Hall/CRC, New York.
27. Liu, S. (2004) On diagnostics in conditionally heteroskedastic time series models under elliptical distributions, *Stochastic Methods and their Applications — Journal of Applied Probability*, 41A, 393–405.
28. Liu, S., Heyde, C.C. (2008) On estimation in conditional heteroskedastic time series models under non-normal distributions, *Statistical Papers*, 49, pp 455–469.
29. Liu, S., Neudecker, H. (2009) On pseudo maximum likelihood estimation for multivariate time series models with conditional heteroskedasticity, *Mathematics and Computers in Simulation*, 79, 2556–2565.
30. Liu, S., Polasek, W. (1999) Maximum likelihood estimation for the VAR-VARCH model: A new approach, In: *Modelling and Decisions in Economics, Essays in Honor of Franz Ferschl*, U. Leopold-Wildburger, G. Feichtinger and K.-P. Kistner, (Editors), Physica-Verlag, Heidelberg, pp. 99–113.
31. Lütkepohl, H. (2005) *New Introduction to Multiple Time Series Analysis*, Springer-Verlag, Berlin.
32. Magnus, J.R., Neudecker, H. (1999) *Matrix Differential Calculus with Applications in Statistics and Econometrics, revised edition*, John Wiley and Sons, Chichester.
33. Mak, T.K., Wong, H., Li, W.K. (1997) Estimation of nonlinear time series with conditional heteroscedastic variances by iteratively weighted least squares, *Computational Statistics & Data Analysis*, 24(2), 169–178.
34. McAleer, M. (2005) Automated inference and learning in modeling financial volatility, *Econometric Theory*, 21, 232–261.
35. McNeil, A.J., Frey, R., Embrechts, P. (2005) *Quantitative Risk Management: Concepts, Techniques, Tools*, Princeton University Press, Princeton.
36. Rachev, S., Mittnik, S. (2000) *Stable Paretian Models in Finance*, John Wiley and Sons, Chichester.
37. Spanos, A. (1994) On modeling heteroskedasticity: the Student's t and elliptical linear regression models, *Econometric Theory*, 10, 286–315.
38. Tiku, M.L., Wong, W.K., Vaughan, D.C., Bian, G. (2000) Time series models with nonnormal innovations: symmetric location-scale distributions, *Journal of Time Series Analysis*, 21(5), 571–596.
39. Tsay, R.S. (2005) *Analysis of Financial Time Series, second edition*, John Wiley and Sons, New York.
40. Tse, Y.K. (1991) Price and volume in the Tokyo stock exchange: an exploratory study, in *Japanese Financial Market Research*, edited by W. Ziemba, T. Bailey and W. Hamao, pp 91–119.
41. Wong, W.K., Bian, G. (2005) Estimating parameters in autoregressive models with asymmetric innovations, *Statistics and Probability Letters*, 71(1), 61–70.