1998

Testing for a unit root in an ar(1) model using three and four moment approximations: symmetric distributions

Moti L Tiku

Wing Keung Wong
Hong Kong Baptist University, awong@hkbu.edu.hk

Citation

Tiku, Moti L, and Wing Keung Wong. "Testing for a unit root in an ar(1) model using three and four moment approximations: symmetric distributions." *Communications in Statistics - Simulation and Computation* 27.1 (1998): 185-198.

TESTING FOR UNIT ROOT IN AR(1) MODEL USING THREE AND FOUR MOMENT APPROXIMATIONS

Moti L Tiku
Mathematics and Statistics Department
McMaster University

and

Wing-Keung Wong
Department of Economics
National University of Singapore

Abstract. Three-moment chi-square and four moment F approximations are given which can be used for testing a unit root in $AR(1)$ model when the innovations have one of a very wide family of symmetric distributions (Student's t).

Keywords. Unit root; Time series; Likelihood function; Modified likelihood; Chi-square distribution; F distribution; Hypothesis testing.

Acknowledgement The senior author would like to thank the National Science and Engineering Research Council of Canada for a research grant.

1. INTRODUCTION

In an $AR(1)$ model $y_t = \phi y_{t-1} + \varepsilon_t$, it is of great interest to test $\phi = 1$ against $\phi < 1$. When ε_t are distributed $N(0, \sigma^2)$, a vast literature on the subject exists; see, for example, Dickey and Fuller (1979), Phillips and Perron (1988), Abadir (1995) and the references given there. The test statistics used are essentially based on the normal-theory estimator $\hat{\phi}_0 = \sum_{t=1}^{n} y_t \, y_{t-1} / \sum_{t=1}^{n} y_{t-1}^2$. Abadir (1995) gives the asymptotic distribution and Vinod and Shenton (1996) give the first four mements of $\hat{\phi}_0$ (and variants of it) when $\phi = 1$. In recent years, however, it has been recognized that the normality assumption is too restrictive from applications point of view; see, for example, Huber (1981) and Tiku et al (1986). Wong et al (1996) considered the situation when ε_t has a distribution which belongs to a wide family of symmetric distributions (Student's t) ranging from Cauchy to normal. They derived the MML (modified maximum likelihood) estimator $\hat{\phi}$ and showed that it is remarkably efficient. They also studied the distribution of $\hat{\phi}$ when $\phi < 1$ and showed that it is normal for large n. In this article, we give three-moment chi-square and four-moment F approximations which give remarkably accurate values for the probability integral and the upper percentage points of the distribution of $\hat{\phi}$ when $\phi = 1$. In particular, these approximations can be used for obtaining the percentage points of the distribution of $\hat{\phi}_0$ when $\phi = 1$. The results given can easily be extended to the model $y_t = \mu + \phi y_{t-1} + \varepsilon_t$.

2. THE MML ESTIMATORS

The MML estimators are based on order statistics of a random sample and are obtained by linearizing the intractable terms in likelihood equations (Tiku et al 1986, Tiku and Suresh 1992). They are known to be asymptotically fully efficient (Bhattacharyya 1985) and almost fully efficient for small sample sizes (Lee et al 1980, Tan 1985, Tiku et al 1986, Tiku and Suresh 1992, and Vaughan 1992a). Consider the model

$$y_t = \phi y_{t-1} + \varepsilon_t \qquad (t = 0, 1, 2, 3, \cdots, n) \tag{1}$$

where $|\phi| < 1$, and $E(\varepsilon_t) = 0$ and $V(\varepsilon_t) = \sigma^2$. Assume that ε_t's are iid and have the distribution

$$f(\varepsilon\,;\,p) \propto \frac{1}{\sigma}\left\{1 + \frac{\varepsilon^2}{k\sigma^2}\right\}^{-p} \qquad (-\inf < \varepsilon < \inf) \tag{2}$$

where $k = 2p - 1$ and $p \geq 2$. It may be noted that the mean and variance of the distribution (2) is zero and σ^2 respectively, and $t = \sqrt{(k/\nu)}\,(\varepsilon/\sigma)$ has a Student's t distribution with $\nu = 2p - 1$ df (degree of freedom). Conditional on y_0, the likelihood equations for estimating ϕ and σ are (Wong et al 1996)

