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Matching preclusion for cube-connected cycles

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Response to Reviewers

1. All minor typos were corrected according to the suggestion of two reviewers.
2. Item 2 of reviewer #1 and item 11 of reviewer #3: We added the missing figure.
3. Item 8 of reviewer #1: We revised the statement of Theorem 2.12.

Matching preclusion for cube-connected cycles ⋆

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Abstract

Matching preclusion is a measure of robustness in the event of edge failure in interconnection networks. The matching preclusion number of a graph G with even order is the minimum number of edges whose deletion results in a graph without perfect matchings and the conditional matching preclusion number of G is the minimum number of edges whose deletion leaves the resulting graph with no isolated vertices and without perfect matchings. We consider matching preclusion of cube-connected cycles network CCC_n. By using the super-edge-connectivity of vertex-transitive graphs, the super cyclically edge-connectivity of CCC_n for $n = 3, 4$ and 5, Hall's Theorem and the strengthened Tutte's Theorem, we obtain the matching preclusion number and the conditional matching preclusion number of CCC_n and classify respective optimal matching preclusion sets.

Key words: Matching preclusion; networks; cube-connected cycles; cyclically edge-connectivity.
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1 Introduction

Throughout this paper, all graphs are assumed to be connected and of even order. For a graph G, $V(G)$ and $E(G)$ are its vertex set and edge set respectively. A *perfect matching* in a graph G is an independent edge set covering all the vertices of G. For $F \subseteq E(G)$, if $G - F$, the subgraph of G by deleting F from it, has no perfect matchings, then we call F a *matching preclusion set.* The *matching preclusion number* of G, denoted by $mp(G)$, is defined to be the smallest cardinality among all matching preclusion sets. Correspondingly, the matching preclusion set of size $mp(G)$ is called an *optimal matching preclusion set* (or in short, *optimal solution*). If G has no perfect matchings, then we set $mp(G) = 0$. The concept of matching preclusion was introduced by Brigham, Harary, Biolin and Yellen [2]. Since some distributed algorithms require each node of the distributed system to be matched with a neighboring partner node, the matching preclusion number measures the robustness of a graph as a communications network topology for them [10]. Meanwhile, matching preclusion number has a theoretical connection with conditional connectivity and "changing and unchanging of invariants". In a network, a vertex with a special matching vertex after edge failure any time implies that tasks running on a fault vertex can be shifted onto its matching vertex. Therefore, under this fault assumption, larger $mp(G)$ signifies higher fault tolerance. In [2], the matching preclusion numbers of the complete graphs, complete bipartite graphs and the hypercubes were computed. Moreover, all the optimal solutions were characterized. Thereafter, the matching preclusion numbers of lots of interesting network graphs are computed and the optimal solutions are established, such as Cayley graphs generated by 2-trees and the hyper Petersen networks [9], Cayley graphs generalized by transpositions and (n, k)-star graphs [10], tori and related Cartesian products [11], (n, k)-bubble-sort graphs [12], augmented cubes [13], burnt pancake graphs [19], balanced hypercubes [21], k-ary n-cubes [30], restricted HL-graphs and recursive circulant $G(2^m, 4)$ [24].

By deleting the edges incident to any vertex in a graph, the resulting subgraph has no perfect matchings. Hence the following result with respect to the upper bound on $mp(G)$ is attained.

Theorem 1.1 ([10]) *For a graph G, $mp(G) \leq \delta(G)$ holds, where $\delta(G)$ is the minimum degree of G.*

As mentioned before, larger $mp(G)$ implies higher fault tolerance. Hence it is desirable for a network G to have $\delta(G)$ as its matching preclusion number. If $mp(G) = \delta(G)$, then we call G *maximally matched*. If all edges in a matching preclusion set are incident with a common vertex, then we call it a *trivial matching preclusion set* (or in short, *trivial solution*). A graph G is called *super matched* if $mp(G) = \delta(G)$ and every optimal solution is trivial. From the known results for the networks, one can see that almost all of them are super matched. Ordinarily, in the event of a random link failure, that all of the links incident to a single vertex fail simultaneously is unlikely to happen. That is, there should be another number higher than $mp(G)$ to measure fault tolerance. Motivated by this, the conditional matching preclusion number $mp_1(G)$ of a graph G was introduced to look for obstruction sets beyond those induced by a single vertex by Cheng, Lesniak and Lipman [6]. More precisely, $mp_1(G)$ is defined to be the minimum number of edges whose deletion results in a graph without isolated vertices and with no perfect matchings. Similarly, if a graph G has no perfect matchings, then we set $mp_1(G) = 0$. For a graph with no isolated vertices, a path $u \to v \to w$, where the degree of both u and w is 1, is a basic obstruction to a perfect matching. So to produce such an obstruction set, one can first pick any path $u \to v \to w$ in the original graph, and then delete all the edges incident to either u or w but not uv and vw if $uw \notin E(G)$, or delete all the edges incident to either u or w but not uv and vw, and also delete uw if $uw \in E(G)$. We define

$$v_e(G) = \min\{d_G(u) + d_G(w) - 2 - y_G(u, w) : u \text{ and } w \text{ are ends of a path of length } 2\},$$

where $d_G(\cdot)$ is the degree function and $y_G(u, w) = 1$ if $uw \in E(G)$ and 0 otherwise.

