2016

L(j,k)-labeling number of Cartesian product of path and cycle

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$L(j,k)$-labeling number of Cartesian product of path and cycle [⋆]

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Abstract

For positive numbers j and k, an $L(j,k)$-labeling f of G is an assignment of numbers to vertices of G such that $|f(u)-f(v)| \geq j$ if $d(u,v)=1$, and $|f(u)-f(v)| \geq k$ if $d(u,v)=2$. The span of f is the difference between the maximum and the minimum numbers assigned by f. The $L(j,k)$-labeling number of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labelings of G. In this article, we completely determine the $L(j,k)$-labeling number $(2j \leq k)$ of the Cartesian product of path and cycle.

Key words: $L(j,k)$-labeling, Cartesian product.
1991 MSC: [2010] 05C78, 05C15

1 Introduction

The *Packet Radio Network* (PRN) is a computer wireless network that transmit data among computers. Its performance is greatly affected by two major types of interference, the *Direct interference* and *Hidden terminal interference*. The $L(j,k)$-labeling, with $j \leq k$, is motivated by the code assignment problem to avoid hidden interference in PRN [1, 14].

Let G be a graph and let $V(G)$ and $E(G)$ be its vertex set and edge set, respectively. For any two vertices u and v, the distance between u and v in G is denoted by $d_G(u,v)$ (or simply $d(u,v)$). All notation not defined in this article can be found in the book [2].

For positive numbers j and k, an $L(j,k)$-*labeling* f of G is an assignment of numbers to vertices of G such that $|f(u)-f(v)| \geq j$ if $uv \in E(G)$, and $|f(u)-f(v)| \geq k$ if $d(u,v)=2$.

⋆ This work is partially supported by Faculty Research Grant, Hong Kong Baptist University.
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The *span* of f is the difference between the maximum and the minimum numbers assigned by f. The $L(j,k)$*-labeling number* of G, $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labeling of G.

$L(j,k)$-labeling numbers of graphs were studied in many articles, please refer to [4–7, 9–12, 15, 21, 22] for $j \geq k$ and [1, 3, 8, 14, 16–19] for $j \leq k$.

Lemma 1.1 *Let j and k be two positive integers with $j \leq k$. If G is a graph and H is an induced subgraph of G, then $\lambda_{j,k}(G) \geq \lambda_{j,k}(H)$.*

Noted that Lemma 1.1 does not hold if H is not an induced subgraph.

2 $L(j,k)$-labeling number of $P_2 \square C_m$

The *Cartesian product* of two graphs G and H is the graph $G \square H$ with vertex set $V(G) \times V(H)$ and two vertices (x, y) and (x_0, y_0) are adjacent if $x = x_0$ and $yy_0 \in E(H)$ or $y = y_0$ and $xx_0 \in E(G)$. Throughout this article, the vertices of the graph $P_n \square C_m$ are displayed as a matrix with n rows and m columns. Moreover. we denote $v_{s,t}$ as the vertex which lies at the s-th row and t-th column where $0 \leq s \leq n-1$ and $0 \leq t \leq m-1$.

In this section, we consider the $L(j,k)$-labeling number of graphs $P_2 \square C_m$, where j and k are two positive numbers with $k \geq 2j$.

Lemma 2.1 **([16])** *The $L(j,k)$-labeling number $\lambda_{j,k}(C_4) = \lambda_{j,k}(P_4) = j + k$ for $k \geq 2j$.*

Lemma 2.2 *Let Y be the graph shown in Fig. 1, we have $\lambda_{j,k}(Y) = j + k$ for $k \geq 2j$.*

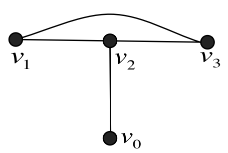


Fig. 1. Graph Y.

Proof. Let f be an $L(j,k)$-labeling of Y. Since vertex v_0 is distance two apart from v_1 and v_3 simultaneously, $|f(v_0) - f(v_1)| \geq k$ and $|f(v_0) - f(v_3)| \geq k$. Moreover, $|f(v_1) - f(v_3)| \geq j$ because v_1 and v_3 are adjacent. Without loss of generality, assume $f(v_1) > f(v_3)$. The following three cases

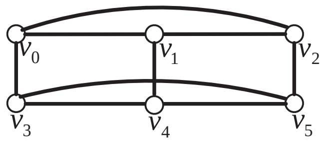
1) $f(v_0) > f(v_1) > f(v_3)$,
2) $f(v_1) > f(v_3) > f(v_0)$,
3) $f(v_1) > f(v_0) > f(v_3)$

all imply the span of f is at least $j + k$.

On the other hand, assigning $f(v_0) = 0$, $f(v_1) = k$, $f(v_2) = j$ and $f(v_3) = j + k$ imply f being a $(j + k)$-$L(j, k)$-labeling of Y. Therefore, $\lambda_{j,k}(Y) = j + k$. □

Theorem 2.3 *The $L(j, k)$-labeling number $\lambda_{j,k}(P_2 \Box C_3) = 3j + k$ for $k \geq 2j$.*

Proof. We define a $(3j + k)$-$L(j, k)$-labeling f by $f(v_0) = 0$, $f(v_1) = j$, $f(v_2) = 2j$, $f(v_3) = 3j + k$, $f(v_4) = 2j + k$ and $f(v_5) = j + k$ (see the right figure). Hence $\lambda_{j,k}(P_2 \Box C_3) \leq 3j + k$.



On the other hand, let $\lambda_{j,k}(P_2 \Box C_3) = \lambda$ and f be a λ-$L(j, k)$-labeling of the graph. Let $v \in f^{-1}(0)$ and $u \in f^{-1}(\lambda)$. For every $w \in V(P_2 \Box C_3) \setminus \{v, u\}$, we have $1 \leq d(w, v) \leq 2$ and $1 \leq d(w, u) \leq 2$ and thus, $f(w) \in [j, \lambda - j]$. Moreover, $P_2 \Box C_3 - \{v, u\}$ is isomorphic to one of the following graphs: C_4, P_4 and Y in Fig. 1. By Lemmas 1.1, 2.1 and 2.2, the length of the interval $[j, \lambda - j]$ is at least $j + k$. Therefore, $\lambda \geq 3j + k$ and the proof is completed. □

Two labels are *t-separated* if the difference between them is at least t. According to the definition of $L(j, k)$-labeling, the labels of adjacent vertices is j-separated and the labels of vertices at distance two is k-separated.

Let σ be a positive real number. The *σ-cycle*, denoted by $S(\sigma)$, is the circle obtained from the closed interval $[0, \sigma]$ by identifying 0 and σ into a single point. For any $x \in \mathbb{R}$, $[x]_\sigma$ denotes the remainder of x upon division of σ.

Lemma 2.4 *Let x, y and r be positive real numbers, then $|[x]_r - [y]_r|$ equals to $[x - y]_r$ or $r - [x - y]_r$.*

Theorem 2.5 *Let m be a positive integer. If $m \equiv 0 \pmod 6$ and $k \geq 2j$, then $\lambda_{j,k}(P_2 \Box C_m) = 2k + j$.*

Proof. Suppose f is an $L(j, k)$-labeling of $P_2 \Box C_m$ and without loss of generality, let $f(v_{0,0}) = 0$. Consider the set $S = \{v_{0,0}, v_{0,1}, v_{1,0}, v_{0,m-1}\}$, which induces a subgraph of $P_2 \Box C_m$ that is isomorphic to $K_{1,3}$. It is easy to verify that $\lambda_{j,k}(K_{1,3}) = 2k + j$ and thus, we have $\lambda_{j,k}(P_2 \Box C_m) \geq \lambda_{j,k}(K_{1,3}) = 2k + j$ by Lemma 1.1.

On the other hand, we define a labeling f as follows:

$$f(v_{0,t}) = [\tfrac{tk}{2}]_{3k} \text{ if } t \text{ is even;}$$
$$f(v_{0,t}) = [\tfrac{(t-1)k}{2}]_{3k} + j \text{ if } t \text{ is odd;}$$
$$f(v_{1,t}) = [f(v_{0,[t+1]_m}) + k]_{3k}, \text{ where } t = 0, 1, \ldots, m - 1.$$

Note that the labels of adjacent vertices at the same row are j-separated and the labels of vertices with distance two apart at the same row are k-separated. Moreover, $f(v_{1,t})$ and $f(v_{0,[t+1]_m})$ are k-separated. As a result, for any vertex $v_{s,t}$, it is sufficient to check the differences between $f(v_{s,t})$ and $f(v_{1-s,t})$ as well as $f(v_{s,t})$ and $f(v_{1-s,[t\pm1]_m})$.

For vertex $v_{1-s,t}$, since $s = 0, 1$, we have $|f(v_{1-s,t}) - f(v_{s,t})| = |f(v_{1,t}) - f(v_{0,t})|$. Consider the following cases.

1) Suppose t is even and $0 \le t \le m-2$.
$|f(v_{1,t})-f(v_{0,t})| = |[[\frac{tk}{2}]_{3k}+j+k]_{3k}-[\frac{tk}{2}]_{3k}| = [j+k]_{3k} = \left\{\begin{array}{ll} 2k-j & \text{if } t \equiv 4 \pmod 6 \\ k+j & \text{otherwise} \end{array}\right\} \ge j.$
2) Suppose t is odd and $0 \le t \le m-2$.
$|f(v_{1,t}) - f(v_{0,t})| = |[[\frac{(t+1)k}{2}]_{3k} + k]_{3k} - [\frac{(t-1)k}{2}]_{3k} - j| = \left\{\begin{array}{ll} 2k-j & \text{if } t \equiv 1 \pmod 6 \\ k+j & \text{otherwise} \end{array}\right\} \ge j.$
3) Suppose $t = m-1$ and hence t is odd.
$|f(v_{1,m-1})-f(v_{0,m-1})| = |[f(v_{0,0})+k]_{3k}-f(v_{0,m-1})| = |k-[\frac{(m-2)k}{2}]_{3k}-j| = |k-2k-j| = k+j$, since $m \equiv 0 \pmod 6$.

For vertices $v_{1-s,[t\pm1]_m}$, it suffices to consider $|f(v_{1,[t\pm1]_m}) - f(v_{0,t})|$ for all $t \in \mathbb{Z}_m$. Also, it is easy to show that $|f(v_{1,[t\pm1]_m}) - f(v_{0,t})|$ is either k or $2k$.

Thus $\lambda_{j,k}(P_2 \square C_m) \le 2k+j$ when $m \equiv 0 \pmod 6$. Therefore, $\lambda_{j,k}(P_2 \square C_m) = 2k+j$. $\square$

Theorem 2.6 *Let m be a positive integer greater than 8. If $m \equiv 3 \pmod 6$ and $k \ge 2j$, then $\lambda_{j,k}(P_2 \square C_m) = 2k+2j$.*

Proof. Firstly, we define a labeling f for $P_2 \square C_m$ as follows:

$$f(v_{s,t}) = \left\{\begin{array}{ll} j[s+t]_2 + k[t]_3 & \text{if } 0 \le t \le m-2, \\ (s+1)j + 2k & \text{if } t = m-1. \end{array}\right.$$

Note that the image of f lies in $[0, 2k+2j]$. The verification of f is an $L(j,k)$-labeling is similar to Theorem 2.5 and we omit it here. Thus, we have $\lambda_{j,k}(P_2 \square C_m) \le 2k+2j$.

We are now going to prove that $\lambda_{j,k}(P_2 \square C_m) \ge 2k+2j$.

Suppose $\lambda_{j,k}(P_2 \square C_m) < 2k+2j$. Let f be an $L(j,k)$-labeling of $P_2 \square C_m$ and $f(v_{0,0}) = 0$. Then $f(v_{0,2}), f(v_{1,1}) \in [k, 2k+2j)$ since $v_{0,0}, v_{0,2}$ and $v_{1,1}$ are mutually distance two apart.

Firstly, we assume that $f(v_{1,1}) \in [k, k+2j)$ and $f(v_{0,2}) \in [2k, 2k+2j)$ (the cases $f(v_{0,2}) \in [k, k+2j)$ and $f(v_{1,1}) \in [2k, 2k+2j)$ are similar). Since $v_{0,2}$, $v_{1,1}$ and $v_{1,3}$ are mutually distance two apart, then $f(v_{1,3}) \in [0, 2j)$. The range of $f(v_{1-s,[t+1]_m})$ can be determined if the ranges of $f(v_{s,t})$ and $f(v_{1-s,[t-1]_m})$ were known. Since m is odd, the range of $f(v_{0,m-1})$ can be obtained by repeating the steps above. Also, the ranges of such labels are $[0, 2j)$, $[k, k+2j)$ and $[2k, 2k+2j)$. Using the ranges of $f(v_{1,m-2})$ and $f(v_{0,m-1})$, we can determine the ranges of $f(v_{1,0})$ and $f(v_{0,1})$ and so on. As a result, the ranges of $f(v_{s,t})$ are characterized as follows:

$$\begin{aligned} f(v_{s,t}) &\in [0, 2j), && \text{if } t \equiv 0 \pmod 3; && (1) \\ f(v_{s,t}) &\in [k, k+2j), && \text{if } t \equiv 1 \pmod 3; && (2) \\ f(v_{s,t}) &\in [2k, 2k+2j), && \text{if } t \equiv 2 \pmod 3, && (3) \end{aligned}$$

where $s = 0, 1$.

Note that $v_{0,t}$ and $v_{1,t}$ are adjacent and thus, $|f(v_{1,t}) - f(v_{0,t})| \ge j$. Moreover, according to (1), we have $f(v_{1,0}) \in [0, 2j)$. Since $f(v_{0,0}) = 0$ and $v_{0,0}$ as well as $v_{1,0}$ are adjacent,

$f(v_{1,0}) \in [j, 2j)$. This implies that $f(v_{1,0}) \geq f(v_{0,0}) + j$.

According to (2), the values $f(v_{0,1}), f(v_{1,1}) \in [k, k + 2j)$. Furthermore, $v_{0,1}$ and $v_{1,0}$ are at distance two together with $f(v_{0,1}) \geq k + j$ implies that $f(v_{0,1}) \in [j + k, k + 2j)$. Since the length of the interval $[j + k, k + 2j)$ is j, hence $f(v_{1,1}) \notin [j + k, k + 2j)$ and therefore, $f(v_{0,1}) \geq f(v_{1,1}) + j$.