$$\frac{\partial\, ln\, L}{\partial\, \phi} = \frac{2p}{k\sigma}\sum_{t=1}^{n} y_{t-1} g(z_t) = 0$$

and

$$\frac{\partial\, ln\, L}{\partial\, \sigma} = -\frac{n}{\sigma} + \frac{2p}{k\sigma}\sum_{t=1}^{n} z_t g(z_t) = 0$$

where $z_t = (y_t - \phi y_{t-1})/\sigma$ and $g(z) = z/\{1 + (1/k)z^2\}$. These equations are, however, intractable. To formulate modified likelihood equations (Tiku 1967, Tiku et al 1986, Tiku and Suresh 1992), we order z_t (for a given ϕ) in order of increasing magnitude and denote the ordered z-values by $z_{(i)} = (y_{[i]} - \phi y_{[i]-1})/\sigma$, $1 \leq i \leq n$. It may be noted that $(y_{[i]}, y_{[i]-1})$ is that pair of (y_i, y_{i-1}) observations which constitutes $z_{(i)}$, $1 \leq i \leq n$. Since complete sums are invariant to ordering

$$\frac{\partial \ln L}{\partial \phi} = \frac{2p}{k\sigma} \sum_{i=1}^{n} y_{[i]-1} g\{z_{(i)}\} = 0 \tag{3}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{2p}{k\sigma} \sum_{i=1}^{n} z_{(i)} g\{z_{(i)}\} = 0 \tag{4}$$

Writing $t_{(i)} = E\{z_{(i)}\}$ $(1 \le i \le n)$ and noting that under very general regularity conditions $z_{(i)}$ converges to $t_{(i)}$ as n tends to infinity, and noting the fact that the function $g(z)$ is almost linear in a small interval $c \le z \le d$ (Tiku 1967, 1968), Wong et al (1996) used the following Taylor expansion to linearize $g\{z_{(i)}\}$:

$$\begin{aligned} g\{z_{(i)}\} &\simeq g\{t_{(i)}\} + [z_{(i)} - t_{(i)}] \left\{ \frac{\partial g(z)}{\partial z} \right\}_{z=t_{(i)}} \\ &= \alpha_i + \beta_i \, z_{(i)} \qquad (\, i = 1, \cdots, n \,) \end{aligned} \tag{5}$$

where

$$\alpha_i = \frac{(2/k) t_{(i)}^3}{[1 + (1/k) \, t_{(i)}^2]^2} \qquad \text{and} \qquad \beta_i = \frac{1 - (1/k) t_{(i)}^2}{[1 + (1/k) \, t_{(i)}^2]^2} \tag{6}$$

It may be noted that $\alpha_i = -\alpha_{n-i+1}$ and $\beta_i = \beta_{n-i+1}$, and $\sum_{i=1}^{n} \alpha_i = 0$; this follows from symmetry. Tables of $t_{(i)}$ are available for $n \le 20$ in Tiku and Kumra (1981) and Vaughan (1992b, 1994). For $n > 20$, the values of $t_{(i)}$ are obtained from the equation

$$\int_{-\inf}^{t_{(i)}} f(z) dz = \frac{i}{n+1} \qquad (1 \le i \le n) \, .$$

Note that $\sqrt{(k/\nu)} z$ has Student's t distribution with $\nu = 2p - 1$ df. For $p = \inf$ (normal innovations), $\alpha_i = 0$ and $\beta_i = 1$ for all $i = 1, 2, \cdots, n$.

The modified likelihood equations are obtained by incorporating (5) in (3) and (4):

$$\frac{\partial \ln L}{\partial \phi} \simeq \frac{\partial \ln L^*}{\partial \phi} = \frac{2p}{k\sigma} \sum_{i=1}^{n} y_{[i]-1} \{\alpha_i + \beta_i \, z_{(i)}\} = 0 \tag{7}$$

and

$$\frac{\partial\, ln\, L}{\partial\, \sigma} \simeq \frac{\partial\, ln\, L^*}{\partial\, \sigma} = -\frac{n}{\sigma} + \frac{2p}{k\sigma} \sum_{i=1}^{n} z_{(i)} \{\alpha_i + \beta_i\, z_{(i)}\} = 0 \tag{8}$$