Theorem 1.2 ([4]) *For a graph G with $\delta(G) \geq 3$, $mp_1(G) \leq v_e(G)$ holds.*

If $mp_1(G) = v_e(G)$, then we call G *conditionally maximally matched.* For a conditionally maximally matched graph G, a conditional matching preclusion set, a set of edges the removal of which results in a subgraph without perfect matchings and without isolated vertices, achieving $v_e(G)$ is called an *optimal conditional matching preclusion set* (or in short, *optimal conditional solution*). Moreover, if the optimal conditional solution is of the form as the basic obstruction set induced above, it is called a *trivial optimal conditional solution* [4]. Pick a cubic graph H for example. It is easy to check that if H has no triangles, then a trivial optimal conditional solution is shown in Fig. 1 (left); if H contains triangles, then a trivial optimal conditional solution is shown in Fig. 1 (right).

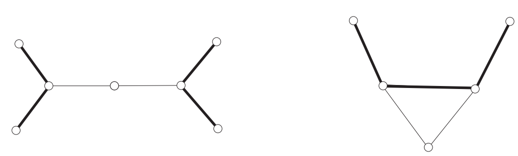

Fig. 1. The thick edges illustrate trivial optimal conditional solutions.

As mentioned earlier, the matching preclusion number measures the robustness of the requirement in the event of link failures, so it is desirable for an interconnection network to be super matched; Similarly, it is desirable to have the property that all the optimal conditional solutions are trivial as well. Such an interconnection network is called *conditionally super matched*. Up to now, the conditional matching preclusion numbers of lots of interesting network graphs were computed and the optimal conditional solutions were established, such as the arrangement graphs [7], alternating group graphs and split-stars [8], Cayley graphs generated by 2-trees and the hyper Petersen networks [9], Cayley graphs generalized by transpositions and (n, k)-star graphs [10], tori and related Cartesian products [11], augmented cubes [13], burnt pancake graphs [19], balanced hypercubes [21], restricted HL-graphs and recursive circulant $G(2^m, 4)$ [24], k-ary n-cubes [30] and hypercube-like interconnection networks [25].

The cube-connected cycles network was introduced by Preparata and Vuillemin earlier in 1981 for using as a network topology in parallel computing [27]. It is the earliest example of what later became known as X-connected cycles, with X being an arbitrary network. The cube-connected cycles network possesses lots of topological properties of Hypercubes but with lower links. In graph theory, the cube-connected cycles CCC_n is an undirected cubic graph, formed by replacing each vertex of the hypercube graph Q_n by a cycle of length n (see CCC_3 in Fig. 2), whose definition will be introduced in Section 2.

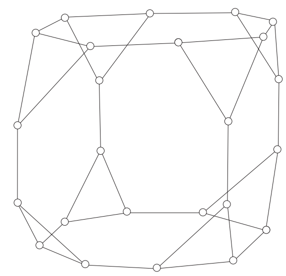

Fig. 2. The network graph CCC_3.

There have been lots of studies on this kind of graphs, such as the embedding into faulty hypercubes [3], the diameter [17], the cycles [18], the crossing number [28], the page number [29] and so on. In this paper, we consider the matching preclusion number and conditional matching preclusion number for the cube-connected cycles and the classification of the corresponding optimal solutions. It should be noted that although CCC_n is a cubic graph, it is not quite easy to solve this problem. The crucial point is that they do not have a recursive structure. Since cube-connected cycles are vertex-transitive graphs, the structural properties of vertex-transitive graphs will play an important role in our proof. For the conditional matching preclusion, the characterization of cyclic edge-cuts is used. Our paper is organized as follows. In Section 2, we present several structural properties of CCC_n. In Section 3, we compute its matching preclusion number and characterize the optimal solutions. In Section 4, we compute its conditional matching preclusion number and characterize the optimal conditional solutions.

2 Preliminaries

In this section, we first present the accurate definition of CCC_n and then study its properties related to cyclic edge-cuts. More precisely, we will show that CCC_n is super cyclically edge-connected for small n. Finally, we establish the structure of cubic graphs with respect to their matching preclusion number and conditional matching preclusion number.

The graph CCC_n has $n \times 2^n$ vertices labelled $(l, \boldsymbol{x})$, where l is an integer between 0 and $n-1$, called the *level* of the vertex, and $\boldsymbol{x}$ is a binary string of length n, called the *row* of $\boldsymbol{x}$. All arithmetic on indices and levels concerning CCC_n is assumed to be modulo n. Two vertices $(l, \boldsymbol{x})$ and $(l', \boldsymbol{y})$ are adjacent if and only if either $\boldsymbol{x} = \boldsymbol{y}$ and $|l - l'| = 1$, or $l = l'$ and $\boldsymbol{x} \stackrel{l}{=} \boldsymbol{y}$. The latter case means that $\boldsymbol{x}$ and $\boldsymbol{y}$ differ in exactly the bit in position l.

As mentioned earlier, CCC_n is formed by replacing each vertex of a hypercube graph Q_n by a cycle of length n. Furthermore, from the definition, one can easily check that all the triangles (resp. quadrangles and pentagons) in CCC_3 (resp. CCC_4 and CCC_5) are exactly those cycles replaced on each vertex in Q_3 (resp. Q_4 and Q_5) in the procedure of creating

CCC_n from Q_n for $n = 3, 4$ and 5. This will be used as a fact time and time again without proof.