Similarly, the values $f(v_{0,2})$ and $f(v_{1,2}) \in [2k, 2k + 2j)$ according to (3). Moreover, since $v_{1,0}$ and $v_{1,2}$ are at distance two, $f(v_{1,2}) \in [j + 2k, 2k + 2j)$. The length of the interval $[j+2k, 2k+2j)$ is j and thus, $f(v_{0,2}) \notin [j+2k, 2k+2j)$. This implies that $f(v_{1,2}) \geq f(v_{0,2})+j$.

Repeat this process, we conclude that

$$f(v_{s,t}) \geq f(v_{1-s,t}) + j \text{ if } (s + t) \equiv 1 \pmod 2. \tag{4}$$

Combining (3) and (4), we have $f(v_{0,m-1}) \in [2k, 2k + j)$ and therefore $f(v_{0,m-1}) - f(v_{0,1}) < (2k + j) - (k + j) = k$, which contradicts with $v_{0,m-1}$ and $v_{0,1}$ being at distance two. As a result, $\lambda_{j,k}(P_2 \square C_m) \geq 2k + 2j$ and the proof completes. □

Theorem 2.7 *Let m be a positive integer greater than 3. If $m \not\equiv 0 \pmod 3$ and $k \geq 2j$, then $\lambda_{j,k}(P_2 \square C_m) = 3k$.*

Proof. Suppose f, if exists, is an $L(j, k)$-labeling with $\lambda_{j,k}(P_2 \square C_m) < 3k$ and $f(v_{0,0}) = 0$. Since vertices $v_{0,0}$, $v_{0,2}$ and $v_{1,1}$ are mutually distance two apart, $f(v_{0,0})$, $f(v_{0,2})$ and $f(v_{1,1})$ are mutually k-separated. Therefore, $f(v_{0,2}), f(v_{1,1}) \in [k, 3k)$. Moreover, vertex $v_{1,3}$ is distance two from $v_{1,1}$ and $v_{0,2}$ and thus, $f(v_{1,3}) \in [0, k)$. Similar to the proof of Theorem 2.6, we conclude that $f(v_{0,[6p]_m}), f(v_{1,[6p+3]_m}) \in [0, k)$, where $p \in \mathbb{Z}$.

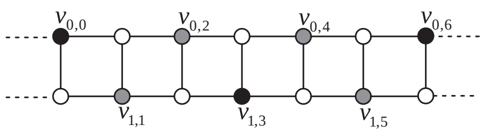


Fig. 2. Labels of black vertices lie in $[0, k)$.

1) When $m \equiv 1, 2 \pmod 6$, we have $f(v_{0,m-2}) \in [0, k)$. It contradicts $v_{0,0}$ and $v_{0,m-2}$ are distance two apart and $f(v_{0,0}) = 0$.
2) When $m \equiv 4, 5 \pmod 6$, we have $f(v_{1,m-1}) \in [0, k)$. It contradicts $v_{0,0}$ and $v_{1,m-1}$ are distance two apart and $f(v_{0,0}) = 0$.

As a result, $\lambda_{j,k}(P_2 \square C_m) \geq 3k$ for $m \not\equiv 0 \pmod 3$.

On the other hand, we define several $3k$-$L(j, k)$-labelings for $P_2 \square C_m$ with different m. For each of the following cases, the defined labeling is a $3k$-$L(j, k)$-labelings for $P_2 \square C_m$ and the verification is routine, hence we omit the details.

1) When $m \equiv 1 \pmod 6$, we define a $3k$-$L(j, k)$-labeling f_1 by

$$f_1(v_{0,t}) = \begin{cases} [\frac{tk}{2}]_{3k} & \text{if } 0 \le t \le m-2, \\ 3k & \text{if } t = m-1. \end{cases}$$

$$f_1(v_{1,t}) = \begin{cases} [k + f_1(v_{0,t+1})]_{3k} & \text{if } 0 \le t \le m-4, \\ [k + f_1(v_{0,[t+1]_m})]_{\frac{7k}{2}} & \text{if } m-3 \le t \le m-1. \end{cases}$$

2) When $m = 8$, we define a $3k$-$L(j,k)$-labeling f_2 as Fig. 3 shows.

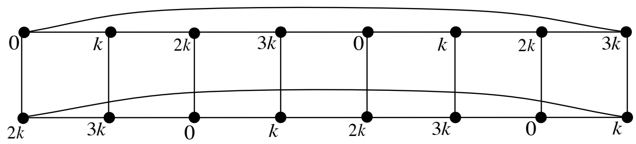


Fig. 3. A $3k$-$L(j,k)$-labeling for $P_2 \square C_8$.

When $m \equiv 2 \pmod 6$ and $m > 8$, we define a $3k$-$L(j,k)$-labeling f_3 for $P_2 \square C_m$ by

$$f_3(v_{0,t}) = \begin{cases} [\frac{t}{2}]_3 k & \text{if } 0 \le t \le m-15, \\ [\frac{t-m+14}{2}]_{\frac{7}{2}} k & \text{if } m-14 \le t \le m-1. \end{cases}$$

$$f_3(v_{1,t}) = \begin{cases} [k + f_2(v_{0,t+1})]_{3k} & \text{if } 0 \le t \le m-15, \\ [k + f_2(v_{0,[t+1]_m})]_{\frac{7k}{2}} & \text{if } m-14 \le t \le m-1. \end{cases}$$

3) When $m = 4$, we define a $3k$-$L(j,k)$-labeling f_4 for $P_2 \square C_4$ as Fig. 4 shows.

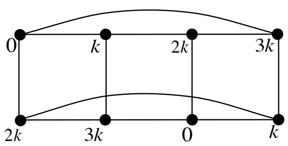


Fig. 4. A $3k$-$L(j,k)$-labeling for $P_2 \square C_4$.

When $m \equiv 4 \pmod 6$ and $m \ge 10$, we define a $3k$-$L(j,k)$-labeling f_5 for $P_2 \square C_m$ by

$$f_5(v_{0,t}) = \begin{cases} [\frac{t}{2}]_3 k & \text{if } 0 \le t \le m-12, \\ [\frac{8}{3} + \frac{2}{3}(t-m+11)]_{\frac{10}{3}} k & \text{if } m-11 \le t \le m-1. \end{cases}$$

$$f_5(v_{1,t}) = \begin{cases} [\frac{3k}{2} + f_4(v_{0,t})]_{3k} & \text{if } 0 \le t \le m-12, \\ [\frac{5k}{3} + f_4(v_{0,t})]_{\frac{10k}{3}} & \text{if } m-11 \le t \le m-1. \end{cases}$$

4) When $m = 5$, we define a $3k$-$L(j,k)$-labeling f_6 for $P_2 \square C_5$ as Fig. 5 shows.

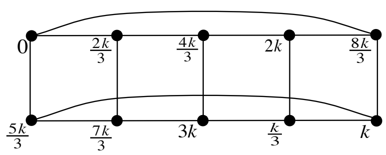


Fig. 5. A $3k$-$L(j,k)$-labeling of $P_2 \square C_5$.

When $m \equiv 5 \pmod 6$ and $m \ge 11$, we define a $3k$-$L(j,k)$-labeling f_7 for $P_2 \square C_m$ by

$$f_7(v_{0,t}) = \begin{cases} [\frac{t}{2}]_3 k & \text{if } 0 \le t \le m-6, \\ [t-m+8]_4 k & \text{if } m-5 \le t \le m-1. \end{cases}$$

$$f_7(v_{1,t}) = \begin{cases} [\frac{3+t}{2}]_3 k & \text{if } 0 \le t \le m-5, \\ [t-m+6]_4 k & \text{if } m-4 \le t \le m-1. \end{cases}$$

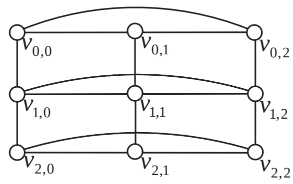
Therefore, we have $\lambda_{j,k}(P_2 \square C_m) \le 3k$. □

3 $L(j,k)$-labeling number of $P_n \Box C_3$

In this section, we discuss the $L(j,k)$-labeling of the special graph $P_n \Box C_3$, where $n \geq 3$ and j,k are two positive numbers with $k \geq 2j$.

Theorem 3.1 *For $k \geq 2j$, $\lambda_{j,k}(P_3 \Box C_3) = 2j + 2k$.*

Proof. Firstly, we define a $(2j+2k)$-$L(j,k)$-labeling for $P_3 \Box C_3$ as follows.
$f(v_{0,0}) = 0,\ f(v_{0,1}) = j,\ f(v_{0,2}) = k;$
$f(v_{1,0}) = 2j+2k,\ f(v_{1,1}) = 2k,\ f(v_{1,2}) = j+2k;$
$f(v_{2,0}) = k,\ f(v_{2,1}) = j+k,\ f(v_{2,2}) = 0.$



On the other hand, let $\lambda = \lambda_{j,k}(P_3 \Box C_3)$.

1) Suppose one of the vertices $v_{1,0}, v_{1,1}$ and $v_{1,2}$ is labeled by 0. Without loss of generality, let $f(v_{1,0}) = 0$ and thus, $\{f(v_{0,1}), f(v_{2,1}), f(v_{0,2}), f(v_{2,2})\} \subset [k, \lambda]$. Note that $v_{1,1}$, $v_{0,2}$ and $v_{2,2}$ are mutually distance two apart. If $f(v_{1,1}) \in [k, \lambda]$, then $\lambda \geq 3k \geq 2k+2j$. Hence, assume $f(v_{1,1}) \in [0, \lambda - 2k]$. Similar for $v_{1,2}$, $v_{2,1}$ and $v_{0,1}$, we may assume $f(v_{1,2}) \in [0, \lambda - 2k]$. Since $f(v_{1,0}) = 0$, we have $\lambda - 2k \geq 2j$. As a result, $\lambda \geq 2k + 2j$.
2) Suppose one vertex in the 0-th row or the 2nd row is labeled by 0. By symmetry and because $v_{1,1}$ is adjacent to $v_{1,2}$, we may assume $f(v_{0,0}) = 0$ and also $f(v_{1,1}) > f(v_{1,2})$. Since $v_{0,0}$ is of distance two from $v_{2,0}$, $v_{1,1}$ and $v_{1,2}$, $\{f(v_{1,1}), f(v_{1,2}), f(v_{2,0})\} \subset [k, \lambda]$.
   (a) Suppose $f(v_{1,1}) > f(v_{2,0}) > f(v_{1,2})$. We have $\lambda - k \geq f(v_{1,1}) - f(v_{1,2}) = f(v_{1,1}) - f(v_{2,0}) + f(v_{2,0}) - f(v_{1,2}) \geq 2k$ and hence, $\lambda \geq 3k \geq 2k + 2j$.
   (b) Suppose $f(v_{2,0}) > f(v_{1,1}) > f(v_{1,2})$. We have $f(v_{1,1}) \in [j+k, \lambda - k]$, $f(v_{1,2}) \in [k, \lambda - k - j]$ and $f(v_{2,0}) \in [j+2k, \lambda]$. Since $v_{1,1}$, $v_{0,2}$ and $v_{2,2}$ are mutually distance two apart, $f(v_{0,2}), f(v_{2,2}) \in [0, \lambda - 2k] \cup [j+2k, \lambda]$.
   If $f(v_{0,2})$ and $f(v_{2,2})$ are both in either $[0, \lambda - 2k]$ or $[j+2k, \lambda]$, then $\lambda \geq 2k + 2j$.
   If $f(v_{2,2}) \in [j+2k, \lambda]$, since $f(v_{2,0}) \in [j+2k, \lambda]$ and $v_{2,2}$ is adjacent to $v_{2,0}$, then $\lambda - (j+2k) \geq j$ and hence $\lambda \geq 2k + 2j$.
   If $f(v_{0,2}) \in [j+2k, \lambda]$, then $f(v_{1,0}) \in [k, \lambda - k]$. This implies that $\{f(v_{1,0}), f(v_{1,1}), f(v_{1,2})\} \subset [k, \lambda - k]$ and thus, $\lambda \geq 2k + 2j$.
   (c) Suppose $f(v_{2,0}) < f(v_{1,2}) < f(v_{1,1})$, then $f(v_{1,1}) \in [2k+j, \lambda]$ and $f(v_{1,2}) \in [2k, \lambda - j]$. Hence $f(v_{2,0}) \in [k, \lambda - j - k]$, $f(v_{0,1}) \in [j, \lambda - j - k]$ and $f(v_{2,1}) \in [0, \lambda - j - k]$. If $f(v_{1,0})$ belongs to one of the intervals $[2k+j, \lambda]$ and $[2k, \lambda - j]$, then $\lambda \geq 2k + 2j$. Thus, assume that $f(v_{1,0}) \in [j, 2k)$. Suppose $f(v_{0,1}) > f(v_{1,0})$ or $f(v_{2,1}) > f(v_{1,0})$. Since $v_{0,1}, v_{2,1}, v_{1,0}$ are mutually distance two apart, either $f(v_{0,1})$ or $f(v_{2,1})$ belongs to $[j+k, \lambda - j - k]$. This implies that $\lambda - j - k - (j+k) \geq 0$ and hence $\lambda \geq 2k + 2j$.
   Suppose $f(v_{0,1}), f(v_{2,1}) < f(v_{1,0})$. Then $f(v_{0,1}) \in [j, k)$ and $f(v_{2,1}) \in [0, k)$ and contradiction occurs because $v_{0,1}, v_{2,1}$ are of distance two. □

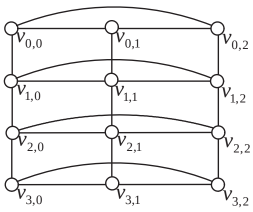
Let f be a labeling of $P_n \Box C_m$. Let $M_f = (f(v_{s,t}))_{n \times m}$ be the *labeling matrix* of $P_n \Box C_m$ under f.

Theorem 3.2 *The $L(j,k)$-labeling number $\lambda_{j,k}(P_4 \Box C_3) = \min\{3k, 3j+2k\}$ for $k \geq 2j$.*

Proof. Firstly, we define a $3k$- or $(3j+2k)$-$L(j,k)$-labeling f for $P_4 \Box C_3$.

For $2j \le k < 3j$, $M_f = \begin{pmatrix} 0 & k & j+k \\ j+2k & 3k & 2k \\ k & 0 & j \\ j+k & 2k & 3k \end{pmatrix}$.

For $k \ge 3j$, $M_f = \begin{pmatrix} j+k & 2j+k & 3j+k \\ 2j & j & 0 \\ j+2k & 2j+2k & 3j+2k \\ 2j+k & j+k & k \end{pmatrix}$.



It is not difficult to verify that f satisfies the constraints of $L(j,k)$-labeling. Therefore, $\lambda_{j,k}(P_4 \square C_3) \le \min\{3k, 3j+2k\}$.