The solutions of (7) and (8) are the MML estimators

$$\hat{\phi} = K + D\hat{\sigma} \qquad \text{and} \qquad \hat{\sigma} = \frac{B + \sqrt{B^2 + 4n\,C}}{2n} \tag{9}$$

where

$$K = \frac{\sum_{i=1}^{n} \beta_i\, y_{[i]}\, y_{[i]-1}}{\sum_{i=1}^{n} \beta_i y_{[i]-1}^2} \qquad , \qquad D = \frac{\sum_{i=1}^{n} \alpha_i\, y_{[i]-1}}{\sum_{i=1}^{n} \beta_i y_{[i]-1}^2}$$

$$B = \frac{2p}{k} \sum_{i=1}^{n} \alpha_i \left\{y_{[i]} - K y_{[i]-1}\right\} \quad \text{and} \quad \begin{aligned} C &= \frac{2p}{k} \sum_{i=1}^{n} \beta_i \left\{y_{[i]} - K y_{[i]-1}\right\}^2 \\ &= \frac{2p}{k} \{ \sum_{i=1}^{n} \beta_i\, y_{[i]}^2 - K \sum_{i=1}^{n} \beta_i\, y_{[i]} y_{[i]-1} \} \end{aligned} \tag{10}$$

It may be noted here that $\beta_i > 0$ $(1 \leq i \leq n)$ for all values of $p > 3$ (Vaughan 1992a). For $p \leq 3$, a few extreme β_i's are negative in which case $\hat{\sigma}$ might cease to be real for some samples. For $p \leq 3$, therefore, whenever $\beta_i < 0$ it is equated to zero and so is the corresponding α_i. The resulting estimator $\hat{\sigma}$ is always real and positive and the estimators $\hat{\phi}$ and $\hat{\sigma}$ generally retain all the optimality properties mentioned above (Vaughan 1992a, Wong et al 1996).

Note that $\hat{\phi}$ is scale invariant. This follows from the fact that K and $D\hat{\sigma}$ are both scale invariant. For studying the distributional properties of $\hat{\phi}$, therefore, σ may be taken to be equal to 1 without any loss of generality.

COMPUTATIONS: In the first place we compute $\hat{\phi}$ from (9) from the order statistics of $y_i - \hat{\phi}_0 y_{i-1}$ $(1 \leq i \leq n)$. We then replace $\hat{\phi}_0$ by $\hat{\phi}$ and use the order statistics of $y_i - \hat{\phi} y_{i-1}$ $(1 \leq i \leq n)$ to compute $\hat{\phi}$ (from 9). Thus, the MML estimates are obtained in two iterations. In all our computations, no more than two iterations were needed for the order statistics of $y_i - \hat{\phi} y_{i-1}$ to stabilize.

3. TESTING FOR UNIT ROOT

For testing $H_0 : \phi = 1$ against $H_1 : \phi < 1$, we use (9) and define the statistic

$$R_1 = \sqrt{n}(2 - \hat{\phi}) \tag{11}$$

Large values of R_1 lead to the rejection of H_0 in favour of H_1. For $p = \text{inf}$ (normal innovations) R_1 reduces to the normal-theory statistic

$$R_0 = \sqrt{n}(2 - \hat{\phi}_0) \quad ; \quad \hat{\phi}_0 = \frac{\sum_{t=1}^{n} y_t \, y_{t-1}}{\sum_{t=1}^{n} y_{t-1}^2} \tag{12}$$

As in Abadir (1995) and Vinod and Shenton (1996) y_0 is taken to be a constant, zero in particular.

It is very difficult to derive the distribution of R_1 or its moments. To investigate the null distribution of R_1 (and R_0), we simulated (from 10,000 runs) their first four moments. Both the distributions turned out to be positively skew. We were pleased to find that their Pearson coefficients $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$ were either close to the Tyle III line or were located in the F-region (Pearson and Tiku 1970, Fig. 1) for all values of n small or large. That makes the following 3-moment chi-square and 4-moment F approximations applicable. The simulated values of the mean, variance and β_1 and β_2 of R_0 and R_1 are given in Table I for $p = 1(.5)$ 5, 7, 10 and inf.