Now we are going to present some structural properties with respect to cyclic edge-cuts of CCC_n. Several notation and terminologies are introduced here. For $X \subset V(G)$, we denote $\overline{X} := V(G) \setminus X$. Then X and $\overline{X}$ form a partition of $V(G)$. We call the set of all edges with one end-vertex in X and the other in $\overline{X}$ *an edge-cut*, denoted by $[X, \overline{X}]$ (or $\partial(X)$). Set $d(X) = |[X, \overline{X}]|$. For a subgraph G' of G, we simply write $\partial(V(G'))$ as $\partial(G')$ and let $d(G') = |\partial(G')|$. An edge-cut is called *trivial* if its removal separates a singleton. A *cyclic edge-cut* of a graph G is an edge-cut such that its removal separates two cycles. If G has a cyclic edge-cut, then it is called *cyclically separable*. For a cyclically separable graph G, the *cyclic edge-connectivity* $c\lambda(G)$ is the cardinality of a minimum cyclic edge-cut of G. A minimum cyclic edge-cut is called *trivial* if it isolates a shortest cycle. We call a graph *super cyclically edge-connected*, if every minimum cyclic edge-cut is trivial. We are going to show that CCC_n is super cyclically edge-connected for $n = 3, 4$ and 5.

Let G be a graph. For a nonempty subset $U \subseteq V(G)$, the maximal subgraph of G containing U is denoted by $G[U]$; For a nonempty subset $X \subseteq E(G)$, the minimal subgraph of G containing X is denoted by $G[X]$. For $U \subset V(G)$, the subgraph $G - U = G[V(G) \setminus U]$.

Suppose G_1 and G_2 are two disjoint graphs. $G_1 + G_2$ is the disjoint union of these two graphs.

The following are some useful results for this paper.

Let Γ be a group and S be an inverse-closed generating set of it. The *Cayley graph* $G = G(\Gamma, S)$ is constructed as follows. Its vertex set is $V(G) = \Gamma$ and for any $x, y \in \Gamma$, x is adjacent to y in G if and only if $xy^{-1} \in S$.

Theorem 2.1 ([1]) *CCC_n is a Cayley graph.*

A graph G is called *vertex-transitive* if for any two vertices x, y in $V(G)$, there exists an automorphism φ of G such that $\varphi(x) = y$. It is known that every Cayley graph is vertex-transitive. Therefore, the corollary below follows immediately.

Corollary 2.2 *CCC_n is a vertex-transitive graph.*

For cubic vertex-transitive graphs, their cyclic edge-connectivities have been investigated earlier in 1995.

Theorem 2.3 ([23]) *Let G be a cubic vertex-transitive or edge-transitive graph with girth g. Then $c\lambda(G) = g$.*

By summarizing the results considering the cycles of CCC_n in [18], we have the following result.

Theorem 2.4 ([18]) *$g(CCC_n) = \min\{n, 8\}$, where $g(CCC_n)$ denotes the girth of this graph.*

By Corollary 2.2, Theorems 2.3 and 2.4, the following theorem arises directly.

Theorem 2.5 $c\lambda(CCC_n) = \min\{n, 8\}$.

Cyclic edge-connectivity relates to cycles in graphs. Hence the following rule for determining a subgraph of a cubic graph containing cycles is needed.

Lemma 2.6 *Let G be a cubic graph and $C = [X, \overline{X}]$ be an edge-cut of G for some set $X \subset V(G)$. If $x = |X| \geq |C| = c$, then $G[X]$ contains a cycle.*

Proof. Let $G' = G[X]$. Then $|E(G')| = \frac{3x-c}{2}$. We have

$$|E(G')| - |V(G')| = \frac{3x - c}{2} - x = \frac{x - c}{2} \geq 0,$$

whence $x \geq c$. Hence G' contains a cycle. This completes the proof. □

In the following, we will use a kind of "conditional edge-connectivity" called "restricted edge-connectivity" to discuss another kind called "super cyclically edge-connectivity". An edge set F of a graph G is a 2-*restricted edge-cut* if $G - F$ is disconnected and each component of $G - F$ contains at least 2 vertices. Let $\lambda^{(2)}(G)$ be the minimum size of all 2-restricted edge-cuts and $\xi_2(G) = \min\{d(U) : |U| = 2 \text{ and } G[U] \text{ is connected}\}$. A graph G is called to be *optimal-$\lambda^{(2)}$* if $\lambda^{(2)}(G) = \xi_2(G)$.

Theorem 2.7 ([15, 31]) *The n-cube Q_n is optimal-$\lambda^{(2)}$. That is, every 2-restricted edge-cut is of size at least $2n - 2$.*

The following several lemmas are still needed.

Lemma 2.8 *For $3 \leq n \leq 8$, any cyclic edge-cut F of CCC_n of size n is independent.*

Proof. By Theorem 2.5, $c\lambda(CCC_n) = n$. Since F is a cyclic edge-cut with the minimum cardinality, $CCC_n - F$ has exactly two components, denoted by G_1 and G_2.