Let f be a λ-$L(j,k)$-labeling of $P_4 \square C_3$.

1) Suppose one of vertices in the 0-th row is labeled by 0. By symmetry we may assume that $f(v_{0,0}) = 0$ and $f(v_{1,1}) > f(v_{1,2})$. Similar to Case 2) of the proof of Theorem 3.1 we obtain $\{f(v_{1,1}), f(v_{1,2}), f(v_{2,0})\} \subset [k, \lambda]$.
   (a) Suppose $f(v_{1,1}) > f(v_{2,0}) > f(v_{1,2})$. Similar to Case 2)(a) of the proof of Theorem 3.1, we have $\lambda \ge 3k$.
   (b) Suppose $f(v_{2,0}) > f(v_{1,1}) > f(v_{1,2})$. Similar to the first part of Case 2)(b) of the proof of Theorem 3.1, we have $f(v_{1,1}) \in [j+k, \lambda-k]$, $f(v_{1,2}) \in [k, \lambda-k-j]$, $f(v_{2,0}) \in [j+2k, \lambda]$ and $f(v_{0,2}), f(v_{2,2}) \in [0, \lambda-2k] \cup [j+2k, \lambda]$.

   If $f(v_{0,2})$ and $f(v_{2,2})$ lie together in either $[0, \lambda-2k]$ or $[j+2k, \lambda]$, then $\lambda \ge 3k$. If any of the intervals $[j+k, \lambda-k]$, $[k, \lambda-k-j]$, $[j+2k, \lambda]$, $[0, \lambda-2k]$ and $[j+2k, \lambda]$ contains different labels of vertices of distance two, then we obtain $\lambda \ge 3k$. Hence, assume that:

   Assumption 1. *Each of those intervals does not contain different labels of vertices of distance two.*
      (i) Suppose $f(v_{2,2}) \in [j+2k, \lambda]$, then $f(v_{0,2}) \in [0, \lambda-2k] \cap [j, \lambda] = [j, \lambda-2k]$. By the assumption above, we have $f(v_{0,2}) < f(v_{1,0}) < f(v_{2,2})$ and thus, $f(v_{1,0}) \in [j+k, \lambda-k]$.

      Similarly, since $v_{3,0}$ and $v_{2,2}$ are of distance two, we have $f(v_{3,0}) < f(v_{2,2})$ and hence, $f(v_{3,0}) \in [0, \lambda-k]$. Moreover, $v_{3,0}$ and $v_{1,0}$ are of distance two. Assume $f(v_{3,0}) < f(v_{1,0})$ and thus, $f(v_{3,0}) \in [0, \lambda-2k]$.

      Since $v_{2,1}$ and $v_{1,0}$ are of distance two, by the assumption above we have $f(v_{2,1}) \notin [j+k, \lambda-k]$. If $f(v_{2,1}) < j+k < f(v_{1,0})$, then $f(v_{2,1}) < \lambda-2k$, which contradicts with the assumption above. Therefore, $f(v_{2,1}) > \lambda-k \ge f(v_{1,0})$ and hence $f(v_{2,1}) \in [j+2k, \lambda]$. As a result, $f(v_{2,0}), f(v_{2,1}), f(v_{2,2}) \in [j+2k, \lambda]$ and we obtain $\lambda - (2k+j) \ge 2j$, which implies $\lambda \ge 3j+2k$.
      (ii) Suppose $f(v_{0,2}) \in [j+2k, \lambda]$, then $f(v_{2,2}) \in [0, \lambda-2k]$. By Assumption 1, we have $f(v_{3,1}) > f(v_{2,2})$ and hence $f(v_{3,1}) \in [k, \lambda]$. Substitute the role of $f(v_{2,2})$ by $f(v_{2,0})$ and $f(v_{1,1})$, we have $f(v_{3,1}) < f(v_{2,0})$ and $f(v_{3,1}) < f(v_{1,1})$, respectively. Therefore, $f(v_{3,1}) \in [k, \lambda-2k]$ and $\lambda \ge 3k$.
   (c) Suppose $f(v_{2,0}) < f(v_{1,2}) < f(v_{1,1})$. Similar to the first part of Case 2)(c) of Theorem 3.1, we have $f(v_{1,1}) \in [2k+j, \lambda]$, $f(v_{1,2}) \in [2k, \lambda-j]$, $f(v_{2,0}) \in [k, \lambda-j-k]$, $f(v_{0,1}) \in [j, \lambda-j-k]$ and $f(v_{2,1}) \in [0, \lambda-j-k]$. Similar to Case (b), we assume that:

Assumption 2. *Each interval stated above does not contain different labels of vertices of distance two.*

Under Assumption 2, we have $f(v_{3,1}) < f(v_{1,1})$ and hence $f(v_{3,1}) \in [0, \lambda - k]$. Moreover, $f(v_{3,1}) \in [0, \lambda-2k-j] \cup [2k, \lambda-k]$ when compare with $f(v_{2,0})$. If $f(v_{3,1}) \in [2k, \lambda - k]$, then $\lambda \geq 3k$. So the remaining case is $f(v_{3,1}) \in [0, \lambda - 2k - j]$. Assume as following:

Assumption 3. $[0, \lambda - 2k - j]$ *does not contain two labels of vertices of distance two.*

If $f(v_{3,2}) \geq 2k$, then $\lambda > 3k$. Suppose $f(v_{3,2}) < 2k$. Since $f(v_{1,2}) \in [2k, \lambda - j]$, we have $f(v_{3,2}) \in [0, \lambda - j - k]$. Under the assumption we need to consider $f(v_{3,2}) < f(v_{2,0})$ and hence $f(v_{3,2}) \in [0, \lambda - j - 2k]$.

By considering the vertices $v_{1,1}$ and $v_{3,1}$, we have $f(v_{2,2}) \in [k, \lambda - k]$. Also, by considering the vertex $v_{3,2}$, we have $f(v_{2,1}) \in [k, \lambda - j - k]$.

Similarly, we have $f(v_{0,2}) < f(v_{1,1})$. If $f(v_{2,2}) < f(v_{0,2})$, then $2k < f(v_{0,2}) < \lambda - k$ and so $\lambda \geq 3k$. The remaining case is $f(v_{2,2}) > f(v_{0,2})$, which implies $f(v_{0,2}) \in [j, \lambda - 2k]$.

Under Assumption 3, $f(v_{0,1}) < f(v_{2,1})$ and hence $f(v_{0,1}) \in [j, \lambda - j - 2k]$.

If $f(v_{0,1}) > f(v_{0,2})$, then $f(v_{0,2}) \in [j, \lambda - j - 2k]$. Therefore, $(\lambda - j - 2k) - j \geq j$ and hence $\lambda \geq 2k + 3j$. If $f(v_{0,1}) < f(v_{0,2})$, then $f(v_{0,2}) \in [2j, \lambda - 2k]$. If $f(v_{2,2}) \in [2j, \lambda - 2k]$, then we have $\lambda > 3k$. Suppose $f(v_{2,2}) > f(v_{0,2})$, which implies $f(v_{2,2}) \in [2j + k, \lambda - k]$. Compare $f(v_{1,1})$ with $f(v_{2,2})$ lead to $f(v_{1,1}) \in [2k + 2j, \lambda]$.

Now, consider the vertex $v_{1,0}$. Initially, $f(v_{1,0}) \in [j, \lambda]$. Compare $f(v_{1,0})$ with $f(v_{0,1})$ implies $f(v_{1,0}) \in [j+k, \lambda]$. If $f(v_{1,0}) \in [k+2j, \lambda-k]$, then $\lambda > 3k$. Suppose $f(v_{1,0}) < f(v_{2,2})$, then $f(v_{1,0}) \in [j + k, \lambda - 2k]$ and hence $\lambda > 3k$. Suppose $f(v_{1,0}) > f(v_{2,2})$, then $f(v_{1,0}) \in [2j + 2k, \lambda]$. Since $v_{1,0}$ and $v_{1,1}$ are adjacent, we have $\lambda \geq 2k + 3j$.

2) Suppose one of the vertices in the 1st row is labeled by 0. By symmetry, we may assume $f(v_{1,0}) = 0$. Similar to Case 1) of Theorem 3.1, we obtain $\{f(v_{0,1}), f(v_{2,1}), f(v_{0,2}), f(v_{2,2}), f(v_{3,0})\} \subset [k, \lambda]$. If $f(v_{1,1})$ or $f(v_{1,2})$ lie in $[k, \lambda]$, then $\lambda \geq 3k$. Therefore, suppose $f(v_{1,1}), f(v_{1,2}) \in [j, \lambda - 2k]$. Without loss of generality, assume $f(v_{1,1}) > f(v_{1,2})$. Hence, $f(v_{1,1}) \in [2j, \lambda - 2k]$ and $f(v_{1,2}) \in [j, \lambda - 2k - j]$. Suppose $f(u) \in [2j, \lambda - 2k]$ for some vertex u. If u is of distance two from $v_{1,1}$, then we obtain $\lambda > 3k$. If u is adjacent to $v_{1,1}$, then $\lambda \geq 2k + 3j$. Thus, assume that $[2j, \lambda - 2k]$ does not contain $f(u)$, where u is of distance at most two from $v_{1,1}$. Similar assumption is made for the interval $[j, \lambda - 2k - j]$. Under these assumptions, since $k \geq 2j$, we have $f(v_{0,1}), f(v_{2,1}), f(v_{0,2}), f(v_{2,2}) \in (\lambda-2k, \lambda]$. To be more specific, $f(v_{0,1}), f(v_{2,1}) \in [j+k, \lambda]$ and $f(v_{0,2}), f(v_{2,2}) \in [2j + k, \lambda]$.

Note that $f(v_{0,0}), f(v_{2,0}) \in [j, \lambda]$. Since $k > j$, $f(v_{0,0})$ and $f(v_{2,0})$ are greater than $f(v_{1,1})$. Thus $f(v_{0,0}), f(v_{2,0}) \in [2j + k, \lambda]$.

If $f(v_{3,1}) \in [2j + k, \lambda]$, then $\lambda \geq 2k + 3j$. Assume $f(v_{3,1})$ less than both $f(v_{2,0})$ and $f(v_{2,2})$, which implies $f(v_{3,1}) \in [0, \lambda - j - k]$. If $f(v_{3,1}) > f(v_{1,1})$, then $\lambda - j - k \geq f(v_{3,1}) \geq 2j + k$ and hence $\lambda \geq 2k + 3j$. If $f(v_{3,1}) < f(v_{1,1})$, then $f(v_{3,1}) \leq \lambda - 3k$ and so $\lambda \geq 3k$.

According to the symmetry of graph $P_4 \square C_3$, we can conclude that $\lambda \geq \min\{3k, 3k + 2j\}$ and the proof is complete. $\square$

Theorem 3.3 *The $L(j,k)$-labeling number $\lambda_{j,k}(P_n \square C_3) = \min\{4j + 2k, 3k + j\}$ for $k \geq 2j$ and $n \geq 5$.*

Proof. Let $\lambda = \min\{4j + 2k, 3k + j\}$. Similar to the proof of Theorem 3.2, we obtain $\lambda_{j,k}(P_5 \square C_3) \geq \lambda$. Since $P_5 \square C_3$ is an induced subgraph of $P_n \square C_3$, we have $\lambda_{j,k}(P_n \square C_3) \geq \lambda_{j,k}(P_5 \square C_3) \geq \lambda$ by Lemma 1.1.

On the other hand, we define a λ-$L(j,k)$-labeling f for graph $P_n \square C_3$.

(1) For $k \geq 3j$, a labeling f for $P_n \square C_m$ is defined as follows: Let $f_0 = 0$, $f_1 = 3j + k$, $f_2 = 2j + 2k$, $f_3 = j$, $f_4 = 2j + k$, $f_5 = 4j + 2k$, $f_6 = 2j$, $f_7 = j + k$ and $f_8 = 3j + 2k$. Define $f(v_{s,t}) = f_i$, for $0 \leq s \leq n-1$, $t = 0, 1, 2$ and $i \equiv s + 3t \pmod 9$. It is not difficult to verify that f is a $(4j + 2k)$-$L(j,k)$-labeling of graph $P_n \square C_3$.

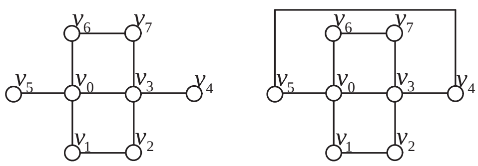
(2) For $k < 3j$, a labeling f for $P_n \square C_3$ is defined as follows: Let $f(v_{0,0}) = 0$, $f(v_{0,1}) = j$, $f(v_{0,2}) = k$, $f(v_{1,0}) = j + 3k$, $f(v_{1,1}) = 3k$ and $f(v_{1,2}) = j + 2k$. Define $f(v_{s,t}) = [f(v_{s-2,t}) + k]_{4k}$ for $2 \leq s \leq n-1$ and $t = 0, 1, 2$. It is not difficult to verify that f is a $(j + 3k)$-$L(j,k)$-labeling of graph $P_n \square C_3$. □

4 $L(j,k)$-number of $P_n \square C_m$

In this section, we consider the graph $P_n \square C_m$ when $n \geq 3$ and $m \geq 4$. We obtain the $L(j,k)$-number of $P_n \square C_m$ in two parts. The first part is to find the lower bounds on $\lambda_{j,k}(P_n \square C_m)$, and the second part is to find the upper bounds on $\lambda_{j,k}(P_n \square C_m)$.

4.1. Lower bounds on $\lambda_{j,k}(P_n \square C_m)$

Firstly, we work on two induced subgraphs H and H' of $P_n \square C_m$ as Fig. 6 shows. Then, we have



(a) Graph H (b) Graph H'

Fig. 6. Graphs H and H'.

Theorem 4.1 *For $k \geq j$, $\lambda_{j,k}(H) = \lambda_{j,k}(H') = 3k + j$.*

Proof. By Lemma 2.5 in [17], $\lambda_{j,k}(H) = 3k + j$.

Note that, the distances between every pair of vertices in H and H' are the same except that of v_4 and v_5. Since the proof of Lemma 2.5 of [17] did not involve the distance of v_4

and v_5, we have $\lambda_{j,k}(H') \geq 3k + j$. Apply the same labeling of H described in [17] to H', we conclude that $\lambda_{j,k}(H') = 3k + j$. $\square$

Since H' is an induced subgraph of $P_n \square C_4$ when $n \geq 3$, and H is an induced subgraph of $P_n \square C_m$ when $n \geq 3$ and $m \geq 5$. By Lemma 1.1 and Theorem 4.1, we obtain the following result.