4. THREE-MOMENT CHI-SQUARE APPROXIMATION

Following Pearson (1959) and Tiku (1963, 1966), let μ_1' be the mean of a random variable X and μ_2, μ_3 and μ_4 be its variance and third and fourth central moments, respectively; $\mu_3 > 0$. If the Pearson coefficients β_1 and β_2 satisfy the condition

$$|\beta_2 - (3 + 1.5\,\beta_1)| \leq 0.5 \tag{13}$$

then the distribution of

$$\chi^2 = \frac{X + a}{b} \tag{14}$$

is effectively a central chi-square with ν degrees of freedom. The values of a and b and ν are obtained by equating the first three moments on both sides of (14):

$$\nu = \frac{8}{\beta_1}\ , \ b = \sqrt{\frac{\mu_2}{2\nu}}\ , \text{ and} \qquad a = b\nu - \mu_1'$$

Realize that for a chi-square distribution $\beta_2 = 3 + 1.5\,\beta_1$ which is called the Type III line; see, for examample, Pearson and Tiku (1970, Fig. 1).

5. FOUR-MOMENT F APPROXIMATION

Let (Tiku and Yip 1978)

$$F = \frac{X + g}{h} \tag{15}$$

where F has the central F distribution with (ν_1, ν_2) degrees of freedom. The values of ν_1, ν_2, g and h are determined by equating the first four moments on both sides of (15):

$$\nu_2 = 2\left[3 + \frac{\beta_2 + 3}{\beta_2 - (3 + 1.5\beta_1)}\right]$$

and

$$\nu_1 = \frac{1}{2}(\nu_2 - 2)\left(-1 + \sqrt{1 + \frac{32(\nu_2 - 4)/(\nu_2 - 6)^2}{\beta_1 - 32(\nu_2 - 4)/(\nu_2 - 6)^2}}\right)$$

$$h = \sqrt{\left\{\frac{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)}{2\nu_2^2(\nu_1 + \nu_2 - 2)}\mu_2\right\}} \quad , \qquad g = \frac{\nu_2}{\nu_2 - 2}h - \mu_1'$$

For the F distribution $\nu_1 > 0$ and $\nu_2 > 0$ and, therefore, for (15) to be valid the (β_1, β_2) values of X should satisfy

$$\beta_1 > \frac{32(\nu_2 - 4)}{(\nu_2 - 6)^2} \qquad \text{and} \qquad \beta_2 > 3 + 1.5\beta_1 \tag{16}$$

It is interesting to note that the inequalities (16) determine the F-region in the (β_1 , β_2)–plane bounded by the χ^2-line and the reciprocal of the χ^2-line (Pearson and Tiku 1970, Fig. 1). Whenever the (β_1 , β_2)–points of X lie within this F-region, the four-moment F approximation (15) provides accurate approximations for the probability integral and the percentage points of X (Tiku and Yip 1978). Thus, the $100(1-\alpha)\%$ point of X is approximately $hF_{1-\alpha}(\nu_1, \nu_2) - g$ where $F_{1-\alpha}(\nu_1, \nu_2)$ is the $100(1-\alpha)\%$ point of the central F distribution with (ν_1, ν_2) df.

We were pleased to find that the (β_1 , β_2) values of both R_1 and R_0 satisfy (16) for all values of p and n. For $p \geq 10$, however, the (β_1 , β_2) values generally satisfy the condition (13). Therefore, the chi-square and F approximations above are applicable. To illustrate the accuracy of these approximations, we give below the simulate values (based on 10,000 Monte Carlo runs) of the probabilities

$$(a) \quad P(R_1 \geq d_1 \,|\, \phi = 1) \qquad \text{and} \qquad (b) \quad P(R_0 \geq d_0 \,|\, \phi = 1)$$

where d_1 and d_0 are the 95% points as determined by (15), or (14) if it is applicable.