Suppose on the contrary that F is not independent. Then there are two edges in F incident with a common vertex v. We may suppose that $v \in V(G_1)$. Since CCC_n is a cubic graph, v is not contained in any cycle of G_1.The set of edges between $V(G_1) \setminus \{v\}$ and $V(G_2) \cup \{v\}$ forms a new cyclic edge-cut with size $|F| - 1$, which contradicts that $c\lambda(CCC_n) = n$. □

Lemma 2.9 *Let F be a minimum edge-cut of a graph G and C be a cycle of G. Then $|F \cap E(C)| \neq 1$.*

Proof. Suppose $F \cap E(C) = \{uv\}$. Then u and v are still connected in $G - F$. This contradicts F being a minimum edge-cut. □

Lemma 2.10 *For $n = 3, 4$ or 5, let F be a minimum cyclic edge-cut of CCC_n. Then either F is trivial, or there exists an n-cycle intersecting with both of the two components of $CCC_n - F$.*

Proof. Note that the edges of CCC_n can be divided into two parts: Some of those are the edges inherited from Q_n and others are come from the n-cycles.

Let G_1 and G_2 be the two components of $CCC_n - F$. Suppose that neither F is trivial nor there exists an n-cycle intersecting with both G_1 and G_2.

Then the edges in F are all inherited from Q_n. By contracting the n-cycles in CCC_n, we can see that F corresponds to an edge-cut of Q_n. Furthermore, since F does not isolate an n-cycle, F corresponds to a 2-restricted edge-cut of Q_n. By Theorem 2.7, $|F| \geq 2n-2$. On the other hand, since F is a minimum cyclic edge-cut of CCC_n, by Lemma 2.5, $|F| = n$. Hence $n \geq 2n-2$, which is impossible for $n = 3, 4$ or 5. This completes the proof. □

In [32], Z. Zhang and B. Zhang have shown that a connected cubic vertex-transitive graph G with $g(G) \geq 7$ is super cyclically edge-connected, and the lower bound of the girth is the best possible. In the next theorem, we are going to show that CCC_n for $n = 3, 4$ and 5, the specific vertex-transitive networks with girth smaller than 7, are also super cyclically edge-connected.

Theorem 2.11 *For $n = 3$, 4 and 5, CCC_n is super cyclically edge-connected.*

Proof. By Theorem 2.5, $c\lambda(CCC_n) = n$. To show that CCC_n is super cyclically edge-connected, it suffices to prove that any cyclic edge-cut F of size n isolates an n-cycle in CCC_n for $n = 3$, 4 and 5. Since F is a cyclic edge-cut with the minimum cardinality, $CCC_n - F$ has exactly two components, each containing cycles, denoted by G_1 and G_2. By Lemma 2.8, F is independent.

Note that each vertex of CCC_n is incident with a unique n-cycle when $3 \leq n \leq 5$.

In the following, we are going to prove that each n-cycle in CCC_n lies entirely in G_1 or G_2. If this holds, then by Lemma 2.10, F is trivial. We consider three cases according to the values of n.

Case 1. $n = 3$. Pick out any 3-cycle C in CCC_3. Since F is independent (by Lemma 2.8), C is contained entirely in G_1 or G_2.

Case 2. $n = 4$. Suppose that there exists a 4-cycle $C = v_1v_2v_3v_4v_1$ (in CCC_4) which intersects both G_1 and G_2. Since F is independent (by Lemma 2.8), at most two edges of C lie in F. By Lemma 2.9, $|F \cap E(C)| = 2$.

Since F is independent, without loss of generality, we may assume that $v_1v_4, v_2v_3 \in F$ and $v_1 \in V(G_1)$. By connectedness (of $CCC_4 - F$), $v_2 \in V(G_1)$ and $v_3, v_4 \in V(G_2)$. Let e_1 and e_2 be edges in G_1 incident with v_1 and v_2, respectively.

Suppose $|V(G_1) \setminus \{v_1, v_2\}| < 4$. That is, $|V(G_1)| \leq 5$. Since G_1 contains a cycle and CCC_4 does not contain 5-cycle, G_1 contains a 4-cycle. Further, G_1 is a 4-cycle (or else, G_1 is a 4-cycle with a pended edge, which contradicts that F is independent). This is impossible since CCC_4 does not contain two adjacent 4-cycles. So $|V(G_1) \setminus \{v_1, v_2\}| \geq 4$. By Lemma 2.6, $G_1 - \{v_1, v_2\}$ contains cycles.

Then $F' = F \cup \{e_1, e_2\} \setminus \{v_1v_4, v_2v_3\}$ is a new cyclic edge-cut of size 4. Note that $CCC_4 - F'$ consists of two components, namely $G_1 - \{v_1, v_2\}$ and $G_2 +$

$\{v_1v_4, v_2v_3, v_1v_2\}$. If all 4-cycles do not intersect both $G_1 - \{v_1, v_2\}$ and $G_2 + \{v_1v_4, v_2v_3, v_1v_2\}$, then F' is trivial by Lemma 2.10, that is, F' isolates a 4-cycle and hence $G_1 - \{v_1, v_2\}$ is a 4-cycle. Thus we obtain two adjacent 4-cycles in CCC_4, a contradiction.

Therefore, there exists a 4-cycle intersecting with both $G_1 - \{v_1, v_2\}$ and $G_2 + \{v_1v_4, v_2v_3, v_1v_2\}$, and we can repeat the above procedure for F' similarly as done for F. Since there are no adjacent 4-cycles in CCC_4, we obtain a minimum cyclic edge-cut F'' with all its edges inherited from Q_n. By Lemma 2.10, F'' is trivial and we arrive at two adjacent quadrangles, a contradiction too. This completes the proof.

Case 3. $n = 5$. Suppose that there exists a 5-cycle $C = v_1v_2v_3v_4v_5v_1$ (in CCC_5) which intersects both G_1 and G_2. Since F is independent (by Lemma 2.8), at most two edges of C lie in F. By Lemma 2.9, $|F \cap E(C)| \neq 1$.