Theorem 4.2 *For $k \geq j$, $m \geq 4$ and $n \geq 3$, $\lambda_{j,k}(P_n \square C_m) \geq 3k + j$.*

Moreover, we obtain a larger lower bound on $\lambda_{j,k}(P_n \square C_m)$ for odd m.

Theorem 4.3 *For odd m with $m \geq 5$ and $k \geq 2j$, $\lambda_{j,k}(P_3 \square C_m) \geq 2j + 3k$.*

Proof. Let $\lambda_{j,k}(P_3 \square C_m) = \lambda$ and f be a λ-$L(j,k)$-labeling of $P_3 \square C_m$. Suppose $\lambda < 2j+3k$. For a vertex $v \in V(P_3 \square C_m)$, then $f(v) \in [0, \lambda] \subseteq [0, 2j + 3k)$.

For convenience, let
$I_0 = [0, j)$, $I_1 = [j, k)$, $I_2 = [k, j + k)$, $I_3 = [j + k, 2k)$, $I_4 = [2k, j + 2k)$, $I_5 = [j + 2k, 3k)$, $I_6 = [3k, j+3k)$ and $I_7 = [j+3k, 2j+3k)$. Also, let $A = I_0 \cup I_2 \cup I_4 \cup I_6$ and $B = I_1 \cup I_3 \cup I_5 \cup I_7$. Note that, since the length of $I_i \cup I_{i+1}$ is less than k, $I_i \cup I_{i+1}$ does not contain two labels of vertices of distance two for $0 \leq i \leq 6$. Similarly, each of I_0, I_2, I_4 and I_6 does not contain two labels of adjacent vertices.

As Fig. 6(a) shows, H is an induced subgraph of $P_3 \square C_m$.

Claim 1 *In H, neither $f(v_0)$ and $f(v_3)$ are both in A nor both in B.*

Proof of Claim 1: Without loss of generality, let $f(v_0) < f(v_3)$.

(1) Suppose $f(v_0)$ and $f(v_3) \in A$. Then $f(v_3) \geq k$. If $f(v_3) \in I_2$, then $f(v_0) \in I_0$. Since v_1, v_5, v_6 are adjacent to v_0 and are of distance two from v_3, the labels $f(v_1)$, $f(v_5)$ and $f(v_6)$ must lie in $[2k, \lambda]$. Since v_1, v_5 and v_6 are mutually distance two, $\lambda \geq 4k \geq 3k+2j$ and contradiction occurs.
   Similarly, if $f(v_3) \in I_4$ or I_6, we can find one vertex that cannot be labeled.
(2) Suppose $f(v_0)$ and $f(v_3)$ lie in the same subinterval of B. Since $|[0, 3k)| < 3k$, one of $f(v_3)$, $f(v_1)$, $f(v_5)$ and $f(v_6)$ must lie in $[3k, \lambda]$. Also, $f(v_0)$ and $f(v_3)$ cannot both in I_7 because $|I_7| < j$.
   (a) Suppose $f(v_0)$ and $f(v_3) \in I_1$. Since $f(v_0) < f(v_3)$, $f(v_3) \in [2j, k)$ and the labels $f(v_1), f(v_5)$ and $f(v_6) \in [2j + k, \lambda]$. Hence $\lambda \geq 3k + 2j$ and contradiction occurs.
   (b) Suppose $f(v_0)$ and $f(v_3) \in I_3$. Similarly $f(v_3) \in [2j + k, 2k)$ and $f(v_1), f(v_5)$ and $f(v_6) \in [0, k) \cup [2j + 2k, \lambda]$. At least two of $f(v_1), f(v_5)$ and $f(v_6)$ lie in $[2j + 2k, \lambda]$. Hence $\lambda \geq 3k + 2j$ and contradiction occurs.
   (c) Suppose $f(v_0)$ and $f(v_3) \in I_5$. Similarly $f(v_3) \in [3j + k, 3k)$ and $f(v_1), f(v_5)$ and $f(v_6) \in [0, 2k) \cup [3j+2k, \lambda]$. At least one of $f(v_1), f(v_5)$ and $f(v_6)$ lies in $[3j+2k, \lambda]$. Hence $\lambda \geq 3k + 2j$ and contradiction occurs.
(3) Suppose $f(v_0)$ and $f(v_3)$ lie in different subintervals of B. That is, $f(v_0) \in I_i$ and $f(v_3) \in I_j$, where $1 \leq i < j$ and $i, j \in \{1, 3, 5, 7\}$. Then $f(v_2), f(v_4)$ and $f(v_7)$ cannot lie in $I_{i-1} \cup I_i \cup I_{i+1}$. Let $\{v_2, v_4, v_7\} = \{x_1, x_2, x_3\}$ and $f(x_1) < f(x_2) < f(x_3)$.

(a) Suppose $f(v_0) \in I_5$. Then $f(v_3) \in I_7$ and $f(x_i) \in [0, 2k)$ for all $i = 1, 2, 3$, which is impossible.

(b) Suppose $f(v_0) \in I_3$. Then $f(v_3) \in I_5 \cup I_7$, and $f(x_1), f(x_2)$ as well as $f(x_3) \in [0, k) \cup [2k + j, \lambda]$. Hence $f(x_2)$ and $f(x_3) \in [2k + j, \lambda]$. Moreover, $2k + j \le f(x_2) \le \lambda - k$ and $f(x_3) \in [3k + j, \lambda]$ imply $f(v_3) \in I_5$. Since v_3 and x_2 are adjacent and $|[2k+j, \lambda-k]| < j$, we have $f(v_3) \in [2j+2k, 3k)$. Also, v_1, v_5, v_6 and v_3 are mutually distance two and thus, $f(v_1)$, $f(v_5)$ and $f(v_6)$ cannot all in $[0, 2k)$. Therefore, one of them is greater than $f(v_3)$ and at least $2j + 3k$, which is impossible.

(c) Suppose $f(v_0) \in I_1$. Then $f(v_3) \in I_3 \cup I_5 \cup I_7$ and hence $k + j \le f(x_1) \le \lambda - 2k$, $2k + j \le f(x_2) \le \lambda - k$ and $f(x_3) \in [3k + j, \lambda]$. Similar to Case (b), we will get a contradiction.

As a result, we proved Claim 1.

Note that $f(v_{1,t})$ lies in either A or B. Since the first row of $P_3 \Box C_m$ is an odd cycle, there exist two adjacent vertices, say $v_{1,t}$ and $v_{1,t+1}$, such that both $f(v_{1,t})$ and $f(v_{1,t+1})$ in A or in B for some $t \in \mathbb{Z}_m$. Since vertices $v_{1,[t-1]_m}$, $v_{1,t}$, $v_{1,t+1}$, $v_{1,[t+2]_m}$, $v_{0,t}$, $v_{0,t+1}$, $v_{2,t}$ and $v_{2,t+1}$ induce a subgraph isomorphic to the graph H, it is impossible by Claim 1. $\square$

Since $P_3 \Box C_m$ is an induced subgraph of $P_n \Box C_m$ where $n \ge 3, m \ge 4$, by Lemma 1.1, we obtain the following corollary.

Corollary 4.4 *For odd $m \ge 5$, $n \ge 3$ and $k \ge 2j$, $\lambda_{j,k}(P_n \Box C_m) \ge 2j + 3k$.*

We continue to investigate the lower bounds on $\lambda_{j,k}(P_n \Box C_m)$ for odd m with $n \ge m$ and for $m \equiv 2 \pmod 4$ and $n \ge \frac{m}{2}$.

Instead of working on $L(j, k)$-number of $P_n \Box C_m$, we study the $L(0, k)$-number of $P_n \Box C_m$. That is, we only consider the distance-two vertices in $P_n \Box C_m$ and easily obtain the following claim.

Claim 2 $\lambda_{j,k}(P_n \Box C_m) \ge \lambda_{0,k}(P_n \Box C_m)$.

For convenience, we define a new graph as follows. We construct a graph with vertex set $V(P_n \Box C_m)$ and two vertices are adjacent if they are at distance two in the graph $P_n \Box C_m$. We call it the *distance-two graph* associated to $P_n \Box C_m$. For $m \equiv 2 \pmod 4$, let $m = 4l + 2$. Since $P_n \Box C_{4l+2}$ is a bipartite graph, let (X, Y) be the bipartition of $P_n \Box C_{4l+2}$. We obtain two distance-two graphs which are constructed from X and Y respectively and are isomorphic to each other. For odd m, we let $m = 2l + 1$ and obtain one distance two graph. These distance-two graphs are isomorphic to the following graph $H(n, 2l + 1)$.

Let the vertex set of the graph $P_n^2 \Box C_{2l+1}$ be $\{u_{i,j} \mid 0 \le i \le n - 1, j \in \mathbb{Z}_{2l+1}\}$, where P_n^2 is the second power of the path P_n. The graph $H(n, 2l + 1)$ is obtained from $P_n^2 \Box C_{2l+1}$ by adding the additional edges $(u_{i,j}, u_{i-1,j+1})$ for odd i with $1 \le i \le n - 1$ and $0 \le j \le 2l$; and $(u_{i,j}, u_{i+1,j+1})$ for odd i with $1 \le i \le n - 2$ and $0 \le j \le 2l$. For a fixed i, the sequence of vertices $u_{i,j}$ according to the natural order is called the i-th row of the graph.

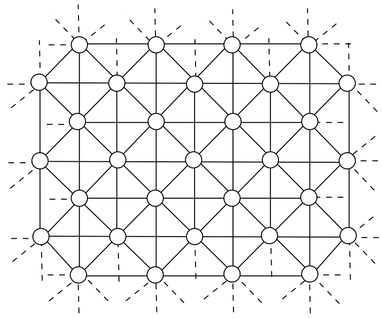


Fig. 7. A part of the distance-two graph associated with $P_n \Box C_m$.

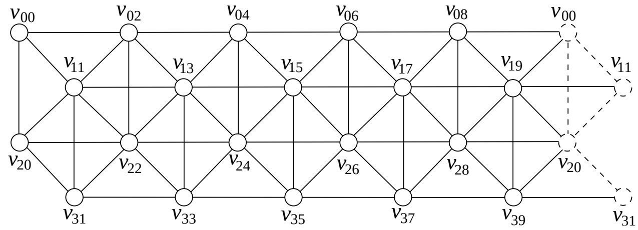


Fig. 8. A component of distance-two graph associated with $P_4 \Box C_{10}$.

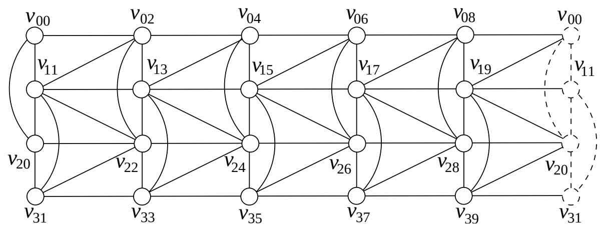


Fig. 9. Another drawing for Fig.8

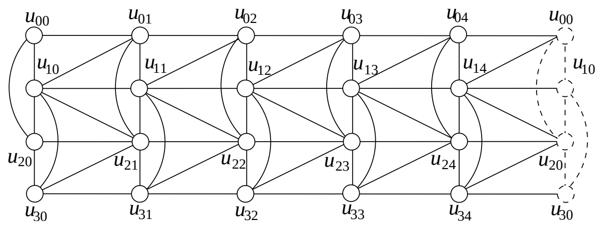


Fig. 10. The graph $H(4, 5)$.

For positive number k, an $L(k)$*-labeling* f of G is an assignment of numbers to vertices of G such that $|f(u) - f(v)| \geq k$ if $uv \in E(G)$. The *span* of f is the difference between the maximum and the minimum numbers assigned by f. Note that, without loss of generality, we may assume that 0 is a value of f. The $L(k)$*-labeling number* of G, denoted by $\lambda_k(G)$, is the minimum span over all $L(k)$-labeling of G.

Now we consider the $L(k)$-labeling problem of $H(n, 2l+1)$ instead of the $L(j,k)$-labeling problem of $P_n \Box C_m$.

Lemma 4.5 *Let $G = (V, E)$ be a graph and let k be a positive number. Suppose $f : V \to [0, \lambda]$ is an $L(k)$-labeling of a graph G, then $\lambda_k \geq (\chi - 1)k$, where χ is the chromatic number of G.*

Proof. Let $a_1, a_2, \ldots, a_s$ be all distinct values of f and we may assume that $a_1 < a_2 < \cdots < a_s$. Let V_j $(= f^{-1}(a_j))$ be the pre-image of a_j under f.

Let S and T be subsets of V. Denote by $[S, T]$ the set of edges with one end in S and the other in T.

Note that $a_{j+1} - a_j \geq k$ if $[V_j, V_{j+1}] \neq \varnothing$. Suppose there is a smallest integer i such that $[V_i, V_{i+1}] = \varnothing$. Then the neighborhood of the set V_{i+1} does not contain any vertex which is labeled by a_i. By relabeling all vertices of V_{i+1} by a_i, we get a new labeling f_1 of G which satisfies that $|f_1(u) - f_1(v)| \geq k$ when $uv \in E$. Perform the above procedure until we get a labeling g such that the distinct values of g, say $b_1, b_2, \ldots, b_r$, have the following properties:
(a) $a_1 = b_1 < b_2 < \cdots < b_r \leq a_s$;
(b) Let $W_i = g^{-1}(b_i)$. Then $[W_j, W_{j+1}] \neq \varnothing$ for each j from 1 to $r-1$.

Hence $\lambda_k \geq a_1 - a_s \geq b_1 - b_r \geq (r-1)k$. Since $r \geq \chi$, the results follows. □

By Lemma 4.5, we can obtain the lower bound on λ_k by working on the chromatic number of $H(n, 2l+1)$ for $n \geq 2l+1$. Note that when s is even, vertices $u_{s,t}$, $u_{s,t+1}$, $u_{s-1,t}$ and $u_{s+1,t}$ induced a K_4 in $H(n, 2l+1)$, thus, $\chi(H(n, 2l+1)) \geq 4$.

Chromatic numbers of $H(n, 2l+1)$:

Our aim is to show that $H(n, 2l+1)$ is not 4-colorable when $n \geq 2l+1$. In what follows we assume that $H(n, 2l+1)$ has a 4-coloring f, which maps the vertices of $H(n, 2l+1)$ to the color set $\{a, b, c, d\}$. The sequence $(f(u_{s,i}), f(u_{s,i+1}), \ldots, f(u_{s,j}))$ of length $j - i + 1$ is called the *color sequence* of the sub-row $u_{s,i}u_{s,i+1} \cdots u_{s,j}$. A sequence of length $2i+2$ of the form $(a, b, c, b, c, \ldots, b, c, d)$ with i copies of (b, c) is denoted by $(a, (b, c)[i], d)$ for short.