Simulated Values of The Probabilities

	Critical Value	Prob	Critical Value	Prob	Critical Value	Prob	Critical Value	Prob
	$p=1$		$p=1.5$		$p=2$		$p=2.5$	
$n=30$								
d_1	5.906	0.0374	6.2003	0.0438	6.397	0.046	6.557	0.051
d_0	—	—	6.7565	0.0490	6.765	0.053	6.797	0.050
$n=50$								
d_1	7.3549	0.0384	7.5928	0.0459	7.752	0.049	7.874	0.050
d_0	—	—	8.0868	0.0501	8.104	0.052	8.132	0.050
$n=100$								
d_1	10.1713	0.0407	10.3274	0.0446	10.460	0.048	10.532	0.049
d_0	—	—	10.7608	0.0468	10.778	0.052	10.764	0.051
	$p=3.0$		$p=3.5$		$p=5$		$p=10$	
$n=30$								
d_1	6.5907	0.0494	6.766	0.050	6.828	0.051	6.806	0.049
d_0	6.7944	0.0501	6.832	0.049	6.858	0.050	6.811	0.049
$n=50$								
d_1	7.9476	0.0506	8.022	0.051	8.146	0.047	8.135	0.051
d_0	8.1344	0.0492	8.162	0.050	8.172	0.046	8.141	0.051
$n=100$								
d_1	10.5875	0.0535	10.684	0.051	10.763	0.051	10.768	0.051
d_0	10.7487	0.0531	10.814	0.049	10.793	0.051	10.776	0.052

It may be noted that two-moment normal approximations do not give accurate values and should not be used.

Relative Efficiency: The expected values and variances of $\hat{\phi}$ and $\hat{\phi}_0$ can easily be calculated from the corresponding values of R_1 and R_0 given in Table I. It is seen that both $\hat{\phi}$ and $\hat{\phi}_0$ have small biases, $\hat{\phi}$ having smaller bias than $\hat{\phi}_0$. However, $\hat{\phi}$ is considerably more efficient than $\hat{\phi}_0$ as an estimator of $\phi(=1)$. For example, we have the following values of the relative efficiency:

$$RE(\hat{\phi}_0) = 100Var(\hat{\phi})/Var(\hat{\phi}_0)$$

n	$p = 1$	$p = 1.5$	$p = 2$	$p = 3$	$p = 5$	$p = 10$	$p = \inf$
30	0	52	69	85	96	99	100
50	0	41	61	79	96	99	100
100	0	31	53	74	93	98	100

Wong et al (1996) showed that $\hat{\phi}$ is considerably more efficient than $\hat{\phi}_0$ for all $|\phi| < 1$.

Power Properties: As expected, the R_1 test has considerably higher power than the R_0 test for testing H_0 against H_1. For $n = 50$, for example, the simulated values (based on 10,000 runs) of the power are given below.

Values of The Power

		$\phi = 1$	$\phi = 0.99$	$\phi = 0.95$	$\phi = 0.90$	$\phi = 0.85$	$\phi = 0.80$	$\phi = 0.70$
$p = 1.0$	R_1	0.038	0.062	0.794	0.982	0.995	0.998	1.00
	R_0	—	—	—	—	—	—	—
$p = 1.5$	R_1	0.046	0.063	0.216	0.673	0.899	0.965	1.000
	R_0	0.050	0.065	0.144	0.332	0.611	0.847	0.990
$p = 2.0$	R_1	0.049	0.064	0.181	0.479	0.770	0.916	0.989
	R_0	0.052	0.064	0.145	0.336	0.597	0.820	0.985
$p = 3.5$	R_1	0.051	0.065	0.152	0.356	0.620	0.829	0.981
	R_0	0.050	0.062	0.143	0.313	0.558	0.781	0.977
$p = 5.0$	R_1	0.047	0.058	0.144	0.325	0.570	0.792	0.981
	R_0	0.046	0.058	0.143	0.316	0.551	0.771	0.975
$p = 10.0$	R_1	0.051	0.064	0.148	0.337	0.584	0.798	0.978
	R_0	0.051	0.062	0.148	0.334	0.578	0.794	0.977

Comment: All the above results also apply if the model is $y_t = \mu + \phi y_{t-1} + \varepsilon_t$. The normal-theory estimator is now

$$\hat{\phi}_0 = \sum_{t=1}^{n} y_t(y_{t-1} - \bar{y}) / \sum_{t=1}^{n} (y_{t-1} - \bar{y})^2$$

and the MML estimators are exactly the same as (9)–(10) with $y_{[i]}$ replaced by $w_{[i]}$ and $y_{[i]-1}$ replaced by $w_{[i]-1}$;

$$w_{[i]} = y_{[i]} - \frac{1}{m} \sum_{i=1}^{n} \beta_i y_{[i]} \qquad\qquad (m = \sum_{i=1}^{n} \beta_i)\,.$$

The techniques given in this paper can also be used for any location-scale distribution of the type $(1/\sigma) f((y - \theta)/\sigma)$. We do not, however, reproduce the details for conciseness.