So we only need to deal with the case $|F \cap E(C)| = 2$. By symmetry, we may assume $v_1v_5, v_2v_3 \in F \cap E(C)$ and $v_1 \in V(G_1)$. By connectedness, $v_2 \in V(G_1)$ and $v_3, v_4, v_5 \in V(G_2)$.

The following discussion is similar to that in Case 2. So we just present the sketch. Suppose $|V(G_1) \setminus \{v_1, v_2\}| < 5$. That is, $|V(G_1)| \leq 6$. Since CCC_5 does not contain 6-cycle, G_1 contains a 5-cycle. Furthermore, G_1 is a 5-cycle and we obtain two adjacent pentagons, a contradiction. But this is impossible since CCC_5 does not contain two adjacent 5-cycles. So $|V(G_1) \setminus \{v_1, v_2\}| \geq 5$. By Lemma 2.6, $G_1 - \{v_1, v_2\}$ contains cycles.

Then we arrive at a cyclic edge-cut F' of size 5, the deletion of which results in two components $G_1 - \{v_1, v_2\}$ and $G_2 + \{v_1, v_2\}$. Note that $G_2 + \{v_1, v_2\}$ is the graph obtained from G_2 by adding v_1, v_2 and all edges incident with at least one of them. If any 5-cycle lies entirely in $G_1 - \{v_1, v_2\}$ or $G_2 + \{v_1, v_2\}$, then by Lemma 2.10, F' is trivial, that is, $G_1 - \{v_1, v_2\}$ is a pentagon. It follows that CCC_5 contains a 4-cycle or 6-cycle, a contradiction. If there exists another 5-cycle intersecting with F, then we do the same procedure of F' as F and arrive at a contradiction too. This completes the proof. □

In the next step, we are going to present the structure of cubic graphs with respect to their matching preclusion number and conditional matching preclusion number. The following result, which can be viewed as kind of a strengthened Tutte's theorem, is needed.

A graph G with at least k vertices is said to be *k-factor-critical* (in short, *k-fc*) if the deletion of any k vertices in G results in a graph with a perfect matching. By the definition, a 1-fc graph is of odd order. We call a vertex set $S \subseteq V(G)$ *matchable* to $G - S$ if the (bipartite) graph H_S, which is obtained from G by contracting each component $c \in \mathcal{C}_{G-S}$ to a singleton and deleting all the edges inside S, contains a matching covering the vertices of S, where $\mathcal{C}_{G-S}$ denotes the set of the components of $G - S$. Note that the set S may be empty.

Theorem 2.12 ([14, p. 41]) *Every graph G, which may be disconnected or of odd order, contains a set $S \subseteq V(G)$ with the following two properties:*

(i) S is matchable to $G - S$;
(ii) every component of $G - S$ is 1-fc.

Moreover, for any such set S, G has a perfect matching if and only if $|S| = |\mathcal{C}_{G-S}|$.

Lemma 2.13 *Let G be a cubic graph with a perfect matching and matching preclusion number $mp(G)$ (resp. conditional matching preclusion number $mp_1(G)$). Let F be an optimal solution (resp. optimal conditional solution). Then there exists $S \subseteq V(G)$ satisfying the followings:*

(i) $G-F-S$ has exactly $|S|+2$ 1-fc components which are denoted by G_i, $1 \le i \le |S|+2$;

(ii) $\sum_{i=1}^{|S|+2} d(G_i) - 2mp(G) \le 3|S|$ (resp. $\sum_{i=1}^{|S|+2} d(G_i) - 2mp_1(G) \le 3|S|$).

Proof. By Theorem 2.12, there exists a set $S \subseteq V(G-F)$ satisfying (i) S is matchable to $G-F-S$, (ii) every component of $G-F-S$ is 1-fc, and $G-F$ has a perfect matching if and only if $|S| = |\mathcal{C}_{G-F-S}|$. By (i), $|\mathcal{C}_{G-F-S}| \ge |S|$.

Since F is an optimal solution (resp. optimal conditional solution), $G-F$ has no perfect matchings. So $|\mathcal{C}_{G-F-S}| \ge |S|+1$. Furthermore, since $|\mathcal{C}_{G-F-S}|$ and $|S|$ have the same parity, $|\mathcal{C}_{G-F-S}| \ge |S|+2$.

For any edge $e \in F$, by the definition of an optimal solution (resp. optimal conditional solution), $G-F+e$ has a perfect matching, where $G-F+e$ stands for the graph by adding e to $G-F$. Hence $c_o(G-F+e-S) \le |S|$ by Tutte's Theorem: *A graph H has a perfect matching if and only if for any $S \subseteq V(H)$, $c_o(H-S) \le |S|$, where $c_o(H-S)$ denotes the number of odd components of $H-S$.* Since every 1-fc component is also an odd component, we have

$$|S| \ge c_o(G-F+e-S) \ge |\mathcal{C}_{G-F-S}| - 2 \ge |S|.$$

Thus, $|\mathcal{C}_{G-F-S}| - 2 = |S|$ and each edge in F connects two 1-fc components. Now we count the number of edges between S and the 1-fc components, denoted by N, in two different ways. On one hand, S can contribute at most $3|S|$ to N. On the other hand, all the 1-fc components send out $\sum_{i=1}^{|S|+2} d(G_i) - 2mp(G)$ (resp. $\sum_{i=1}^{|S|+2} d(G_i) - 2mp_1(G)$) edges to N. Thus

$$\sum_{i=1}^{|S|+2} d(G_i) - 2mp(G) \le N \le 3|S| \text{ (resp. } \sum_{i=1}^{|S|+2} d(G_i) - 2mp_1(G) \le N \le 3|S|).$$

This completes the proof. □

3 Matching Preclusion

In this section, we compute the matching preclusion number of CCC_n and characterize the optimal solutions. Before proving the main result of this section, we present several useful results about vertex-transitive graphs.