Lemma 4.6 *Let f be a 4-coloring of $H(n, 2l+1)$. Suppose there is a color sequence $(a, (b, c)[t], a)$ occurring at the s-th row, where $t \geq 1$. Then there is a color sequence $(a, (b, c)[t-1], a)$ occurring at either the $(s+2)$-th or $(s-2)$-th row.*

Proof. Without loss of generality, we may assume $(a, (b, c)[t], a)$ is the color sequence of the sub-row $u_{s,0}u_{s,1} \cdots u_{s,2t+1}$.

Suppose s is even. Then $\{f(u_{s-1,0}), f(u_{s+1,0})\} = \{c, d\}$. Without loss of generality, we assume $f(u_{s-1,0}) = c$ and $f(u_{s+1,0}) = d$. Then it is easy to see that the colors assigned to some vertices of $(s-2)$-th, $(s-1)$-th and $(s+1)$-th rows are fixed and they are shown in Fig. 11.

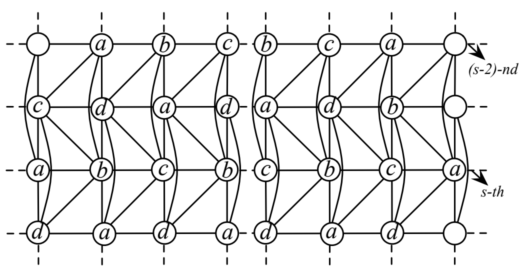


Fig. 11. A partial coloring of $H(n, 2l+1)$ for even s.

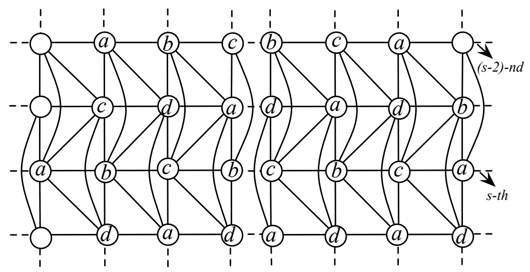


Fig. 12. A partial coloring of $H(n, 2l+1)$ for odd s.

Hence, the color sequence of the sub-row $u_{s-2,1}u_{s-2,2} \cdots u_{s-2,2t}$ is $(a, (b, c)[t-1], a)$.

Suppose s is odd. Then $\{f(u_{s-1,1}), f(u_{s+1,1})\} = \{c, d\}$. Without loss of generality, we assume $f(u_{s-1,1}) = c$ and $f(u_{s+1,1}) = d$. Similar to the above argument, $(a,(b,c)[t-1],a)$ is the color sequence of $u_{s-2,1}u_{s-2,2}\cdots u_{s-2,2t}$ as Fig. 12 shows. □

Note that when s is odd, we can reflect the graph about a vertical axis and the labeling will remain the same as those when s is even.

As an example, if $t = 1$, then there exist two adjacent vertices with same color in the $(s-2)$-th row as Fig. 13 and 14 show. A contradiction occurs!

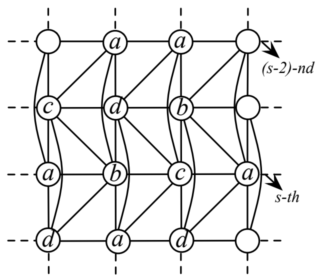


Fig. 13. A partial coloring for s is even.

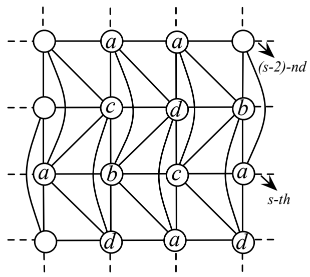


Fig. 14. A partial coloring for s is odd.

Corollary 4.7 *Let f be a 4-coloring of $H(n, 2l+1)$. Suppose there is a color sequence $(a,(b,c)[t],a)$ occurring at the s-th row, where $t \geq 1$. If l is large enough, then there is a contradiction at either the $(s+2t)$-th or the $(s-2t)$-th row.*

Proof. Same as the proof of Lemma 4.6, if we assume $f(u_{s-1,0}) = c$ and $f(u_{s+1,0}) = d$ when s is even, then the color sequences of $u_{s-1,0}u_{s-1,1}\cdots u_{s-1,2t}$ and $u_{s-2,1}u_{s-2,2}\cdots u_{s-2,2t}$ are $(c,d,(a,d)[t-1],b)$ and $(a,(b,c)[t-1],a)$, respectively. Suppose $t \geq 2$. Note that, since $f(u_{s-1,1}) = d$, it impliess that $f(u_{s-3,1}) = c$. This property will be kept at every two rows. By induction, this color sequence of the sub-row of the s-th row will yield a contradiction at the $(s-2t)$-th row. Similarly, if we assume $f(u_{s-1,0})$ is d, then it will yield a contradiction at the $(s+2t)$-th row.

Note that the same result will be obtained when s is odd. □

Lemma 4.8 *Let f be a 4-coloring of $H(n, 2l+1)$. Suppose there is a color sequence $(c,(d,a)[t],d,b)$ occurring at the s-th row, where $t \geq 1$. Then there is a color sequence $(c,(d,a)[t-1],d,b)$ occurring at either the $(s+2)$-th or $(s-2)$-th row.*

Proof. Suppose $(c,(d,a)[t],d,b)$ is the color sequence of the sub-row $u_{s,0}u_{s,1}\cdots u_{s,2t+2}$, where s is even. Then $\{f(u_{s-1,2t+1}), f(u_{s+1,2t+1})\} = \{a, c\}$. Without loss of generality, assume $f(u_{s-1,2t+1}) = a$. Hence, the color sequences of the sub-rows $u_{s-1,0}\cdots u_{s-1,2t+1}$ and $u_{s-2,1}\cdots u_{s-2,2t+1}$ are $(a,(b,c)[t],a)$ and $(c,(d,a)[t-1],d,b)$, respectively.

Similarly, we have $\{f(u_{s-1,1}), f(u_{s+1,1})\} = \{a, b\}$ when s is odd. Without loss of generality, assume $f(u_{s-1,1}) = a$. Hence, the color sequences of the sub-rows $u_{s-1,0}\cdots u_{s-1,2t+1}$ and $u_{s-2,1}\cdots u_{s-2,2t+1}$ are $(a,(b,c)[t],a)$ and $(c,(d,a)[t-1],d,b)$, respectively. □

Corollary 4.9 *Let f be a 4-coloring of $H(n, 2l+1)$. Suppose there is a color sequence $(a, (b, c)[t], b, d)$ occurring at the s-th row, where $t \geq 1$. If l is large enough, then there is a contradiction at either the $(s+2t+1)$-th or $(s-2t-1)$-th row.*

Proof. The argument is similar to Corollary 4.7. □

We call the sequence of color that cause a contradiction as a *bad sequence* of color. According to the above argument, we obtain two kinds of bad sequence as Corollaries 4.7 and 4.9 show. Without loss of generality, let those two kinds of bad sequence of color occur in the s-th row.

Furthermore, if there exists some kind of relationship of sub-sequences of color between two adjacent rows, say the s-th and the $(s-1)$-th rows, then the sub-sequences of color for the s-th and the $(s-1)$-th rows will cause a contradiction at some row. The following Lemmas demonstrate two kinds of relationship of sub-sequences of color between the s-th and the $(s-1)$-th rows.

Lemma 4.10 *Let s be even. Suppose f is a 4-coloring of $H(n, 2l+1)$ and $t \geq 2$ satisfying*

(1) $f(u_{s-1,0}) = f(u_{s-1,2t-1}) = a$;
(2) $f(u_{s,i}) = a$ for all even i with $2 \leq i < 2t-1$.

Then the coloring f yields a contradiction at the $(s-x)$-th row for some x, where $2 \leq x \leq 2t-1$.

Proof. Without loss of generality, we may assume $f(u_{s,0}) = b$ and $f(u_{s,1}) = c$.

Suppose there is an integer $h \in [2, t]$ such that $f(u_{s,2h-1}) = b$, we choose the smallest h that satisfies the condition.

Claim 3 *$f(u_{s-1,2j}) = f(u_{s+1,2j-1}) = b$ and $f(u_{s-1,2j-1}), f(u_{s+1,2j}) \in \{c, d\}$, $1 \leq j \leq h-1$.*

Proof of Claim 3: From our initial assumption, it is easy to see that $f(u_{s+1,0}) = d$, $f(u_{s+1,1}) = b$, and $f(u_{s-1,1}) = d$. Since $f(u_{s,2}) = a$ and $f(u_{s+1,1}) = b$, we have $f(u_{s+1,2}) \in \{c, d\}$. By the assumption, $f(u_{s,3}) \in \{c, d\}$ and so $f(u_{s-1,2}) = b$. Thus, the Claim holds when $j = 1$.

Suppose the Claim holds for some j with $1 \leq j \leq h-2$. Since $f(u_{s+1,2j}), f(u_{s+1,2j+1}) \in \{c, d\}$, $f(u_{s,2j+2}) = a$ and $f(u_{s+1,2j+1}) = b$. Hence $f(u_{s-1,2j+1}) \in \{c, d\}$. Since $f(u_{s,2j+3}) \in \{c, d\}$ and $f(u_{s-1,2j+2}) = b$, we have $f(u_{s+1,2j+2}) \in \{c, d\}$. This complete the proof of Claim 3.

(1) Suppose the color d does not appear in the color sequence of the sub-row $u_{s,0} \cdots u_{s,2h-1}$, then this color sequence is $(b, (c, a)[h-1], b)$. Similar to the proof of Lemma 4.6, the coloring f will yield a contraction at the $(s-(2h-2))$-th row since $f(u_{s-1,0}) = a$.
(2) Suppose d appears in the color sequence of the sub-row $u_{s,0} \cdots u_{s,2h-1}$. Let $k \in [2, h-1]$ be the largest integer such that $f(s, 2k-1) = c$. If $k < h-1$, then the color sequence of the sub-row $u_{s,2k-1} \cdots u_{s,2h-1}$ is $(c, (a, d)[h-k-1], a, b)$. If $k = h-1$, let $j \in [2, h-1]$

be the largest integer such that $f(u_{s,2j-1}) = d$, then the color sequence of the sub-row $u_{s,2j-1} \cdots u_{s,2h-1}$ is $(d, (a, c)[h - j - 1], a, b)$. By Lemma 4.8 and Claim 3, f will yield a contraction at the $(s - y)$-th row for some x, where $2 \le y \le 2t - 2$.

Now we assume that there is no color b appearing in the color sequence of the sub-row $u_{s,3} \cdots u_{s,2t-1}$. Then $f(u_{s,2i-1}) \in \{c, d\}$ for $2 \le i \le t$, and Claim 3 still works for $h > t$.

Hence the color sequence of the sub-row $u_{s-1,0}u_{s-1,1} \cdots u_{s-1,2t-1}$ is of the form (a, b_1, b, b_2, b, ..., b, b_{t-1}, b, a), where $b_1 = d$ and $b_j \in \{c, d\}$ for $2 \le j \le t - 1$. Let k be the largest integer in $[1, t - 1]$ such that $b_k = d$.

(1) If $k < t - 1$, then the color sequence of $u_{s-1,2k-1} \cdots u_{s-1,2t-1}$ is $(d, (b, c)[t - k - 1], b, a)$. Since $f(u_{s,2k}) = a$, $f(u_{s-1,2k+1}) = c$ and $f(u_{s-1,2k}) = b$, we have $f(u_{s-2,2k}) = c$. Similar to the proof of Lemma 4.8, we get a contradiction at the $(s - 1 - (2t - 2k - 1))$-th row.
(2) Suppose $k = t - 1$. If there is no color c appearing in the color sequence, then the color sequence of $u_{s-1,0}u_{s-1,1} \cdots u_{s-1,2t-1}$ is $(a, (d, b)[t - 1], a)$. By Lemma 4.6, we get a contradiction at the $(s - 1 - (2t - 2))$-th row since $f(u_{s,1}) = c$.
If there is a $b_j = c$, then we let j be the largest integer in $[2, t-2]$ such that $b_j = c$. Thus, the color sequence of $u_{s-1,2j-1} \cdots u_{s-1,2t-1}$ is $(c, (b, d)[t - j - 1], b, a)$. Since $f(u_{s,2j}) = a$, $f(u_{s-1,2j-1}) = c$ and $f(u_{s-1,2j}) = b$, we have $f(u_{s-2,2j}) = d$. Similar to the proof of Lemma 4.8, we get a contradiction at the $(s - 1 - (2t - 2j - 1))$-th row.

As a result, f will yield a contraction at the $(s - x)$-th row for some x, where $2 \le x \le 2t - 1$ and the proof completes. ☐

Note that in Lemma 4.10, if $f(u_{s+1,0}) = f(u_{s+1,2t-1}) = a$, then the coloring f will cause a contradiction in the $(s + x)$-th row, where $2 \le x \le 2t - 1$.

Lemma 4.11 *Suppose s is even. Let a, b, c, d represent four different colors. For consecutive vertices $u_{s,0}, u_{s,1}, \ldots, u_{s,t}$, if $f(u_{s,0}) = f(u_{s,t}) = f(u_{s\mp1,1}) = a$ and $f(u_{s,i}) \in \{b, c, d\}$, where $1 \le i \le t - 1$ and $t \ge 3$, then*

(1) $f(u_{s\pm1,t-2}) = a$ if t is even;
(2) $f(u_{s\mp1,t-2}) = a$ if t is odd.

Proof. We only consider the case when $f(u_{s-1,1}) = a$, and the proof is similar when $f(u_{s+1,1}) = a$.

Note that $u_{s,i}$, $u_{s,i+1}$, $u_{s-1,i}$ and $u_{s+1,i}$ induced a K_4. Thus,

$$\{f(u_{s,i}), f(u_{s,i+1}), f(u_{s-1,i}), f(u_{s+1,i})\} = \{a, b, c, d\}.$$

Since $f(u_{s,i})$ and $f(u_{s,i+1})$ are not equal to a for $1 \le i \le t - 2$, $a \in \{f(u_{s-1,i}), f(u_{s+1,i})\}$. Since $f(u_{s-1,1}) = a$, color a must appear in odd column of the $(s - 1)$-st row and even column of the $(s + 1)$-st row, respectively. That is, $f(u_{s-1,i}) = a$ and $f(u_{s+1,j}) = a$ for odd i and even j with $1 \le i, j \le t - 2$. Hence we have the result. ☐

Now we are going to prove the main theorem.