REFERENCES

Abadir K.M. (1995) The limiting distribution of the t ratio under a unit root. *Econometric Theory* 11, 775-793.

Bhattacharyya, G.K. (1985) The asymptotics of maximum likelihood and related estimators based on Type II censored data. *J. Amer. Statist. Assoc.* 80, 398-404.

Dickey, D.A. and W.A. Fuller (1979) Distribution of the estimators for autoregressive time series with unit root. *Journal of American Statistical Association* 74, 427-431.

Huber, P.J. (1981) *Robust Statistics.* John Wiley, New York.

Lee, K.R., C.H. Kapadia and B.B. Dwight. (1980) On estimating the scale parameter of Rayleigh distribution from censored samples. *Statist. Hefte* 21, 14-20.

Pearson, E.S. (1959) Note on an approximation to the distribution of noncentral chi-square. *Biometrika* 46, 364.

Pearson, E.S. and M.L. Tiku (1970) Some notes on the relationship between the distributions of central and noncentral F. *Biometrika* 57, 175-179.

Phillips, P.C.B. and P. Perron (1988) Testing for a unit root in time series regression. *Biometrika* 75, 335-346.

Tan W.Y. (1985) On Tiku's robust procedure – a Bayesian insight. *J. Statist. Plann. and Inf.* 11, 329-340.

Tiku, M.L. (1963) Chi-square approximations for the distributions of goodness-of-fit statistics U_N^2 and W_N^2. *Biometrika* 52, 630-633.

Tiku, M.L. (1966) Distribution of the derivative of the likelihood function. *Nature* 20, 766.

Tiku, M.L. (1967) Estimating the mean and standard deviation from a censored normal sample. *Biometrika* 54, 155-65.

Tiku, M.L. (1968) Estimating the parameters of normal and logistic distributions from censored samples. *Aust. J. Statist.* 10, 64-74.

Tiku, M.L. and D.Y.N. Yip (1978) A four-moment approximation based on the F distribution. *Aust. J. Statist.* 20, 257-261.

Tiku, M.L. and S. Kumra (1981) Expected values and variances and covariances of order statistics for a family of symmetrical distributions (Student's t). *Selected Tables in Mathematical Statistics 8, American mathematical Society, Providence,* RI: 141-270.

Tiku, M.L. and R.P. Suresh (1992) A new method of estimation for location and scale parameters. *J. Stat. Plann. and Inf.* 30, 281-292.

Tiku, M.L., W.Y. TAN and N. Balakrishnan (1986) *Robust Inference.* New York, Marcel Dekker.

Vaughan, D.C. (1992a) On the Tiku-Suresh method of estimation. *Commun. Stat. Theory Meth.* 21, 451-69.

Vaughan, D.C. (1992b) Expected values, variances and covariances of order statistics for Student's t distribution with two degrees of freedom. *Commun. Stat. Simul.* 21, 391-404.

Vaughan, D.C. (1994) The exact values of the expected values, variances and covariances of the order statistics from the Cauchy distribution. *J. Stat. Comput. Simul.* 49, 21-32.

Vinod, H.D. and L.R. Shenton (1996) Exact moments for autoregressive and random walk models for a zero or stationary initial value. *Econometric Theory* 12, 481-499.

Wong, W.K., M.L. Tiku and G. Bian (1996) Time series models with nonnormal innovations, symmetric location-scale distributions, Working Paper, Department of Mathematics and Statistics, McMaster University, Canada.