Theorem 3.1 ([20, Lemma 5.5.26]) *Let G be a k-regular vertex-transitive graph. Then G is k-edge-connected.*

A k-regular graph G is said to be *super-edge-connected* (or simply *super-*λ) if every minimum edge-cut is the set of edges incident with a common vertex. The following result gives a sufficient and necessary condition for a graph to be super-λ.

Theorem 3.2 ([22]) *Let G be a k-regular connected vertex-transitive graph which is neither a complete graph nor a cycle. Then G is super-λ if and only if it does not contain k-cliques.*

Hall's theorem is also needed to determine the matching preclusion number of bipartite $CCC_n's$. The method we use here is similar to that in [4, 5]. To the completeness of this paper, we present it.

Theorem 3.3 (Hall's Theorem [26]) *Let G be a bipartite graph with bipartition W and B. Then G has a perfect matching if and only if $|W| = |B|$ and for any $U \subseteq W$, $|N_G(U)| \geq |U|$ holds, where $N_G(U)$ denotes the neighborhood of U.*

A k-fc graph of order $n \geq k$ is said to be *trivial* if $n = k$ and *nontrivial* otherwise.

Lemma 3.4 ([16]) *For $k \geq 1$, every nontrivial k-fc graph is $(k+1)$-edge-connected.*

We can deduce from the above lemma that a 1-fc graph with at least two vertices is 2-edge-connected and hence contains cycles. So a 1-fc graph is trivial if and only if it is a singleton.

Now we are ready to determine the matching preclusion number of CCC_n and characterize the optimal solutions.

Theorem 3.5 *$mp(CCC_n) = 3$. Moreover, all optimal solutions are trivial when $n \geq 4$. Any optimal solution of CCC_3 is either trivial, or a trivial cyclic 3-edge-cut (edge-cut of size 3), or a set of thick edges shown in the configuration of Fig. 1 (right).*

Proof. Let F be an optimal solution of CCC_n. Then there exists $S \subseteq V(CCC_n)$ satisfying the conclusion of Lemma 2.13. We shall keep the notation introduced in Lemma 2.13.

By Corollary 2.2 and Theorem 3.1, CCC_n is 3-edge-connected. That is, $d(G_i) \geq 3$. Substituting this into the inequality shown in Lemma 2.13, we have $mp(CCC_n) \geq 3$. Combining this with Theorem 1.1, we obtain that $mp(CCC_n) = 3$.

For the characterization of the optimal solutions, there are two cases according to the parity of n.

Case 1. n is even.

In this case, CCC_n is bipartite. Assume that W and B are the bipartition of CCC_n. Firstly we show that each optimal solution is an edge-cut.

Since F is an optimal solution, $CCC_n - F$ has no perfect matchings. By Hall's Theorem, there exists $U \subseteq W$ such that $|N_{CCC_n - F}(U)| \leq |U| - 1$. On the other hand, since F is a matching preclusion set with the smallest cardinality, for every $e \in F$, $CCC_n - F + e$ has a perfect matching. Also by Hall's Theorem, we have $|N_{CCC_n-F+e}(U)| \geq |U|$. Adding one edge e to $CCC_n - F$ will increase the neighbors of U at most one, so $|N_{CCC_n-F+e}(U)| \leq |N_{CCC_n-F}(U)| + 1$.

Combining three inequations above, we obtain that $|U| = |N_{CCC_n-F}(U)|+1 = |N_{CCC_n-F+e}(U)|$. This implies that e is incident with a vertex in U. Denote $U' = N_{CCC_n-F}(U)$. The edges sending out from U are divided into two parts: One lies in F and one goes into U'. By $|F| = 3$, U sends exactly $3|U| - 3$ edges to U'. Since $|U'| = |U| - 1$, there are no edges connecting U' to $W - U$. Therefore, F is an edge-cut.

Since CCC_n is triangle-free, it follows directly that F is a trivial edge-cut by Theorem 3.2, that is, F isolates a singleton.

Case 2. n is odd.

Since $d(G_i) \geq 3$ for each i and $mp(CCC_n) = 3$, we obtain that

$$3|S| \leq \sum_{i=1}^{|S|+2} d(G_i) - 2mp(CCC_n) \leq 3|S|.$$

Thus $d(G_i) = 3$ for each i. Moreover, S is an independent set.

(A) Suppose $n \geq 5$. From Lemma 2.13 we know that $CCC_n - F - S$ consists of $|S| + 2$ 1-fc components G_i $(1 \leq i \leq |S|+2)$, i.e., $CCC_n - F - S = \sum_{i=1}^{|S|+2} G_i$, the disjoint union of G_i's. We want to show that $S = \varnothing$. Suppose not. Then $|S| + 2 \geq 3$. Choose a 1-fc component of $CCC_n - F - S$ arbitrarily, say G_1.