Theorem 4.12 *The graph $H(n, 2l+1)$ is not 4-colorable if $n \geq 2l+1$.*

Proof. Let f be a proper 4-coloring of $H(n, 2l+1)$ with the color set $\{a, b, c, d\}$, where $n \geq 2l+1$. A vertex is called *x-vertex* if it is colored by x. Let s be an even integer such that $s = l$ or $s = l+1$. Consider the s-th row of $H(n, 2l+1)$, which is a colored odd cycle $C \cong C_{2l+1}$. There is at least one color, say a, appearing in C odd times. After removing all a-vertices from C, the resulting graph is a disjoint union of paths. More precisely, if the number of a-vertices is m (where m is odd), then the cycle $C = u_1P_1u_2 \cdots u_mP_mu_1$, where P_i are paths of length at least one and $f(u_i) = a$ for all $i = 1, \dots, m$. Moreover, P_i does not contain any a-vertex and there are even number of odd paths P_j. This implies that there exists at least one even path.

Suppose we consider a section $u_iP_iu_{i+1}$ of the cycle C so that P_i is the shortest even path among all paths P_j's, where $u_{m+1} = u_1$. For convenience, we may rename the vertices so that $i = 1$ and $u_1P_1u_2 = u_{s,0}u_{s,1} \cdots u_{s,t}u_{s,t+1}$, where $t \geq 2$ and even. Hence, this sub-row satisfies the condition of Lemma 4.11.

(1) If P_1 is colored by two colors, then the color sequence of $u_1P_1u_2$ satisfies the condition of Corollary 4.7. Hence there is a contradiction at the $(s \pm t)$-th row.

(2) If P_1 is colored by three colors b, c and d, then at least one color, say b, appears in P_1 even times. Let $v_1, \dots, v_k$ be all b-vertices in P_1 so that $P_1 = Q_0v_1Q_1v_2 \cdots v_kQ_k$, where Q_j's are sub-paths. Note that Q_0 and Q_k may be empty paths. Moreover, there exists at least one even path Q_r.
   (a) If there is an r that is neither 0 nor k, then the color sequence of $v_rQ_rv_{r+1}$ satisfies the condition of Corollary 4.7.
   (b) If $r = 0$ or k, then Q_r is the only even sub-path in P_1. Because of the symmetry under reflection, we may assume $r = 0$ (s may be odd). For convenience, we rename the vertices so that $v_1Q_1v_2 = v_1x_1x_2 \cdots v_2$. Without loss of generality, let $f(x_1) = c$. In order to avoid the contraction, the color sequence of P_1 must be

$$((d, c)[\beta_0], b, c, (d, c)[\beta_1], b, c, (d, c)[\beta_2], b, \dots, b, c, (d, c)[\beta_k]),$$

   for some $\beta_i \geq 0$. Let h be the largest index such that $\beta_h \geq 1$. If $h = k$, then the color sequence of $v_kQ_ku_{i+1}$ is $(b, c, (d, c)[\beta_k], a)$, which satisfies the condition of Corollary 4.9. If $h < k$, then the color sequence of $Q_hv_{h+1}Q_{h+1} \cdots v_kQ_ku_{i+1}$ is $(d, c, (b, c)[\gamma], a)$ for some positive γ, which satisfies the condition of Corollary 4.9.

By Corollary 4.7 or Corollary 4.9, we get a contradiction at the $(s \pm x)$-th row, where $0 \leq x \leq t-1$.

Therefore, if $t \leq l$, then we are done. In the following we assume that $t \geq l+1$, then P_1 is the only even path. Without loss of generality, we may assume $f(u_{s-1,1}) = a$ and thus, $f(u_{s-1,t-1}) = a$ by Lemma 4.11.

Suppose that each odd path P_j contains t_j vertices, where $2 \leq j \leq m$ and $t_j \geq 1$. Firstly, we assume that there is a $t_j \geq 3$.

Claim 4 *Suppose there exist two paths P_i and P_j such that $t_i, t_j \geq 2$ and $t_k = 1$ for $i < k < j$. Let $u_iP_iu_{i+1} = u_{s,q}u_{s,q+1} \cdots u_{s,\alpha}$ and $u_jP_ju_{j+1} = u_{s,p}u_{s,p+1} \cdots u_{s,\beta}$ for some q,* *where* $\alpha = t_i + 1 + q$, $p = \alpha + 2(j-i-1)$ *and* $\beta = p + t_j + 1$. *If* $f(u_{s-1,\alpha-2}) = a$, *then* $f(u_{s+1,p+1}) = a$.

Proof of Claim 4: If $f(u_{s+1,p+1}) \neq a$, then $f(u_{s-1,p+1}) = a$. With the color sequence of the sub-row $u_{s,\alpha-2}u_{s,\alpha-1}u_{i+1}P_{i+1}u_{i+2}\cdots u_j u_{s,p+1}$ satisfying the conditions of Lemma 4.10, there exists a contradiction in some row. Hence , the claim is proved.

Suppose $P_i = u_{s,p+1}u_{s,p+2}\cdots u_{s,p+t_i}$, where $t_i \geq 2$. For convenience, we rename the vertices $u_{s+1,p+1} = y_i$ and $u_{s-1,p+t_i-1} = x_i$. Note that $x_1 = u_{s-1,t-1}$ and $y_1 = u_{s+1,1}$. Thus, $f(x_1) = a$ and $f(y_1) \neq a$ under the assumption $f(u_{s-1,1}) = a$.

Let j be the smallest index such that P_j is an odd path of length greater than 1. By Claim 4, we have $f(y_j) = a$ and by Lemma 4.11, we have $f(x_j) = a$. Repeat this process, we can conclude that $f(y_k) = a$ and $f(x_k) = a$ for all defined x_k and y_k with $k \neq 1$.

Suppose the path P_ℓ is the odd path such that ℓ is the largest index with $t_\ell \geq 3$. According to the discussion above, $f(x_\ell) = f(y_\ell) = a$. Then the color sequence of the sub-row $P_\ell u_{\ell+1}\cdots u_m P_m u_1 u_{s,1}$ with $f(x_\ell) = f(u_{s-1,1}) = a$ satisfies the conditions of Lemma 4.10. Let the length of $u_{\ell+1}\cdots u_m P_m u_1 u_{s,1}$ be L. Then $L \leq l - t_\ell + 1 \leq l-2$. By Lemma 4.10, we obtain a contradiction at the $(s-x)$-th row for some x, where $2 \leq x \leq L+1 \leq l-1$.

Finally, we assume that all odd paths are of length one. The length of $\widetilde{C} = C - \{u_{s,2}, \ldots, u_{s,t-2}\}$ is $x = 2l - t + 4 \leq l + 3$. By Lemma 4.10, we know that it has a contradiction at the $(s-y)$-th row (if any), where $y \leq x-1$. However, if $y \leq l$, then we are done. Thus, we only required to deal with $x = l+2$ and $l+3$.

1. Suppose $x = l+3$, then the length of P_1 is $l+1$. By the proof of (1) and (2), if there is no contradiction, then the color sequence of $u_1P_1u_2$ must be $(a, (b,c)[\frac{l+1}{2}], a)$. Furthermore, the color sequence of the sub-row $\widetilde{C}$ is $\{b, c, a, p_2, a, \cdots, p_m, a, b\}$, where $p_i \in \{b, c, d\}$. According to the proof of Lemma 4.10, there exists a contradiction at the $(s-z)$-th row (if any), where $2 \leq z \leq l+1$. Hence if $2 \leq z \leq l$, then we are done. Thus, we have to deal with $z = l+1$. Similar to the proof of Lemma 4.10, all p_i's are c. Therefore, the color sequence of the sub-row $P_m u_1 u_{s,1} u_{s,2}$ is $\{c, a, b, c\}$, which yields a contradiction by Corollary 4.7.
2. Suppose $x = l+2$, then the length of P_1 is $l+2$. Similar to the argument above, the color sequence of the sub-row $\widetilde{C}$ is $\{b, (c,a)[\frac{l}{2}], A\}$, where $A \in \{c, d\}$. Similar to the above discussion on $u_1P_1u_2$, there exists a contradiction in the $(s-y)$-th row for some y, where $0 \leq y \leq l+2$. Hence if $2 \leq y \leq l$, then we are done. Thus we only deal with $y = l+1$ and $l+2$. Since $f(u_{s,t-2}) = b$, $f(u_{s,t-1}) = c$ and $f(u_{s,1}) = A \in \{c, d\}$, which implies $y \neq l+2$. Consider $y = l+1$. Similar to the proof of Corollary 4.9, the only color sequence of $u_1P_1u_2$ is $\{a, d, c, (b,c)[\frac{l}{2}, a]\}$. But the color sequence of $P_m u_1 u_{s,1} u_{s,2}$ is $\{c, a, d, c\}$, which yields a contradiction by Corollary 4.7.

As a result, we prove that 4 colors are not sufficient to color the graph $H(n, 2l+1)$ when $n \geq 2l+1$. $\square$

By Lemma 4.5, we obtain following corollary.

Corollary 4.13 *For $n \geq 2l+1$ and $l \geq 2$, $\lambda_k(H(n, 2l+1)) \geq 4k$.*

Note that the result does not hold when $n \leq 2l$.

It is easy to verify that $L(0,k)$-labeling problem of $P_n \square C_m$ is equivalent to the $L(k)$-labeling of $H(n, 2l+1)$ where $m = 4l+2$ or $m = 2l+1$. By Claim 2 and Corollary 4.13, we obtain following theorems.

Theorem 4.14 *For $l \geq 2$, and $n \geq 2l+1$, $\lambda_{j,k}(P_n \square C_{4l+2}) \geq 4k$ and $\lambda_{j,k}(P_n \square C_{2l+1}) \geq 4k$.*

4.2. Upper bounds on $\lambda_{j,k}(P_n \square C_m)$

In this subsection, we will only write a detail proof for the most complicated case. The verifications of other cases are omitted. Interested reader may refer to [20].

We consider the case when m is a multiple of 4 first.

Theorem 4.15 *For $m \equiv 0 \pmod 4$, $n \geq 3$ and $k \geq 2j$, $\lambda_{j,k}(P_n \square C_m) \leq 3k + j$.*

Proof. We define a $(j+3k)$-$L(j,k)$-labeling f for $P_n \square C_m$ when $m \equiv 0 \pmod 4$ as follows.

Let $f_0 = 0, f_1 = j, f_2 = k, f_3 = j+k$; $g_0 = j+2k, g_1 = 3k, g_2 = j+3k, g_3 = 2k$. Then, for $s \in \mathbb{Z}_n, t \in \mathbb{Z}_m$,
$f(v_{s,t}) = f_i$, if s is even, where $i \equiv t+s \pmod 4$;
$f(v_{s,t}) = g_i$, if s is odd, where $i \equiv t+s-1 \pmod 4$.

Note that the image of f contains in $[0, j+3k]$ and it can be verified that f satisfies the constraints of $L(j,k)$-labeling. That is, f is a $(j+3k)$-$L(j,k)$-labeling of $P_n \square C_m$ if $m \equiv 0 \pmod 4$. □

Now we consider the case when m is not a multiple of 4.

For odd m and $3 \leq n \leq m-1$, the upper bounds on $\lambda_{j,k}(P_n \square C_m)$ are obtained in the following two theorems.

Theorem 4.16 *For $k \geq 2j$, $3 \leq n \leq m-1$ and $m \equiv 1 \pmod 4$, $\lambda_{j,k}(P_n \square C_m) \leq 3k + 2j$.*

Proof. Since $P_n \square C_m$ is a subgraph of $P_{m-1} \square C_m$, it suffices to find a $(3k+2j)$-$L(j,k)$-labeling f for $P_{m-1} \square C_m$.

For $0 \leq t \leq m-1$, let $S_t = (S_t^{\alpha})_{\alpha=0}^{m-2} = (f(v_{0,t}), f(v_{1,t-1}), f(v_{2,t-2}), \ldots, f(v_{m-2,t-m+2}))$ denote a sequence of colors defined below, where the indices are taken in modulo m. In particular, $f(v_{\alpha,t-\alpha}) = S_t^{\alpha}$.

For $t \equiv 0 \pmod 4$ and $0 \leq t \leq m-4$, $S_t = ((k,0)[\frac{m+t-1}{4}], k, 2k, (3k, 2k)[\frac{m-t-5}{4}])$;
For $t \equiv 1 \pmod 4$ and $0 \leq t \leq m-4$,
$S_t = ((j+k, j)[\frac{t-1}{4}], j+k, j+2k, (j+3k, j+2k)[\frac{2m-t-5}{4}])$;

For $t \equiv 2 \pmod 4$ and $0 \le t \le m-4$, $S_t = ((2k,3k)[\frac{m+t+1}{4}],(0,k)[\frac{m-t-3}{4}])$;
For $t \equiv 3 \pmod 4$ and $0 \le t \le m-4$, $S_t = ((j+2k,j+3k)[\frac{t+1}{4}],(j,j+k)[\frac{2m-t-3}{4}])$;
For $t = m-3$, $S_t = ((2j+2k,2j+3k)[\frac{m-1}{2}])$;
For $t = m-2$, $S_t = ((j+2k,j+3k)[\frac{m-1}{4}],(0,k)[\frac{m-1}{4}])$;
For $t = m-1$, $S_t = ((j,j+k)[\frac{m-1}{2}])$.
It is easily to see that the image of f lies in $[0,3k+2j]$.

For an arbitrary label S_t^α, because of the symmetry of distance, we need to verify the labels $S_t^{\alpha+1}$, $S_{t+1}^{\alpha+1}$, S_{t+1}^{α}, S_{t+2}^{α}, $S_{t+2}^{\alpha+1}$ and $S_{t+2}^{\alpha+2}$ corresponding to the vertices. Of course, it is no need to check the labels when the corresponding vertices do not exist.

For $S_t^{\alpha+1}$, it is obvious that $|S_t^{\alpha+1} - S_t^\alpha| \ge k$.

For S_{t+2}^{α}, $S_{t+2}^{\alpha+1}$ and $S_{t+2}^{\alpha+2}$, they lie in S_{t+2}. We need to compare the two sequences S_t and S_{t+2}.