Table I: The simulated values of μ, σ^2, and β_1 and β_2 of R_0 and R_1

n		Mean	Var	β_1	β_2	Mean	Var	β_1	β_2
		$p = 1$				$p = 1.5$			
30	R_0	—	—	—	—	5.7388	0.2682	3.8871	9.2576
	R_1	5.4887	0.0516	11.1708	30.7200	5.5049	0.1385	9.2339	23.2553
50	R_0	—	—	—	—	7.2872	0.1696	4.2007	10.2569
	R_1	7.0859	0.0215	13.7876	36.2597	7.0916	0.0702	6.1192	15.5394
100	R_0	—	—	—	—	10.1590	0.0916	4.6004	10.1320
	R_1	10.0150	0.0072	17.5700	38.2890	10.0127	0.0287	7.5301	20.4420
		$p = 2$				$p = 2.5$			
30	R_0	5.7588	0.2730	3.6985	9.5893	5.7629	0.2748	3.4887	8.4181
	R_1	5.5749	0.1882	6.5534	16.2716	5.6428	0.2209	5.0709	11.7930
50	R_0	7.2928	0.1702	3.7902	9.0622	7.3011	0.1762	3.9102	9.0919
	R_1	7.1304	0.1041	5.4380	13.0050	7.1849	0.1288	6.0126	14.2328
100	R_0	10.1677	0.0934	4.1994	9.3170	10.1687	0.0926	4.8093	11.0821
	R_1	10.0363	0.0493	6.3671	15.3403	10.0688	0.0590	6.0359	14.7356
		$p = 3$				$p = 3.5$			
30	R_0	5.7637	0.2731	3.4604	8.3738	5.7731	0.2924	4.3261	10.1604
	R_1	5.6540	0.2311	4.9797	11.5588	5.7536	0.2678	4.7326	10.9559
50	R_0	7.3062	0.1758	3.9848	9.2696	7.3111	0.1868	4.3091	9.9673
	R_1	7.2286	0.1386	4.9011	11.8297	7.2431	0.1589	4.8786	11.2411
100	R_0	10.1656	0.0912	5.0621	12.1412	10.1725	0.1032	5.2057	11.0322
	R_1	10.0936	0.0675	5.8366	14.5538	10.1185	0.0836	6.0691	13.3377
		$p = 4$				$p = 4.5$			
30	R_0	5.7771	0.2877	3.4583	8.0111	5.7774	0.3023	4.3361	10.0078
	R_1	5.7625	0.2709	3.7087	8.4560	5.7657	0.2864	4.5096	10.2688
50	R_0	7.3038	0.1739	4.0331	9.6776	7.3079	0.1795	3.9521	9.1516
	R_1	7.2878	0.1575	4.3920	10.6644	7.2958	0.1680	4.2322	9.8300
100	R_0	10.1742	0.0991	4.7483	10.5139	10.1765	0.0968	4.0575	8.8664
	R_1	10.1382	0.0845	4.8821	10.5965	10.1439	0.0853	4.2665	9.1028
		$p = 5$				$p = 7$			
30	R_0	5.7775	0.2950	3.6252	8.3788	5.7743	0.2828	3.5969	8.6320
	R_1	5.7695	0.2840	3.8134	8.8158	5.7700	0.2787	3.7885	9.0781
50	R_0	7.3140	0.1857	4.0882	9.2250	7.3176	0.1845	3.6729	8.3634
	R_1	7.3051	0.1773	4.2352	9.3813	7.3128	0.1799	3.7192	8.4439
100	R_0	10.1759	0.0992	4.7672	10.9409	10.1706	0.0939	4.5356	9.9487
	R_1	10.1663	0.0919	4.7938	10.8064	10.1658	0.0900	4.3960	9.7693
		$p = 10$				$p = \infty$			
30	R_0	5.7739	0.2752	3.0630	7.4837	5.7829	0.2880	3.2171	7.6168
	R_1	5.7724	0.2731	3.1130	7.5842	5.7829	0.2880	3.2171	7.6168
50	R_0	7.3108	0.1778	3.9508	9.3170	7.3070	0.1814	3.8017	8.5576
	R_1	7.3087	0.1758	4.0004	9.3802	7.3070	0.1814	3.8017	8.5576
100	R_0	10.1705	0.0921	4.0238	9.0099	10.1751	0.1003	5.3447	11.8654
	R_1	10.1685	0.0904	3.9430	8.8117	10.1751	0.1003	5.3447	11.8654