Let G' be the subgraph induced by $\overline{V(G_1)}$. Then G' is of order greater than $|S| + |V(G_2)| + |V(G_3)| \geq 3$. By Lemma 2.6, G' contains cycles. Since $d(G_1) = 3$, $\partial(G_1)$ is not a cyclic edge-cut by Theorem 2.5. So G_1 contains no cycles. Hence G_1 is a singleton. Since the 1-fc component is chosen arbitrarily, all 1-fc components are singletons. By deleting the three edges in F, the resulting graph is bipartite (with bipartition $(S, \bigcup_{i=1}^{|S|+2} V(G_i))$). But this is impossible since there are at least $2^n > 3$ disjoint odd cycles in CCC_n. So $S = \varnothing$ and hence $CCC_n - F = G_1 + G_2$. Therefore, the three edges in F connect the two components and hence form a 3-edge-cut. Since $n \geq 5$, CCC_n does not contain triangles. By Theorem 3.2, it is trivial, that is, it isolates a singleton.

(B) Suppose $n = 3$.

If S is empty, then there are two 1-fc components G_1 and G_2 and the three edges in F connect the two components. Hence F is a 3-edge-cut. If one of G_1 and G_2 is a singleton, then F is trivial. If both G_1 and G_2 are non-trivial, then they both contain cycles and hence F forms a cyclic 3-edge-cut. Furthermore, by Theorem 2.11, F isolates a triangle.

Now we assume that $|S| \geq 1$. Note that each vertex in CCC_3 lies in a triangle and all the triangles are disjoint. For each vertex $s \in S$, let sx_1x_2s be a triangle in CCC_3. Since S is independent, $x_1, x_2 \in \overline{S}$. Suppose x_1 and x_2 lie in the same component, say G_i for some i. Since G_i is a 1-fc, G_i contains a cycle. Since $|\overline{V(G_i)}| \geq 2|S| + 1 \geq 3$, $CCC_3 - V(G_i)$ contains a cycle by Lemma 2.6. Hence $\partial(G_i)$ is a cyclic edge-cut. By Theorem 2.11, G_i is a triangle or $CCC_3 - V(G_i)$ is a triangle. This is impossible since all the triangles in CCC_3 are disjoint. So x_1 and x_2 lie in different components, say G_1 and G_2, respectively. Suppose one of G_1 and G_2, say G_1, is not a singleton. Then G_1 contains a cycle. By the same argument above, this is impossible. So G_1 and G_2 are singletons. By a similar argument above for the other vertices in S, we are left to the case that each vertex in S corresponds to two singleton 1-fc components. Hence we have $|S| + 2 \geq 2|S|$, i.e., $|S| \leq 2$.

Suppose $|S| = 2$. By the discussion above, $CCC_3 - S$ consists of four singletons. Hence CCC_3 contains only 6 vertices which is impossible. So $|S| = 1$ and there are exactly three 1-fc components G_1, G_2 and G_3. Since $\sum_{i=1}^{3} |V(G_i)| + 1 = 24$, there is a 1-fc component containing at least 9 vertices. Without loss of generality, we may assume that $|V(G_1)| \geq 9$. This implies that $|V(CCC_3)| - |V(G_1)| \geq 3$ and $d(G_1) = 3$, by Lemma 2.6, G_1 and the subgraph G' induced by $\overline{V(G_1)}$ contain cycles. Therefore, $\partial(G_1)$ is a cyclic 3-edge-cut. By Theorem 2.11, G' is a triangle. Hence G_2 and G_3 are singletons. Thus, we get the structure of F shown in Fig. 1 (right). □

4 Conditional Matching Preclusion

In this section, the conditional matching preclusion number of CCC_n is computed, and further, the conditional optimal solutions are classified. By the results in the above section, we can see the following.

Theorem 4.1 *$mp_1(CCC_3) = 3$ and any optimal conditional solution is either trivial or a trivial cyclic 3-edge-cut (three edges isolating a triangle).*

Hence we restrict our discussion to $n \geq 4$. The following is our main result.

Theorem 4.2 *For $n \geq 4$, $mp_1(CCC_n) = 4$. For $n \geq 6$, the optimal conditional solutions are trivial; the optimal conditional solutions for CCC_4 are either trivial or the set of thick edges shown in Fig. 3; the optimal conditional solutions for CCC_5 are either trivial or the set of thick edges shown in Fig. 4.*

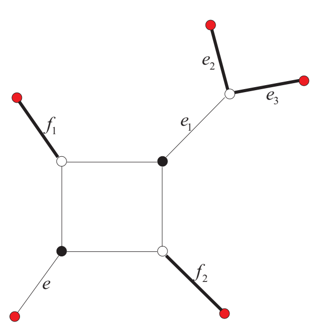

Fig. 3. The set of thick edges is an optimal conditional solution for CCC_4.

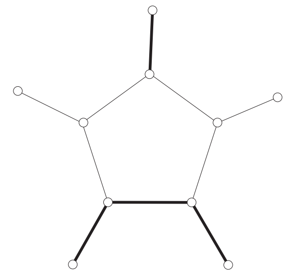

Fig. 4. The set of thick edges is an optimal conditional solution for CCC_5.

Proof. Similarly to the proof in Theorem 3.5, we have $mp_1(G) \geq 3$. By Theorem 1.2, we have $mp_1(G) \leq 4$. If $mp_1(G) = 3$, then by Theorem 3.5, the optimal conditional solutions isolate a vertex. This contradicts the definition of the conditional matching preclusion number. Therefore, $mp_1(CCC_n) = 4$.