(1) Suppose $t, t+2 \in [0,m-4]$.
  (a) If $t \equiv 0 \pmod 4$, then
  $S_t = ((k,0)[\frac{m+t-1}{4}],k,2k,(3k,2k)[\frac{m-t-5}{4}])$ and
  $S_{t+2} = ((2k,3k)[\frac{m+t+3}{4}],(0,k)[\frac{m-t-5}{4}]) = ((2k,3k)[\frac{m+t-1}{4}],2k,3k,(0,k)[\frac{m-t-5}{4}])$.
  (b) If $t \equiv 2 \pmod 4$, then it is similar to the case above.
  (c) If $t \equiv 1 \pmod 4$, then
  $S_t = ((j+k,j)[\frac{t-1}{4}],j+k,j+2k,(j+3k,j+2k)[\frac{2m-t-5}{4}])$ and
  $S_{t+2} = ((j+2k,j+3k)[\frac{t+3}{4}],(j,j+k)[\frac{2m-t-5}{4}])$
  $= ((j+2k,j+3k)[\frac{t-1}{4}],j+2k,j+3k,(j,j+k)[\frac{2m-t-5}{4}])$.
  (d) If $t \equiv 3 \pmod 4$, then it is similar to the case above.
(2) If $t \in [0,m-4]$ and $m-3 \le t+2 \le m-1$, then $t = m-5$ or $t = m-4$. We have the following two subcases.
  (a) Suppose $t = m-5$, that is $t+2 = m-3$. Note that $t \equiv 0 \pmod 4$. Hence
  $S_{m-5} = ((k,0)[\frac{m+t-1}{4}],k,2k,(3k,2k)[\frac{m-t-5}{4}]) = ((k,0)[\frac{m-3}{2}],k,2k)$ and
  $S_{m-3} = ((2j+2k,2j+3k)[\frac{m-3}{2}],2j+2k,2j+3k)$.
  (b) Suppose $t = m-4$, that is $t+2 = m-2$. Note that $t \equiv 1 \pmod 4$. Hence
  $S_{m-4} = ((j+k,j)[\frac{t-1}{4}],j+k,j+2k,(j+3k,j+2k)[\frac{2m-t-5}{4}])$
  $= ((j+k,j)[\frac{m-5}{4}],j+k,j+2k,(j+3k,j+2k)[\frac{m-1}{4}])$ and
  $S_{m-2} = ((j+2k,j+3k)[\frac{m-5}{4}],j+2k,j+3k,(0,k)[\frac{m-1}{4}])$.
(3) If $t+2 \in [0,m-4]$ and $m-3 \le t \le m-1$, then $t = m-2$ or $t = m-1$. We have the following two subcases.
  (a) If $t = m-2$, then $[t+2]_m = 0$ and
  $S_{m-2} = ((j+2k,j+3k)[\frac{m-1}{4}],0,k,(0,k)[\frac{m-5}{4}])$ and
  $S_0 = ((k,0)[\frac{m-1}{4}],k,2k,(3k,2k)[\frac{m-5}{4}])$.
  (b) If $t = m-1$, then $[t+2]_m = 1$ and
  $S_{m-1} = (j,j+k,(j,j+k)[\frac{m-3}{2}])$ and
  $S_1 = (j+k,j+2k,(j+3k,j+2k)[\frac{m-3}{2}])$.
(4) If $m-3 \le t, t+2 \le m-1$, then $t = m-3$. Hence
  $S_{m-3} = ((2j+2k,2j+3k)[\frac{m-1}{2}])$ and
  $S_{m-1} = ((j,j+k)[\frac{m-1}{2}])$.

For all cases, we have $|S^{\alpha}_{t+2} - S^{\alpha}_t| \geq k$, $|S^{\alpha+1}_{t+2} - S^{\alpha}_t| \geq k$ and $|S^{\alpha+2}_{t+2} - S^{\alpha}_t| \geq k$.

For labels $S^{\alpha+1}_{t+1}$ and S^{α}_{t+1}, since they lie in S_{t+1}, we compare the sequences S_t and S_{t+1} as follows.

(1) Suppose $t, t+1 \in [0, m-4]$. We have the following subcases.
  (a) If $t \equiv 0 \pmod 4$, then
  $S_t = ((k,0)[\frac{t}{4}], k, 0, (k,0)[\frac{m-5}{4}], k, 2k, (3k,2k)[\frac{m-t-5}{4}])$ and
  $S_{t+1} = ((j+k,j)[\frac{t}{4}], j+k, j+2k, (j+3k, j+2k)[\frac{m-5}{4}], j+3k, j+2k,$
  $(j+3k, j+2k)[\frac{m-t-5}{4}])$.
  (b) If $t \equiv 1 \pmod 4$, then
  $S_t = ((j+k,j)[\frac{t-1}{4}], j+k, j+2k, (j+3k, j+2k)[\frac{2m-t-5}{4}])$
  $= ((j+k,j)[\frac{t-1}{4}], j+k, j+2k, (j+3k, j+2k)[\frac{m-1}{4}], (j+3k, j+2k)[\frac{m-t-4}{4}])$ and
  $S_{t+1} = ((2k,3k)[\frac{m+t+2}{4}], (0,k)[\frac{m-t-4}{4}])$
  $= ((2k,3k)[\frac{t-1}{4}], 2k, 3k, (2k,3k)[\frac{m-1}{4}], (0,k)[\frac{m-t-4}{4}])$.
  (c) If $t \equiv 2 \pmod 4$, then
  $S_t = ((2k,3k)[\frac{m+t+1}{4}], (0,k)[\frac{m-t-3}{4}]) = ((2k,3k)[\frac{t+2}{4}], (2k,3k)[\frac{m-1}{4}], (2k,3k)[\frac{m-t-3}{4}])$
  and
  $S_{t+1} = ((j+2k, j+3k)[\frac{t+2}{4}], (j, j+k)[\frac{m-1}{4}], (j, j+k)[\frac{m-t-3}{4}])$.
  (d) If $t \equiv 3 \pmod 4$, then
  $S_t = ((j+2k, j+3k)[\frac{t+1}{4}], (j, j+k)[\frac{m-1}{4}], j, j+k, (j, j+k)[\frac{m-t-6}{4}])$ and
  $S_{t+1} = ((k,0)[\frac{t+1}{4}], (k,0)[\frac{m-1}{4}], k, 2k, (3k,2k)[\frac{m-t-6}{4}])$.

(2) If $t \in [0, m-4]$ and $m-3 \leq t+1 \leq m-1$, then $t = m-4$. Hence
  $S_{m-4} = ((j+k, j)[\frac{m-5}{4}], j+k, j+2k, (j+3k, j+2k)[\frac{m-1}{4}])$ and
  $S_{m-3} = ((2j+2k, 2j+3k)[\frac{m-5}{4}], 2j+2k, 2j+3k, (2j+2k, 2j+3k)[\frac{m-1}{4}])$.

(3) If $t+1 \in [0, m-4]$ and $m-3 \leq t \leq m-1$, then $t = m-1$ and $[t+1]_m = 0$. Hence
  $S_{m-1} = ((j, j+k)[\frac{m-1}{4}], j, j+k, (j, j+k)[\frac{m-5}{4}])$ and
  $S_0 = ((k,0)[\frac{m-1}{4}], k, 2k, (3k,2k)[\frac{m-5}{4}])$.

(4) If $m-3 \leq t, t+1 \leq m-1$, then $t = m-3$ or $t = m-2$.
  (a) If $t = m-3$, then
  $S_{m-3} = ((2j+2k, 2j+3k)[\frac{m-1}{4}], (2j+2k, 2j+3k)[\frac{m-1}{4}])$ and
  $S_{m-2} = ((j+2k, j+3k)[\frac{m-1}{4}], (0,k)[\frac{m-1}{4}])$.
  (b) If $t = m-2$, then $S_{m-2} = ((j+2k, j+3k)[\frac{m-1}{4}], (0,k)[\frac{m-1}{4}])$ and
  $S_{m-1} = ((j, j+k)[\frac{m-1}{4}], (j, j+k)[\frac{m-1}{4}])$.

For all cases, we have $|S^{\alpha}_{t+1} - S^{\alpha}_t| \geq j$ and $|S^{\alpha+1}_{t+1} - S^{\alpha}_t| \geq j$. □

Theorem 4.17 *For $k \geq 2j$, $3 \leq n \leq m-1$ and $m \equiv 3 \pmod 4$, $\lambda_{j,k}(P_n \Box C_m) \leq 3k + 2j$.*

Proof. Similar to the proof of Theorem 4.16, since $P_n \Box C_m$ is a subgraph of $P_{m-1} \Box C_m$, it suffices to find a $(3k+2j)$-$L(j,k)$-labeling f for $P_{m-1} \Box C_m$.

For $0 \leq t \leq m-1$, let $S_t = (S^{\alpha}_t)^{m-2}_{\alpha=0} = (f(v_{0,t}), f(v_{1,t-1}), f(v_{2,t-2}), \ldots, f(v_{m-2,t-m+2}))$ denote a sequence of colors defined below, where the indices are taken in modulo m.

For $t \equiv 0 \pmod 4$ and $0 \leq t \leq m-4$,
$S_t = ((j+k, j)[\frac{m+1+t}{4}], j+k, j+2k, (j+3k, j+2k)[\frac{m-t-7}{4}])$;

For $t=1$, $S_t=(2k,2j+3k,(0,k)[\frac{m-3}{2}])$;
For $t\equiv 1 \pmod 4$ and $5\le t\le m-4$, $S_t=((2k,3k)[\frac{t+3}{4}],(0,k)[\frac{2m-t-5}{4}])$;
For $t\equiv 2 \pmod 4$ and $0\le t\le m-4$, $S_t=((j+2k,j+3k)[\frac{m+t+3}{4}],(j,j+k)[\frac{m-t-5}{4}])$;
For $t\equiv 3 \pmod 4$ and $0\le t\le m-4$, $S_t=((k,0)[\frac{t+1}{4}],k,2k,(3k,2k)[\frac{2m-t-7}{4}])$;
For $t=m-3$, $S_t=(0,j+k,(j,j+k)[\frac{m-3}{2}])$;
For $t=m-2$, $S_t=((j+2k,j+3k)[\frac{m+1}{4}],(0,k)[\frac{m-3}{4}])$;
For $t=m-1$, $S_t=(k,2j+2k,(2j+3k,2j+2k)[\frac{m-3}{2}])$.

It follows that the image of f lies in $[0,3k+2j]$. Hence f is a $(3k+2j)$-$L(j,k)$-labeling of $P_{m-1}\Box C_m$.

For even m and $3\le n\le\frac{m-2}{2}$, we have the following result.

Theorem 4.18 *For $k\ge 2j$, $m\equiv 2 \pmod 4$ and $3\le n\le\frac{m-2}{2}$, $\lambda_{j,k}(P_n\Box C_m)\le 3k+j$.*

Proof. Let $f_0=0$, $f_1=j$, $f_2=k$, $f_3=k+j$; $g_0=2k$, $g_1=2k+j$, $g_2=3k$, $g_3=3k+j$.

Suppose s is even and $0\le s\le\frac{m-4}{2}$. For $[t-s]_m\in[0,m-3-2s]$, define $f(v_{s,t})=f_i$; for $[t-s]_m\in[m-2-2s,m-1]$, define $f(v_{s,t})=g_i$, where $i\equiv[t-s]_m \pmod 4$.

Suppose s is odd and $0\le s\le\frac{m-4}{2}$. For $[t-s]_m\in[0,m-3-2s]$, define $f(v_{s,t})=g_i$; for $[t-s]_m\in[m-2-2s,m-1]$, define $f(v_{s,t})=f_i$, where $i\equiv[t-s]_m+2 \pmod 4$.

It can be verified that f is a $(3k+j)$-$L(j,k)$-labeling of $P_n\Box C_m$. Hence $\lambda_{j,k}(P_n\Box C_m)\le 3k+j$ for $3\le n\le\frac{m-2}{2}$. □

By Theorems 4.2, 4.18 and 4.15, we have following corollaries.

Corollary 4.19 *For $m\equiv 0 \pmod 4$, $n\ge 3$ and $k\ge 2j$, $\lambda_{j,k}(P_n\Box C_m)=3k+j$.*

Corollary 4.20 *For $3\le n\le\frac{m-2}{2}$, $m\ge 6$ and $m\equiv 2 \pmod 4$, $\lambda_{j,k}(P_n\Box C_m)=3k+j$.*

By Corollary 4.4, Theorems 4.16 and 4.17, we obtain the following result.

Corollary 4.21 *For $k\ge 2j$, $3\le n\le m-1$ and m is odd, $\lambda_{j,k}(P_n\Box C_m)=3k+2j$.*

Now we consider odd m with $n\ge m$. The upper bounds on $P_n\Box C_m$ are shown in the following two theorems.

Theorem 4.22 *For $m\equiv 1 \pmod 4$, $n\ge m\ge 5$ and $k\ge 2j$, $\lambda_{j,k}(P_n\Box C_m)\le 4k$.*

Proof.

Case 1: When $m=5$, we give a labeling h_5 of $P_6\Box C_5$ as follows.

$$M_{h_5}=\begin{pmatrix} 0 & k & 2k & 3k & 4k \\ 2k & 3k & 4k & 0 & k \\ 4k & 0 & k & 2k & 3k \\ k & 2k & 3k & 4k & 0 \\ 3k & 4k & 0 & k & 2k \\ 0 & k & 2k & 3k & 4k \end{pmatrix}.$$

Clearly, h_5 is a $4k$-$L(j,k)$-labeling of $P_6\square C_5$. For $n=5$, since $P_5\square C_5$ is a subgraph of $P_6\square C_5$, we have the result. If $n\geq 7$, then the labeling f of the graph is defined by $f(v_{s,t})=f(v_{i,t})$ for $s\equiv i \pmod 6$, where $s\in\mathbb{Z}_n, 0\leq t\leq 5$. It is the labeling obtained by repeating the rows of M_{h_5} in a period of 5. Clearly, the image of f lies in $[0,4k]$ and f satisfies the constraints of $L(j,k)$-labeling.

Case 2: When $m=9$, similar to Case 1, we define a labeling f of $P_n\square C_9$ as follows.

Let $f_i=ik/2$, $0\leq i\leq 8$. Define
$f(v_{s,t})=f_i$, if s is even, where $i\equiv t+s \pmod 9$;
$f(v_{s,t})=f_i$, if s is odd, where $i\equiv t+s+4 \pmod 9$,
where $0\leq s\leq n-1$, $t\in\mathbb{Z}_9$.

It is not difficult to verify that f is a $4k$-$L(j,k)$-labeling of $P_n\square C_9$.

Case 3: When $m\geq 13$, we define an $L(j,k)$-labeling of $P_n\square C_m$ as follows.