Now we are going to characterize the optimal conditional solutions. Let F be an optimal conditional solution. Then F is of size 4 and $CCC_n - F$ has no perfect matchings. There are two cases according to the parity of n.

Case 1. n is even.

In this case, CCC_n is a bipartite graph. By a similar argument and keeping the notation defined in Case 1 of the proof of Theorem 3.5 (the only difference is $|F| = 4$ now), we first obtain that $U' = N_{CCC_n - F}(U)$, $|U| = |U'| + 1$, $U \cup U'$ sends out five edges. Furthermore, by the definition of U and U', $F = [U, B \backslash U']$. This implies that $d(U \cup U') = 5$ and $C = \partial(U \cup U')$ is an edge-cut of size 5. Hence $|U| \geq 2$ and $|W \setminus U| \geq 1$.

If $|U| = 2$ or $|W \setminus U| = 1$, then F is trivial. We are done.

So now we assume that $|U| \geq 3$ and $|W \setminus U| \geq 2$. By Lemma 2.6, we can easily check that C is a cyclic edge-cut of size 5. By Theorem 2.5, $n = 4$. If C is independent, then using a similar proof as in Case 2 of the proof of Theorem 2.11, we obtain a contradiction. So there exist two edges e_2 and e_3 in C such that they are incident with a vertex w. Since F is a conditional matching preclusion set (in other words, F does not isolate a singleton), there is an edge $e_1 \notin C$ incident with w. Then $C' = (C \setminus \{e_2, e_3\}) \cup \{e_1\}$ is a cyclic edge-cut of size 4 in CCC_4. By Theorem 2.11, C' is a trivial cyclic 4-edge-cut. More precisely, let $F = \{e_2, e_3, f_1, f_2\}$, $C' = \{e_1, f_1, f_2, e\}$, where $e \in C$ is an edge connecting a vertex in U' and a vertex in $W \setminus U$. Let Q be the 4-cycle isolated by C'. If $w \in V(Q)$, then $e_2, e_3 \in E(Q)$ and hence C is not a cyclic edge-cut. So $w \notin V(Q)$. Moreover, e, e_1, f_1 and f_2 are incident with 4 vertices of Q. Now the edge-cut C separates CCC_4 into two parts. One of these parts is a subgraph induced by the vertices of Q and w. Without loss of generality, we may assume that this part is $CCC_4[U \cup U']$. Since $|U| = |U'| + 1$, the white vertices in Fig. 3 lie in U. Recall that $F = [U, B \setminus U']$. So we have Fig. 3.

Case 2. n is odd and $n \geq 5$.

By Lemma 2.13, there exists $S \subseteq V(G)$ satisfying the conclusions (i) and (ii).

By putting $d(G_i) \geq 3$ and $mp_1(CCC_n) = 4$ into (ii), we obtain that there is at most one edge in the subgraph of CCC_n induced by S.

(A) Suppose S is independent. Then $\sum_{i=1}^{|S|+2} d(G_i) - 2mp_1(G) = 3|S|$, that is, $\sum_{i=1}^{|S|+2} d(G_i) - 8 = 3|S|$. It follows that at least one 1-fc component, say G_1, satisfies that $d(G_1) \geq 4$. Furthermore, since CCC_n is cubic, $d(G_1)$ and $|V(G_1)|$ are of the same parity. Then $d(G_1) \geq 5$. Since $d(G_i) \geq 3$ for $2 \leq i \leq |S|+2$, $d(G_1) = 5$ and $d(G_i) = 3$. This implies that G_1 is not trivial and hence G_1 contains a cycle. Since $d(G_i) = 3$ for $2 \leq i \leq |S|+2$, $\partial(G_i)$ is not a cyclic edge-cut by Theorem 2.5. Thus G_i is a singleton for $2 \leq i \leq |S|+2$.

If $|S| = 0$, then $CCC_n - F = G_1 + G_2$. This is impossible since $|F| = 4$ and G_2 is a singleton.

Suppose $|S| = 1$. Then $CCC_n - F = G_1 + G_2 + G_3$, where G_2 and G_3 are singletons. Since CCC_n is triangle-free, the edges in F cannot connect the two trivial components. It follows directly that F is trivial.

If $|S| \geq 2$, then there are at least 5 vertices outside G_1. Also $d(\overline{G_1}) = d(G_1) = 5$, so there is a cycle outside G_1 by Lemma 2.6. Therefore, $\partial(G_1)$ is a cyclic edge-cut of size 5 in CCC_n. By Theorem 2.5, $n = 5$. Furthermore, by Theorem 2.11, $CCC_5[\overline{G_1}]$ or G_1 is a 5-cycle. If $CCC_5[\overline{G_1}]$ is a 5-cycle, then we obtain the structure shown in Fig. 4. If the latter holds, then recall that each G_i is a singleton for $2 \leq i \leq |S|+2$. Then $CCC_n - E(G_1) - F$ is bipartite. This is impossible since there are at least $2^5 = 32$ disjoint odd cycles of length 5 in it.

(B) Suppose S is not independent. Then $CCC_n[S]$ contains exactly one edge, say e. Therefore, $d(G_i) = 3$ for each i. Furthermore, each G_i is a singleton by Theorem 3.2. Then $CCC_n - (F \cup \{e\})$ is bipartite, which is impossible. □

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