Let $f_i=ik/2$, $0\leq i\leq 8$. Note that $|f_x-f_y|=\frac{|x-y|k}{2}$. Define

$$f(v_{0,t})=\begin{cases} f_i \text{ if } 0\leq t\leq m-10, \text{ where } i\equiv t \pmod 4;\\ f_i \text{ if } m-9\leq t\leq m-1, \text{ where } i\equiv t-m \pmod 9.\end{cases}$$

$$f(v_{1,t})=\begin{cases} f_{i+5} \quad \text{if } 0\leq t\leq m-10, \text{ where } i\equiv t \pmod 4;\\ f_{[i+5]_9} \text{ if } m-9\leq t\leq m-1, \text{ where } i\equiv t-m \pmod 9.\end{cases}$$

For $2\leq s\leq n-1, t\in\mathbb{Z}_m$,
$f(v_{s,t})=f(v_{s-2,[t+2]_m})=f(v_{0,[t+s]_m})$, if s is even;
$f(v_{s,t})=f(v_{s-2,[t+2]_m})=f(v_{1,[t+s-1]_m})$, if s is odd.

Note that the image of f lies in $[0,4k]$ and some routine steps will show that f is a $4k$-$L(j,k)$-labeling of $P_n\square C_m$. □

Theorem 4.23 *For $k\geq 2j$, $n\geq m\geq 7$ and $m\equiv 3 \pmod 4$, $\lambda_{j,k}(P_n\square C_m)\leq 4k$.*

Proof. We prove this upper bound by defining a $4k$-$L(j,k)$-labeling for $P_n\square C_m$.

Case 1: Suppose $m=7$. We define a labeling h_7 of $P_7\square C_7$ as follows:

$$M_{h_7}=\begin{pmatrix} 0 & k/2 & k & 3k/2 & 2k & 3k & 4k\\ 3k/2 & 2k & 5k/2 & 3k & 4k & 0 & k\\ 3k & 7k/2 & 4k & 0 & k & 2k & 5k/2\\ k/2 & k & 3k/2 & 2k & 3k & 4k & 0\\ 2k & 5k/2 & 3k & 4k & 0 & k & 3k/2\\ 7k/2 & 4k & 0 & k & 2k & 5k/2 & 3k\\ k & 3k/2 & 2k & 3k & 4k & 0 & k/2 \end{pmatrix}.$$

Clearly h_7 is an $L(j,k)$-labeling. Note that, it is easily to see that $f(v_{s,t})=h_7(v_{i,[t+\lfloor\frac{s}{3}\rfloor]_7})$, where $i\equiv s \pmod 3$, $0\leq s\leq 6$ and $0\leq t\leq 6$.

For $n\geq 8$, we define $f(v_{s,t})=h_7(v_{i,[t+\lfloor\frac{s}{3}\rfloor]_7})$, where $i\equiv s \pmod 3$, $0\leq s\leq n-1$ and $0\leq t\leq 6$. Hence the labels of the s-th row is those of the $(s-3)$-rd row by shifting one unit to the left cyclically. Moreover, the labeling is periodic with a period of 21. Since the labels of the first three rows of the matrix M_{h_7} satisfy the constraints of the $L(j,k)$-labeling, f is a $4k$-$L(j,k)$-labeling of $P_n \Box C_7$.

Case 2: Suppose $m = 11$. We define a labeling h_{11} of $P_5 \Box C_{11}$ as follows:

$$M_{h_{11}} = \begin{pmatrix} 0 & k/2 & k & 2k & 3k & 4k & 0 & k & 2k & 3k & 4k \\ 3k/2 & 2k & 3k & 4k & 0 & k & 2k & 3k & 4k & 0 & k \\ 3k & 4k & 0 & k & 2k & 3k & 4k & 0 & k & 2k & 5k/2 \\ 0 & k & 2k & 3k & 4k & 0 & k & 2k & 3k & 7k/2 & 4k \\ 2k & 3k & 4k & 0 & k & 2k & 3k & 4k & 0 & k/2 & k \\ 4k & 0 & k & 2k & 3k & 4k & 0 & k & 3k/2 & 2k & 3k \end{pmatrix}.$$

For $n \geq 11$, we assign $f(v_{s,t}) = h_{11}(v_{i,[t+3\lfloor \frac{s}{4} \rfloor]_{11}})$, where $i \equiv s \pmod 4$, $0 \leq s \leq n-1$ and $0 \leq t \leq 11$. Hence the labels of s-th row is the $(s-4)$-th row by shifting three units to the left cyclically. Similar to Case 1, f is a $4k$-$L(j,k)$-labeling of $P_n \Box C_{11}$.

Case 3: Suppose $m = 15$. We define a $4k$-$L(j,k)$-labeling of $P_n \Box C_{15}$ as follows:

Let $f_i = ik$, $0 \leq i \leq 4$. Define
$f(v_{0,t}) = f_i$, where $i \equiv t \pmod 5$,
$f(v_{s,t}) = f(v_{s-1,[t+2]_5})$ for $1 \leq s \leq n-1, 0 \leq t \leq 14$.
It is not difficult to show that f is the $4k$-$L(j,k)$-labeling.

Case 4: Suppose $m = 19$. We define a $4k$-$L(j,k)$-labeling of $P_n \Box C_{19}$ as follows:

Firstly, we give a labeling h_{19} of $P_7 \Box C_{19}$ such that the labeling matrix $M_{h_{19}} = (A\ A\ B)$, where

$$A = \begin{pmatrix} 0 & k & 2k & 3k & 4k \\ 3k & 4k & 0 & k & 2k \\ k & 2k & 3k & 4k & 0 \\ 4k & 0 & k & 2k & 3k \\ 2k & 3k & 4k & 0 & k \\ 0 & k & 2k & 3k & 4k \\ 3k & 4k & 0 & k & 2k \end{pmatrix} \qquad B = \begin{pmatrix} 0 & k/2 & k & 3k/2 & 2k & 5k/2 & 3k & 7k/2 & 4k \\ 5k/2 & 3k & 7k/2 & 4k & 0 & k/2 & k & 3k/2 & 2k \\ k & 3k/2 & 2k & 5k/2 & 3k & 7k/2 & 4k & 0 & k/2 \\ 7k/2 & 4k & 0 & k/2 & k & 3k/2 & 2k & 5k/2 & 3k \\ 2k & 5k/2 & 3k & 7k/2 & 4k & 0 & k/2 & k & 3k/2 \\ 0 & k/2 & k & 3k/2 & 2k & 5k/2 & 3k & 7k/2 & 4k \\ 5k/2 & 3k & 7k/2 & 4k & 0 & k/2 & k & 3k/2 & 2k \end{pmatrix}$$

Clearly, h_{19} is a $4k$-$L(j,k)$-labeling.

For $n \geq 19$, we repeat the labeling of graph $P_7 \Box C_{19}$ with a period of 5. That is, assign $f(v_{s,t}) = h_{19}(v_{i,t})$, where $i \equiv s \pmod 5$, $5 \leq s \leq n-1$.

Similar to Case 1, f is a $4k$-$L(j,k)$-labeling of $P_n \Box C_{19}$.

Case 5: Suppose $m = 23$. We define a labeling h_{23} of $P_7 \Box C_{23}$ by $M_{h_{23}} = (A\ B\ B)$, where A and B are defined in Case 4.

For $n \geq 23$, we repeat the labeling of graph $P_7 \Box C_{23}$ with a period of 5. That is, assign $f(v_{s,t}) = h_{23}(v_{i,t})$, where $i \equiv s \pmod 5$, $5 \leq s \leq n-1$.

Similar to Case 1, f is a $4k$-$L(j,k)$-labeling of $P_n \Box C_{23}$.

Case 6: Suppose $m \equiv 3 \pmod 4$ and $m \geq 27$. We define a $4k$-$L(j,k)$-labeling as follows:

Let $f_i = \frac{ik}{2}$, $0 \leq i \leq 8$. Define

$$f(v_{0,t}) = \begin{cases} f_i & \text{if } 0 \leq t \leq m-28, \text{ where } i \equiv t \pmod 4; \\ f_i & \text{if } m-27 \leq t \leq m-1, \text{ where } i \equiv t-m \pmod 9. \end{cases}$$

$$f(v_{1,t}) = \begin{cases} f_{i+5} & \text{if } 0 \le t \le m-28, \text{ where } i \equiv t \pmod 4; \\ f_{[i+5]_9} & \text{if } m-27 \le t \le m-1, \text{ where } i \equiv t-m \pmod 9). \end{cases}$$

For $2 \le s \le n-1, t \in \mathbb{Z}_m$,
$f(v_{s,t}) = f(v_{0,[t+s]_m})$, if s is even;
$f(v_{s,t}) = f(v_{1,[t+s-1]_m})$, if s is odd.

Note that f is similar to the labeling when $m \equiv 1 \pmod 4$ and $m \ge 13$. Here we omit the verification. □

For even m with $n \ge \frac{m}{2}$, we have following result.

Theorem 4.24 *For $k \ge 2j$, $m \equiv 2 \pmod 4$ and $n \ge \frac{m}{2} \ge 3$, $\lambda_{j,k}(P_n \square C_m) \le 4k$.*

Proof. We consider four cases as follows.

Case 1: When $m = 6$, we give a labeling h_6 of $P_9 \square C_6$ as follows:

$$M_{h_6} = \begin{pmatrix} 0 & k/2 & k & 2k & 3k & 4k \\ 3k/2 & 2k & 3k & 4k & 0 & k \\ 3k & 4k & 0 & k & 2k & 5k/2 \\ 0 & k & 2k & 3k & 7k/2 & 4k \\ 2k & 3k & 4k & 0 & k/2 & k \\ 4k & 0 & k & 3k/2 & 2k & 3k \\ k & 2k & 5k/2 & 3k & 4k & 0 \\ 3k & 7k/2 & 4k & 0 & k & 2k \\ 0 & k/2 & k & 2k & 3k & 4k \end{pmatrix}.$$

Clearly, h_6 is a $4k$-$L(j,k)$-labeling of $P_9 \square C_6$.

For $3 \le n \le 8$, the graph $P_n \square C_6$ is a subgraph of $P_9 \square C_6$ and hence we have the result. For $n \ge 10$, we define a labeling f by $f(v_{s,t}) = f(v_{i,t})$, $s \equiv i \pmod 8$, $9 \le s \le n-1$ and $0 \le t \le 5$. It is the labeling obtained by repeating the rows of M_{h_6} in a period of 8. Hence f is a $4k$-$L(j,k)$-labeling of $P_n \square C_6$.

Case 2: Suppose $m = 10$. We define a $4k$-$L(j,k)$-labeling as follows.

Let $f_i = ik$, $0 \le i \le 4$. Define $f(v_{0,t}) = f_i$ for $t \equiv i \pmod 5$, $f(v_{s,t}) = f(v_{s-1,[t+2]_{10}})$ for $1 \le s \le n-1, 0 \le t \le 9$.

Here, it is easy to show that f is a $4k$-$L(j,k)$-labeling of $P_n \square C_{10}$.

Case 3: Suppose $m = 14$. We define a $4k$-$L(j,k)$-labeling of $P_n \square C_{14}$ as follows:

Let $g_i = ik$, $0 \le i \le 4$; and $f_i = ik/2$, $0 \le i \le 8$.
For even s, define $f(v_{s,t}) = \begin{cases} g_{[t+\frac{s}{2}]_5}, & \text{if } 0 \le t \le 4; \\ f_{[t+s]_9}, & \text{if } 5 \le t \le 13. \end{cases}$
For odd s, define $f(v_{s,t}) = \begin{cases} g_{[t+\frac{s+1}{2}+2]_5}, & \text{if } 0 \le t \le 4; \\ f_{[t+s-1]_9}, & \text{if } 5 \le t \le 13. \end{cases}$

It is not difficult to verify that f is a $4k$-$L(j,k)$-labeling of $P_n \square C_{14}$.

Case 4: Suppose $m \geq 18$. We define a $4k$-$L(j,k)$-labeling as follows.

Let $f_i = ik/2$, $0 \leq i \leq 8$. Define

$$f(v_{0,t}) = \begin{cases} f_i \text{ if } 0 \leq t \leq m-19 \text{ and } t \equiv i \pmod 4; \\ f_i \text{ if } m-18 \leq t \leq m-1 \text{ and } t-m+18 \equiv i \pmod 9. \end{cases}$$

$$f(v_{1,t}) = \begin{cases} f_{i+5} & \text{if } 0 \leq t \leq m-19 \text{ and } t \equiv i \pmod 4; \\ f_{[i+5]_9} & \text{if } m-18 \leq t \leq m-1 \text{ and } t-m+18 \equiv i \pmod 9. \end{cases}$$

For $2 \leq s \leq n-1$, $t \in \mathbb{Z}_m$,
$f(v_{s,t}) = f(v_{0,[t+s]_m})$, if s is even;
$f(v_{s,t}) = f(v_{1,[t+s-1]_m})$, if s is odd.

It can be verified that f is a $4k$-$L(j,k)$-labeling of $P_n \square C_m$. □

Hence, by Theorems 4.24 and 4.14, we obtain the following corollary.

Corollary 4.25 *For $l \geq 2$ and $n \geq 2l+1$, $\lambda_{j,k}(P_n \square C_{4l+2}) = 4k$.*

Moreover, by Theorems 4.14, 4.23, and 4.22, we obtain the following corollary.

Corollary 4.26 *For $l \geq 2$ and $n \geq 2l+1$, $\lambda_{j,k}(P_n \square C_{2l+1}) = 4k$.*

The following table summarized all the $\lambda_{j,k}(P_n \square C_m)$ with $k \geq 2j$ that obtained in this article:

Condition on m		Condition on n	$\lambda_{j,k}(P_n \square C_m)$
$m = 3$		$n = 2$	$3j + k$
		$n = 3$	$2j + 2k$
		$n = 4$	$\min\{3k, 3j + 2k\}$
		$n \geq 5$	$\min\{4j + 2k, 3k + j\}$
$m \geq 4$	$m \equiv 0 \pmod 6$		$2k + j$
	$m \equiv 3 \pmod 6$, $m \neq 3$	$n = 2$	$2k + 2j$
	$m \not\equiv 0 \pmod 3$		$3k$
	$m \equiv 0 \pmod 4$	$n \geq 3$	$3k + j$
	$m \equiv 2 \pmod 4$	$3 \leq n < m/2$	$3k + j$
		$n \geq m/2$	$4k$
	m is odd	$3 \leq n \leq m-1$	$3k + 2j$
		$n \geq m$	$4k$

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